Circuit Analysis
vis-à-vis
Electromagnetic Field Analysis

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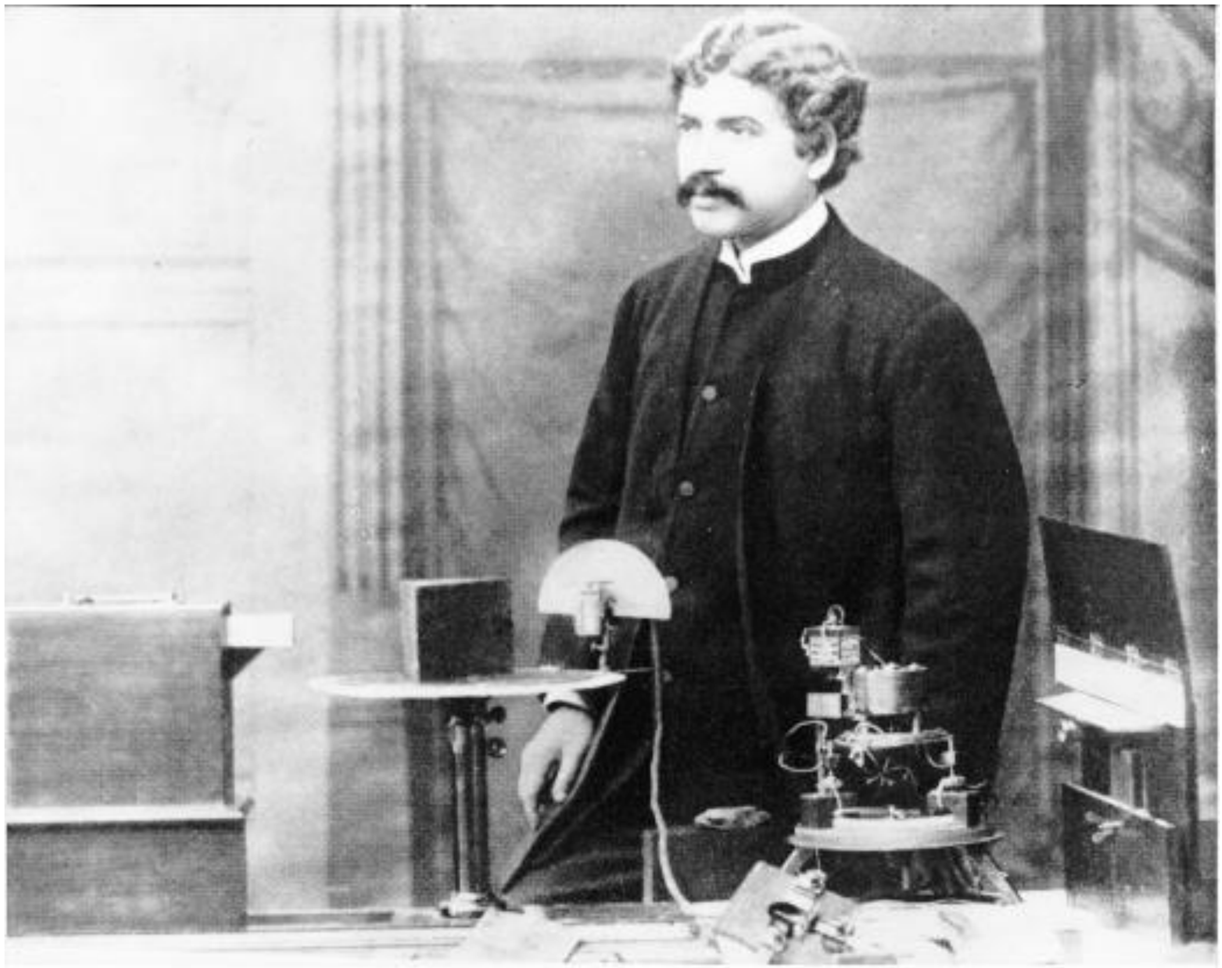
"For whatever may be said about the importance of aiming at depth rather than width in our studies, and however strong the demand of the present age may be for specialists, there will be work, not only for those who build up particular sciences and write monographs on them, but for those who open up such communications between the different groups of builders as will facilitate a healthy interaction between them."

— James Clerk Maxwell

J.C. Bose (1858-1937) at the Royal Institution, London, 1897



In 1895 Bose gave his first public demonstration of electromagnetic waves, using them to ring a bell remotely and to explode some gunpowder. In 1896 the Daily Chronicle of England reported: ***"The inventor (J.C. Bose) has transmitted signals to a distance of nearly a mile and herein lies the first and obvious and exceedingly valuable application of this new theoretical marvel.*"**

“Popov in Russia was doing similar experiments, but had written in December 1895 that he was still entertaining the hope of remote signaling with radio waves.”

“The first successful wireless signaling experiment by Marconi on Salisbury Plain in England was not until May 1897.”

Source: D. T. Emerson, “The work of Jagadis Chunder Bose: 100 years of mm-wave research,” *IEEE Trans. Microwave Th. Tech.* December 1997, 45, No. 12 (2267-2273)

There is only one nature — the division into science and engineering is a human imposition, not a natural one. Indeed, the division is a human failure; it reflects our limited capacity to comprehend the whole.

— Sir William Cecil Dampier

Electromagnetic theory → Science

Circuit theory → Engineering

Maxwell's equations

$$\nabla . \vec{D} = \rho$$

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

"Simple enough to imprint on a T-shirt, and yet rich enough to provide new insights throughout a lifetime of study"

— J. R. Whinnery

"The teaching of Electromagnetics," *IEEE Trans. Education*, Vol. 33, pp. 3-7 (1990)

Whinnery's T-shirt has enough space to accommodate both Maxwell's equations and electromagnetic boundary conditions

$$\nabla . \vec{D} = \rho$$

$$\nabla . \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_S$$

$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

General boundary conditions

$$\rho_s = Lt\rho dh$$
$$dh \to 0$$

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_S$$
$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$
$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$$
$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

$$\vec{J}_S = Lt\vec{J}dh$$
$$dh \to 0$$

Dielectric (1)-dielectric (2) interface

Both time-dependent and time-independent

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = 0$$
$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$
$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = 0$$
$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

Conductor (1)-dielectric (2) interface

Time-independent

$$\vec{D}_2.\vec{a}_n = \rho_S$$
$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$
$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = 0$$
$$\vec{a}_n \times \vec{E}_2 = 0$$

Time-dependent

$$\vec{D}_2.\vec{a}_n = \rho_S$$
$$\vec{B}_2.\vec{a}_n = 0$$
$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$$
$$\vec{a}_n \times \vec{E}_2 = 0$$

Electromagnetic theory and circuit theory are the two sides of the same coin

Electromagnetic theory		Circuit theory
$\vec{J} = \sigma\vec{E}$	**Ohm's law**	$V = IR$
$\vec{S} = \vec{E} \times \vec{H}$ Poynting vector	**Joule's law**	I^2R power loss
$\vec{a}_n \times (E_2 - E_1) = 0$ Electromagnetic boundary condition	**Law of parallel resistances**	$\frac{1}{R_{\text{equivalent}}} = \frac{1}{R_1} + \frac{1}{R_2}$

<u>Electromagnetic theory</u>

<u>Circuit theory</u>

Dispersion relation of a hollow-pipe waveguide

Boundary-value problem

Transmission line equivalent circuit

$$\omega^2 - \beta^2 c^2 - \omega_c^2 = 0$$

(dispersion relation of a waveguide)

Electromagnetic theory

Circuit theory

Dispersion relation of a helical slow-wave structure of a travelling-wave tube

Boundary-value problem

$$\beta^2 = \omega^2 LC$$

Transmission line equivalent circuit

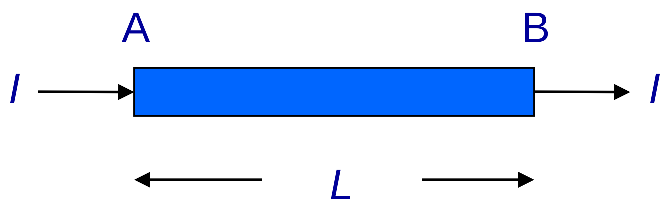
$$\frac{k\cot\psi}{\gamma} = \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)} \right)^{1/2}$$

(Helix in free space)

(dispersion relation of a helix in free-space)

Ohm’s Law

$$z = 0 \qquad z = l$$



$$\vec{E} = -\nabla V = \frac{-\partial V}{\partial x}\vec{a}_x + \frac{-\partial V}{\partial y}\vec{a}_y + \frac{-\partial V}{\partial z}\vec{a}_z$$

(wire of length *L* offering a resistance to the current flow)

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$$

$$\vec{E} = -\frac{\partial V}{\partial z}\vec{a}_z$$

$$\vec{J} = \sigma\vec{E}$$

(Ohm’s electromagnetic law)

$$E = -\frac{\partial V}{\partial z} \longrightarrow J = \sigma E$$

$$J = -\sigma\frac{\partial V}{\partial z}$$

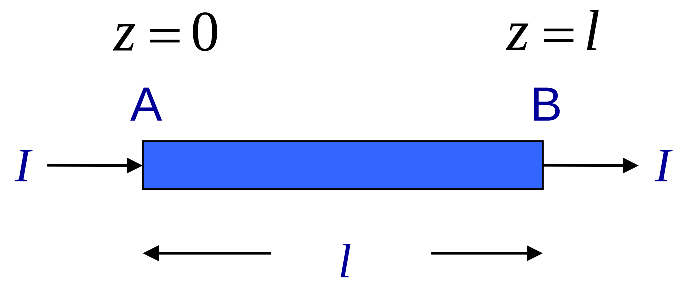
$$-\frac{\partial V}{\partial z} = \frac{J}{\sigma}$$

$$-\frac{\partial V}{\partial z} = \frac{J}{\sigma}$$

$$-\int_{V_B}^{V_A} dV = \int_{l}^{0} \frac{J}{\sigma} dz$$

$$\downarrow$$

$$V_A - V_B = V = \frac{J}{\sigma} l$$



A is the cross-sectional area of the linear conductor (a piece of wire offering a resistance to the current flow)

$$V = \frac{I/A}{\sigma} l = \frac{I}{\alpha A} l = I \frac{1}{\sigma} \frac{l}{A} = IR \quad \longleftarrow \quad \frac{1}{\sigma}\frac{l}{A} = R$$

$$V = IR$$

(Ohm's circuital law)

Joule's Law

Take a wire of circular cross section:

$$J_z = \sigma E_z \longleftarrow J_z\vec{a}_z = \sigma E_z \vec{a}_z \longleftarrow \vec{J} = \sigma\vec{E}$$

Poynting vector

$$\vec{E} = E_z\vec{a}_z$$

$$\vec{S} = \vec{E}\times\vec{H}$$

$$E_z = \frac{J_z}{\sigma}$$

$$\vec{H} = H_\theta \vec{a}_\theta$$

$$\vec{S} = E_z\vec{a}_z \times H_\theta\vec{a}_\theta = -E_z H_\theta \vec{a}_r$$

$$H_\theta = \frac{I}{2\pi a}$$

(Ampere's law)

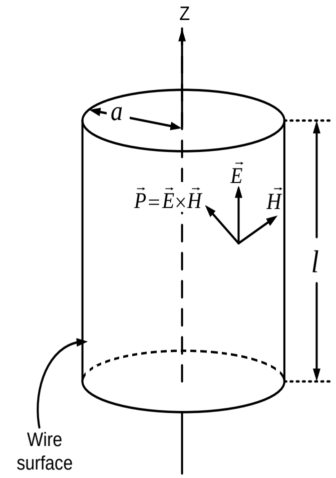
$$\vec{S} = -E_z H_\theta \vec{a}_r = -\frac{J_z}{\sigma}\frac{I}{2\pi a}\vec{a}_r$$

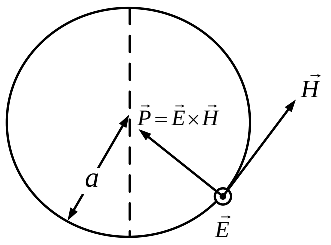
Power density flow into the wire

$$= \frac{J_z}{\sigma}\frac{I}{2\pi a}$$

$$J_z = \frac{I}{\alpha}$$

$$= \frac{1}{\sigma} J_z \frac{I}{2\pi a} = \frac{1}{\sigma}\frac{I}{\alpha}\frac{I}{2\pi a}$$

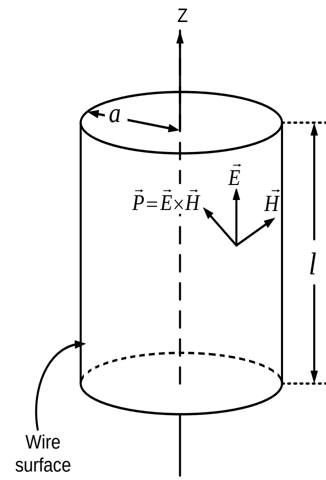


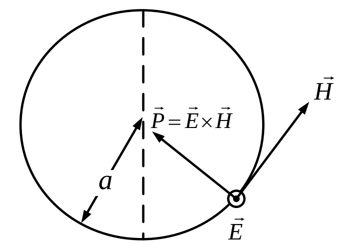


Power density flow into the wire

$$= \frac{J_z}{\sigma}\frac{I}{2\pi a} \longleftarrow J_z = \frac{I}{\alpha}$$

$$= \frac{1}{\sigma} J_z \frac{I}{2\pi a} = \frac{1}{\sigma}\frac{I}{\alpha}\frac{I}{2\pi a}$$





Power flow over / (length) into the wire $= (\frac{1}{\sigma}\frac{I}{\alpha}\frac{I}{2\pi a})(2\pi a l) = I^2(\frac{1}{\sigma}\frac{l}{\alpha}) = I^2 R \longleftarrow \frac{1}{\sigma}\frac{l}{\alpha} = R$

Joule's circuit loss $= I^2 R$

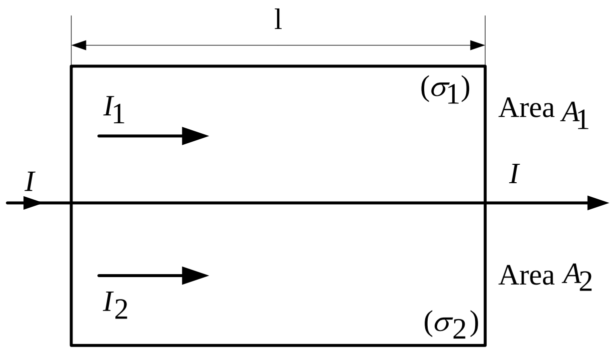
Circuit law of parallel resistances with the help of the boundary condition that the tangential component of electric field is continuous at the interface between two media

For this purpose, the said boundary condition is applied at the interface between two rectangular conducting slabs in contact of the same length *l*, conductivities σ_1 and σ_2 and cross-sectional areas A_1 and A_2 respectively.

Current densities J_1 and J_2 are related to electric fields E_1 and E_2 in the slabs through Ohm's law while the current *I* fed into the slabs in contact is divided in currents I_1 and I_2 through the slabs.

$$J_1 = \sigma_1 E_1 \qquad J_2 = \sigma_2 E_2$$

$$E_1 = \frac{J_1}{\sigma_1} \qquad E_2 = \frac{J_2}{\sigma_2}$$



E_1 and E_2, which are tangential at the interface, are continuous at the interface:

$$E_1 = E_2$$ **(boundary condition)**

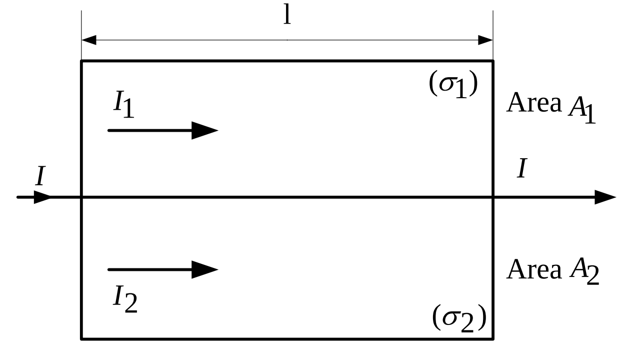
$$\downarrow$$

$$\frac{J_1}{\sigma_1} = \frac{J_2}{\sigma_2}$$

$$\downarrow$$

$$\frac{I_1}{A_1\sigma_1} = \frac{I_2}{A_2\sigma_2}$$

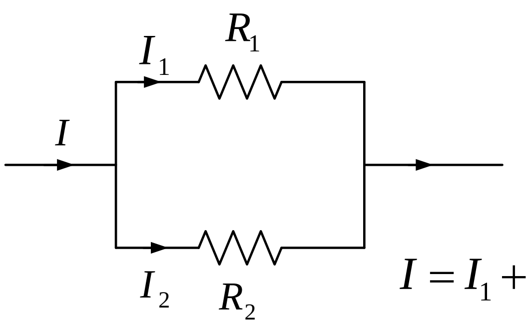
$$I = I_1 + I_2 \longrightarrow$$



$$\downarrow$$ **Multiplying by *l***

$$I_1 \frac{l}{A_1\sigma_1} = I_2 \frac{l}{A_2\sigma_2}$$

$$I = I_1 + I_2 \longrightarrow$$



$$I = I_1 + I_2$$

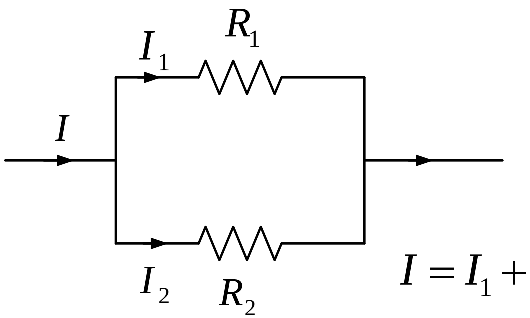
$$R_1 = \frac{1}{\sigma_1}\frac{l}{A_1}$$

$$R_2 = \frac{1}{\sigma_2}\frac{l}{A_2}$$

$$\downarrow \longleftarrow$$

$$I_1 R_1 = I_2 R_2$$

$$I_1 R_1 = I_2 R_2 \longrightarrow \frac{I_1}{I_2} = \frac{R_2}{R_1}$$



$$I = I_1 + I_2$$

$$\downarrow$$

$$\frac{I_1}{I_1 + I_2} = \frac{R_2}{R_2 + R_1} = \frac{R_2}{R_1 + R_2}$$

$$I = I_1 + I_2$$

$$\downarrow$$

$$\frac{I_1}{I} = \frac{R_2}{R_1 + R_2}$$

$$\downarrow$$

$$I_1 = \frac{R_2}{R_1 + R_2} I$$

Multiplying by R_1

$$\longrightarrow I_1 R_1 = I \frac{R_1 R_2}{R_1 + R_2} = I R_{\text{equivalent}}$$

$$\downarrow$$

$$R_{\text{equivalent}} = \frac{R_1 R_2}{R_1 + R_2}$$

$$\downarrow$$

$$\frac{1}{R_{\text{equivalent}}} = \frac{1}{R_1} + \frac{1}{R_2}$$

(Law of parallel resistances)

We have seen that the circuit theory concepts, namely, Ohm's law, Joule's law and law of parallel resistances can be appreciated by electromagnetic field theory as well.

Will electromagnetic field theory and circuit theory yield one and the same dispersion relation of a hollow-pipe waveguide?

Will electromagnetic field theory and circuit theory yield one and the same dispersion relation of a helix used as a slow-wave structure of a travelling-wave tube? ?

Transmission-line equivalent of a rectangular waveguide excited typically in the TE modes

E.C Jordan: Electromagnetic wave and Radiating Systems. Prentice-Hall of India, New Delhi, 1986. Chapter 8.

B.N. Basu: Engineering Electromagnetics Essentials. Universities Press, Hyderabad, 2015. Chapters 9 and 10.

Rectangular waveguide

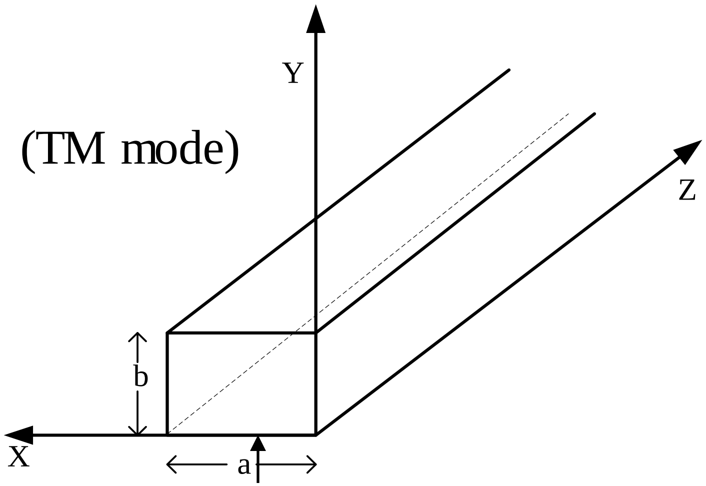
The TE mode is also known as the H mode since it is associated with a non-zero value of the axial magnetic field:

The TM mode is also known as the E mode since it is associated with a non-zero value of the axial electric field:

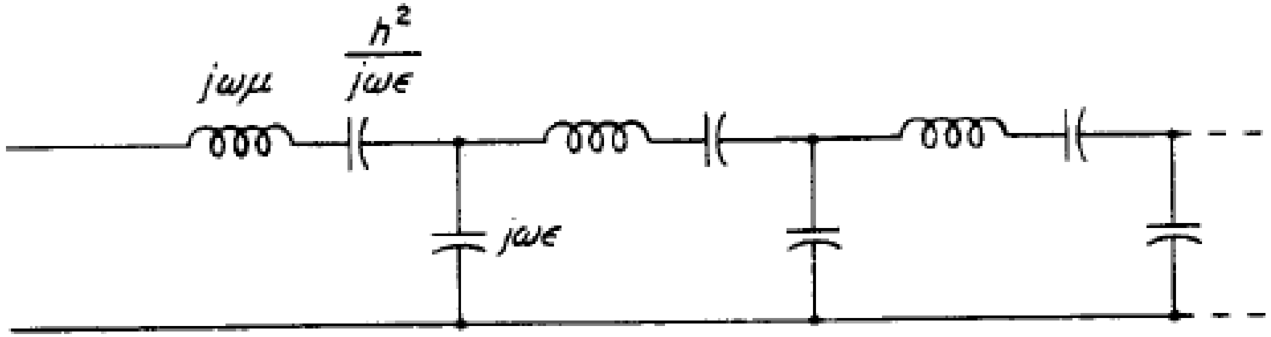
Wave equations

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} \quad (H_z \neq 0, E_z = 0) \quad \text{(TE mode)}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2} \quad (E_z \neq 0, H_z = 0) \quad \text{(TM mode)}$$

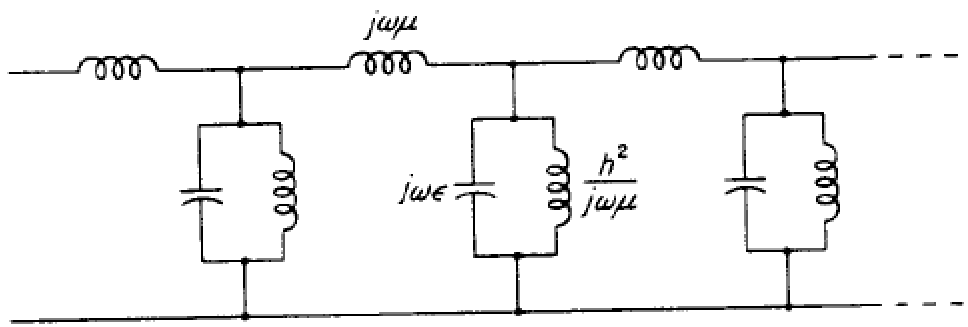


Rectangular waveguide with its right side wall located at $x = 0$**; left side at** $x = a$**; bottom wall at** $y = 0$ **and top wall at** $y = b$**.**



Equivalent transmission-line circuit representation for TM waves.

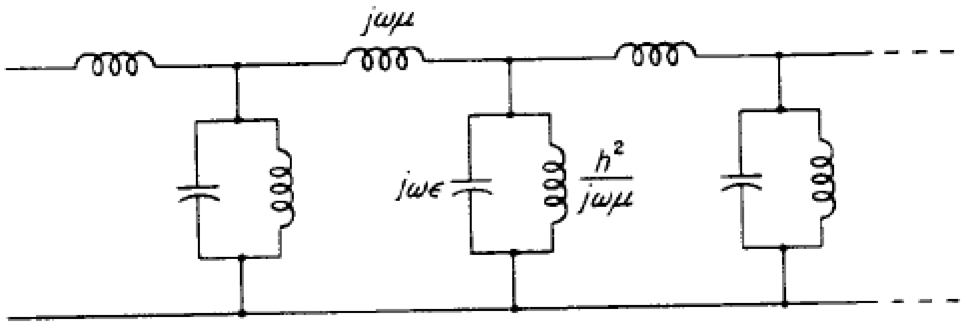
$$\boxed{h^2 = k^2 - \beta^2}$$



Equivalent transmission-line circuit representation for TE waves.

E.C Jordan: Electromagnetic wave and Radiating Systems. Prentice-Hall of India, New Delhi, 1986. Chapter 8.

$$\boxed{h^2 = k^2 - \beta^2}$$



Equivalent transmission-line circuit representation for TE waves.

E.C Jordan: Electromagnetic wave and Radiating Systems. Prentice-Hall of India, New Delhi, 1986. Chapter 8, p.266.

Field quantities varying as

$$\exp\ j(\omega t - \beta z)$$

μ **and** ε **may be taken as** μ_0 **and** ε_0**, respectively.**

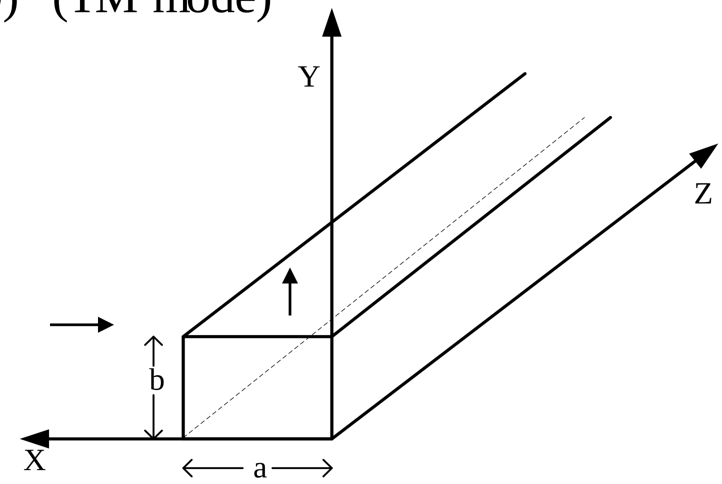
Rectangular waveguide

Wave equations

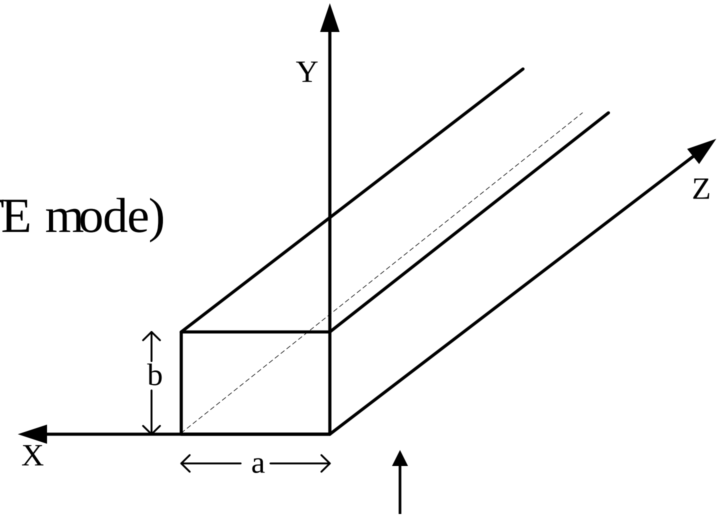
$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} \quad (H_z \neq 0, E_z = 0) \quad \text{(TE mode)}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2} \quad (E_z \neq 0, H_z = 0) \quad \text{(TM mode)}$$

Rectangular waveguide with its right side wall located at $x = 0$**; left side at** $x = a$**; bottom wall at** $y = 0$ **and top wall at** $y = b$**.**



TE mode



Rectangular waveguide with its right side wall located at $x = 0$**; left side at** $x = a$**; bottom wall at** $y = 0$ **and top wall at** $y = b$**.**

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} \quad (H_z \neq 0, E_z = 0) \quad (\text{TE mode})$$

(wave equation)

Field quantities varying as

$$\exp\, j(\omega t - \beta z)$$

$$\partial / \partial t = j\omega$$

$$\partial / \partial z = -j\beta$$

$$\left.\begin{aligned} \frac{\partial^2 H_z}{\partial z^2} &= (-j\beta)(-j\beta) H_z = j^2 \beta^2 H_z = -\beta^2 H_z \\ \frac{\partial^2 H_z}{\partial t^2} &= (j\omega)(j\omega) H_z = j^2 \omega^2 H_z = -\omega^2 H_z \end{aligned}\right\}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} - \beta^2 H_z = -\mu_0 \varepsilon_0 \omega^2 H_z$$

$$k = \omega(\mu_0 \varepsilon_0)^{1/2}$$

(free - space propagation constant)

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (k^2 - \beta^2) H_z = 0$$

(wave equation)

Transmission line equivalent of a waveguide

TE mode:

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon_0 E_x$$

$$E_x = \frac{-j\omega\mu_0}{k^2 - \beta^2}\frac{\partial H_z}{\partial y}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\omega^2\mu_0\varepsilon_0}{k^2 - \beta^2}\frac{\partial H_z}{\partial y}$$

Introduce scalar magnetic potential *U* such that $H_y = -\frac{\partial U}{\partial y}$

$$\frac{\partial H_z}{\partial y} + \frac{\partial}{\partial z}\frac{\partial U}{\partial y} = \frac{\omega^2\mu_0\varepsilon_0}{k^2 - \beta^2}\frac{\partial H_z}{\partial y}$$

U has the dimension of current

$$\frac{\partial H_z}{\partial y} + \frac{\partial}{\partial y}\frac{\partial U}{\partial z} = \frac{\omega^2\mu_0\varepsilon_0}{k^2 - \beta^2}\frac{\partial H_z}{\partial y}$$

$$H_z + \frac{\partial U}{\partial z} = \frac{\omega^2\mu_0\varepsilon_0}{k^2 - \beta^2}H_z$$

$$\frac{\partial U}{\partial z} = \left(\frac{\omega^2\mu_0\varepsilon_0}{k^2 - \beta^2} - 1\right)H_z$$

TE mode:

$$\frac{\partial U}{\partial z} = \left(\frac{\omega^2 \mu_0 \varepsilon_0}{k^2 - \beta^2} - 1 \right) H_z$$ **(rewritten)** $\longleftarrow$ $$h^2 = k^2 - \beta^2$$

$\downarrow$

$$\frac{\partial U}{\partial z} = \left(\frac{\omega^2 \mu_0 \varepsilon_0}{h^2} - 1 \right) H_z$$

$\downarrow$

$$\boxed{\frac{\partial U}{\partial z} = -\left(\frac{h^2}{j\omega\mu_0} + j\omega\varepsilon_0 \right) \left(\frac{j\omega\mu_0}{h^2} \right) H_z}$$ **(to be recalled later)**

TE mode:

$$\frac{\partial E_y}{\partial z} = j\omega\mu_0 H_x$$

$$E_y = \frac{j\omega\mu_0}{k^2-\beta^2}\frac{\partial H_z}{\partial x}$$

$$\frac{\partial}{\partial z}\frac{j\omega\mu_0}{k^2-\beta^2}\frac{\partial H_z}{\partial x} = j\omega\mu_0 H_x$$

$$H_x = -\frac{\partial U}{\partial x}$$

$$h^2 = k^2 - \beta^2$$

$$\frac{\partial}{\partial z}\frac{j\omega\mu_0}{h^2}\frac{\partial H_z}{\partial x} = -j\omega\mu_0\frac{\partial U}{\partial x}$$

$$\frac{\partial}{\partial x}\frac{j\omega\mu_0}{h^2}\frac{\partial H_z}{\partial z} = -j\omega\mu_0\frac{\partial U}{\partial x}$$

$$\frac{j\omega\mu_0}{h^2}\frac{\partial H_z}{\partial z} = -j\omega\mu_0 U$$

$$\boxed{\frac{\partial}{\partial z}\frac{j\omega\mu_0}{h^2}H_z = -j\omega\mu_0 U}$$

(to be recalled later)

TE mode:

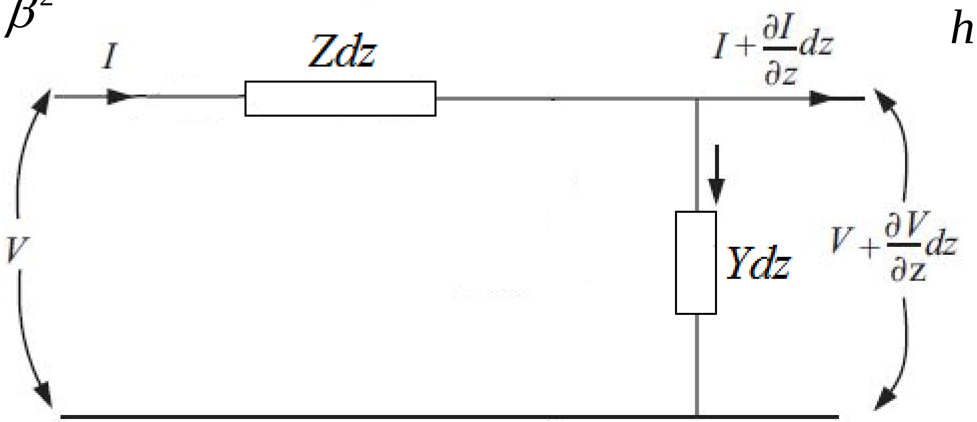
$$\boxed{\frac{\partial}{\partial z}\frac{j\omega\mu_0}{h^2}H_z = -j\omega\mu_0 U}$$

$\longleftarrow$ **Recalled** $\longrightarrow$

$$\boxed{\frac{\partial U}{\partial z} = -\left(\frac{h^2}{j\omega\mu_0} + j\omega\varepsilon_0\right)\left(\frac{j\omega\mu_0}{h^2}\right)H_z}$$

$$h^2 = k^2 - \beta^2$$

$$h^2 = k^2 - \beta^2$$



$$\frac{\partial V_1}{\partial z} = -ZI_1 \longleftarrow V_1 - (V_1 + \frac{\partial V_1}{\partial z}dz) = I_1 Zdz$$

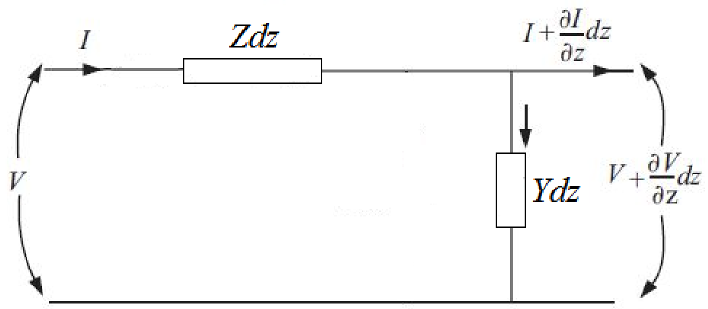
$$\frac{\partial I_1}{\partial z} = -YV_1 \longleftarrow I_1 - (I_1 + \frac{\partial I_1}{\partial z}dz) = V_1 Ydz$$

$$\longrightarrow \frac{\partial^2 V_1}{\partial z^2}(= -Z\frac{\partial I_1}{\partial z}) = ZYV_1$$

$$\gamma^2 = \sqrt{ZY}$$

$$\frac{\partial^2 V_1}{\partial z^2}(= -Z\frac{\partial I_1}{\partial z}) = ZYV_1 \longrightarrow \frac{\partial^2 V_1}{\partial z^2} = \gamma^2 V_1$$ **(transmission line equation)**

TE mode:



$$\frac{\partial V_1}{\partial z} = -ZI_1$$

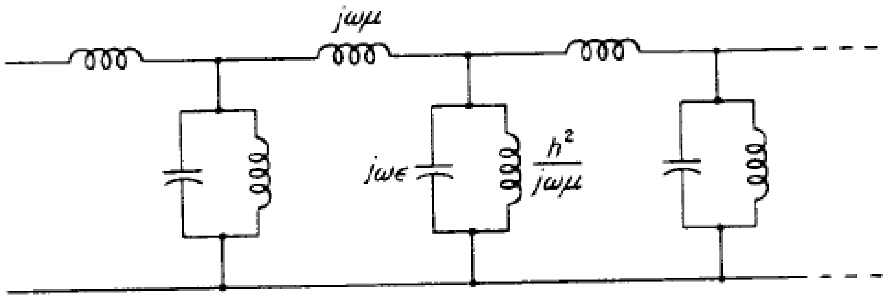
$$\boxed{\frac{\partial}{\partial z}\frac{j\omega\mu_0}{h^2}H_z = -j\omega\mu_0 U}$$ **(recalled)**

$$h^2 = k^2 - \beta^2$$

$$\left.\begin{aligned} V_1 &= \frac{j\omega\mu_0}{h^2}H_z \\ I_1 &= U \\ Z &= j\omega\mu_0 \end{aligned}\right\}$$

U has the dimension of current and $j\omega\mu_0/h^2$ has the dimension of voltage

TE mode:



Reactance per unit length of a capacitor

$$\frac{\partial I_1}{\partial z} = -YV_1$$

$$\boxed{\frac{\partial U}{\partial z} = -\left(\frac{h^2}{j\omega\mu_0} + j\omega\varepsilon_0\right)\left(\frac{j\omega\mu_0}{h^2}\right)H_z}$$ **(recalled)**

Reactance per unit length of an inductor

$$h^2 = k^2 - \beta^2$$

$$\left.\begin{aligned} V_1 &= \frac{j\omega\mu_0}{h^2}H_z \\ I_1 &= U \\ Y &= \frac{h^2}{j\omega\mu_0} + j\omega\varepsilon_0 \end{aligned}\right\}$$

Waveguide dispersion relation

The waveguide cutoff frequency corresponds to $Y = 0$

$$Y = \frac{h^2}{j\omega\mu_0} + j\omega\varepsilon_0 = 0 \quad \longleftarrow \quad h^2 = k^2 - \beta^2$$

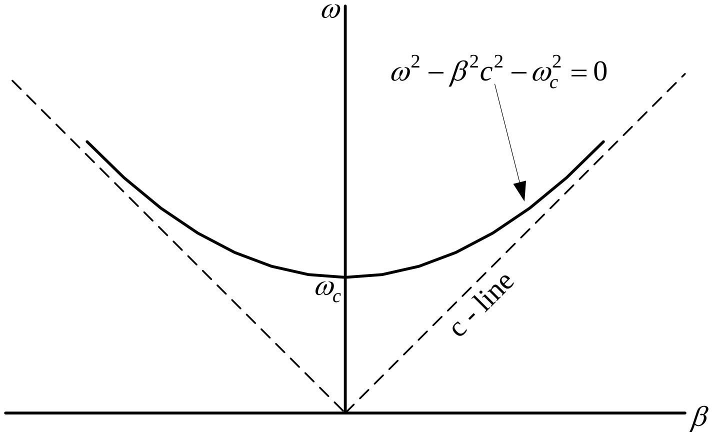
$$\Big\downarrow \quad \longleftarrow \quad \omega = \omega_c$$

$$\boxed{k^2 - \beta^2 - \omega_c^2\mu_0\varepsilon_0 = 0}$$

$$\Big\downarrow \quad \longleftarrow \quad \left.\begin{aligned} k &= \omega / c \\ c^2 &= \frac{1}{\mu_0\varepsilon_0} \end{aligned}\right\}$$

$$\boxed{\omega^2 - \beta^2c^2 - \omega_c^2 = 0}$$

(waveguide dispersion relation)



Characteristic impedance of the transmission line equivalent of a waveguide

$$\boxed{\omega^2-\beta^2c^2-\omega_c^2=0}$$

(dispersion relation)

$$\omega^2-\beta^2c^2=\omega_c^2$$

$$\boxed{Z=j\omega\mu_0}\qquad \boxed{Y=\frac{h^2}{j\omega\mu_0}+j\omega\varepsilon_0}$$

$$Z_0=\sqrt{\frac{Z}{Y}}=\sqrt{\frac{j\omega\mu_0}{\frac{h^2}{j\omega\mu_0}+j\omega\varepsilon_0}}=\sqrt{\frac{j\omega\mu_0}{j\omega\varepsilon_0(\frac{h^2}{j\omega\mu_0}\frac{1}{j\omega\varepsilon_0}+1}}\longleftarrow h^2=k^2-\beta^2$$

$$Z_0=\sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{1}{1-\frac{h^2}{\omega^2\mu_0\varepsilon_0}}}==\sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{1}{1-\frac{k^2-\beta^2}{\omega^2\mu_0\varepsilon_0}}}$$

$$\left.\begin{aligned}k^2c^2&=\omega^2\\ \omega^2-\beta^2c^2&=\omega_c^2\end{aligned}\right\}$$

$$Z_0=\sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{1}{1-\frac{k^2c^2-\beta^2c^2}{\omega^2}}}=\sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{1}{1-\frac{\omega^2-(\omega^2-\omega_c^2)}{\omega^2}}}$$

$$Z_0=\sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{1}{1-(\omega_c^2/\omega^2)}}$$

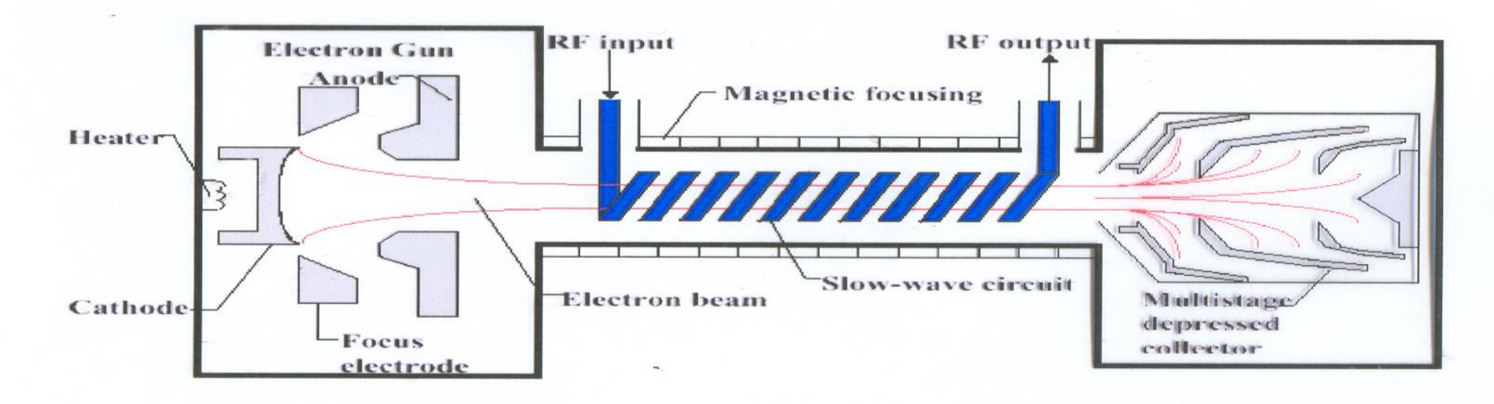
(characteristic impedance of the transmission line equivalent of a waveguide)

Transmission-line equivalent of a helical slow-wave structure

B.N. Basu, "Electromagnetic Theory and Applications in Beam-Wave Electronics," World Scientific Publishing Co. Inc., Singapore, New Jersey, London, Hong Kong (1996).

Helical slow-wave structure of a travelling-wave tube

- **Electromagnetic field Analysis**
- **Equivalent circuit analysis**



Sheath-helix model

Actual helix replaced by a circular cylindrical sheath that has

- **Infinitesimal thickness**
- **Radius equal to the mean radius of the actual helix of a finite thickness**
- **Anisotropic conductivity: infinite conductivity and zero conductivity in directions parallel and perpendicular to the helix winding direction, respectively**

Valid for large number of turns per guide wave length: $\boxed{\dfrac{\lambda_g}{p} >> 1}$

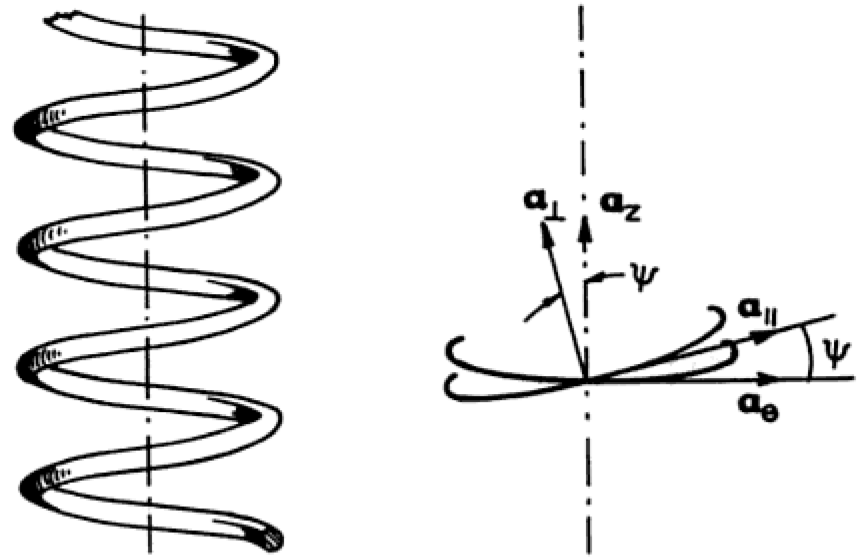
$$\frac{v_p}{c} = \frac{p}{2\pi a} \qquad v_p = f\lambda_g \qquad c = f\lambda \qquad \tan\psi = \frac{p}{2\pi a}$$

$$\downarrow$$

$$\frac{\lambda_g}{p} >> 1$$

$$\downarrow$$

$$\boxed{\frac{\lambda}{2\pi a} >> 1}$$



$$\vec{a}_\theta \times \vec{a}_z = \vec{a}_r$$

$$\vec{a}_{//} \times \vec{a}_\perp = \vec{a}_r$$

$$\nabla^2 E_z - \mu_0\varepsilon_0 \frac{\partial^2 E_z}{\partial t^2} = 0$$

$$\nabla^2 H_z - \mu_0\varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} = 0$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} \right)\left(E_z, H_z\right) = 0$$

RF quantities vary as

$$\exp\ j(\omega t - \beta z)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right)\left(E_z, H_z\right) - (\beta^2 - \omega^2\mu_0\varepsilon_0)\left(E_z, H_z\right) = 0$$

$$\frac{\partial}{\partial \theta} = 0$$ **(non-azimuthally varying mode)**

$$\gamma^2 = \beta^2 - \omega^2\mu_0\varepsilon_0$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right)\left(E_z, H_z\right) - \gamma^2\left(E_z, H_z\right) = 0$$

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\gamma = \sqrt{\beta^2 - k^2}$$

$$\omega^2\mu_0\varepsilon_0 = \frac{\omega^2}{c^2} = k^2$$

$$k = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c}$$

$$\gamma^2 = \beta^2 - \omega^2\mu_0\varepsilon_0 = \beta^2 - \frac{\omega^2}{c^2} = \beta^2 - k^2$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)(E_z,H_z)-\gamma^2(E_z,H_z)=0$$

For slow waves: $v_p < c$

Field expressions

$$\gamma=\sqrt{\beta^2-k^2}$$

$$E_z = AI_0(\gamma r)+BK_0(\gamma r)$$

$$H_z = CI_0(\gamma r)+DK_0(\gamma r)$$

$$\beta=\frac{\omega}{v_p} \qquad \frac{\omega}{v_p}>\frac{\omega}{c}$$

$$\beta > k$$

With the help of Maxwell's equations

$$E_\theta=-\frac{j\omega\mu_0}{\gamma}[CI_1(\gamma r)-DK_1(\gamma r)]$$

$$H_\theta=\frac{j\omega\varepsilon_0}{\gamma}[AI_1(\gamma r)-BK_1(\gamma r)]$$

$$E_r=\frac{j\beta}{\gamma}[AI_1(\gamma r)-BK_1(\gamma r)]$$

$$H_r=\frac{j\beta}{\gamma}[CI_1(\gamma r)-DK_1(\gamma r)]$$

$I_0(x)$: zeroth order modified Bessel function of the first kind

$K_0(x)$: zeroth order modified Bessel function of the second kind

$I_1(x)$: first order modified Bessel function of the first kind

$K_1(x)$: first order modified Bessel function of the second kind

$$I_0'(x)=I_1(x)$$

$$K_0'(x)=-K_1(x)$$

$$\gamma=\sqrt{\beta^2-k^2}$$

$$x=\gamma r$$

$$k=\omega\sqrt{\mu_0\varepsilon_0}=\frac{\omega}{c}$$

Field expressions

$$E_{z1} = A_1 I_0(\gamma r) + B_1 K_0(\gamma r)$$
$$H_{z1} = C_1 I_0(\gamma r) + D_1 K_0(\gamma r)$$

$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma}[C_1 I_1(\gamma r) - D_1 K_1(\gamma r)]$$

$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma}[A_1 I_1(\gamma r) - B_1 K_1(\gamma r)]$$

$$K_0(0) \rightarrow \infty$$

$$B_1 = 0$$

$$D_1 = 0$$

$$E_{z2} = A_2 I_0(\gamma r) + B_2 K_0(\gamma r)$$
$$H_{z2} = C_2 I_0(\gamma r) + D_2 K_0(\gamma r)$$

$$E_{\theta 2} = -\frac{j\omega\mu_0}{\gamma}[C_2 I_1(\gamma r) - D_2 K_1(\gamma r)]$$

$$H_{\theta 2} = \frac{j\omega\varepsilon_0}{\gamma}[A_2 I_1(\gamma r) - B_2 K_1(\gamma r)]$$

$$I_0(\infty) \rightarrow \infty$$

$$A_2 = 0$$

$$C_2 = 0$$

Field expressions

$$B_1 = 0 \qquad A_2 = 0$$

$$D_1 = 0 \qquad C_2 = 0$$

$$E_{z1} = A_1 I_0(\gamma r) \qquad E_{z2} = B_2 K_0(\gamma r)$$

$$H_{z1} = C_1 I_0(\gamma r) \qquad H_{z2} = D_2 K_0(\gamma r)$$

$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma r) \qquad E_{\theta 2} = \frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma r)$$

$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma r) \qquad H_{\theta 2} = \frac{-j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma r)$$

A_1, C_1, B_2, D_2 : **four non-zero field constants**

Boundary conditions at the mean helix radius = sheath-helix radius $r = a$

For electromagnetic field analysis

$$E_{z1} = E_{z2}$$

$$E_{\theta 1} \cos\psi + E_{z1} \sin\psi = 0$$

$$E_{\theta 2} \cos\psi + E_{z2} \sin\psi = 0$$

$$H_{\theta 1} \cos\psi + H_{z1} \sin\psi = H_{\theta 2} \cos\psi + H_{z2} \sin\psi$$

For circuit analysis

$$E_{z1} = E_{z2}$$

$$E_{\theta 1} = E_{\theta 2}$$

$$H_{\theta 2} - H_{\theta 1} = \frac{I_z}{2\pi a}$$

$$H_{z1} - H_{z2} = \frac{I_\theta}{2\pi a}$$

Electromagnetic Field analysis

Boundary conditions at the sheath-helix radius $r = a$:

$$E_{\theta 2}\cos\psi + E_{z2}\sin\psi = 0$$

$$H_{\theta 1}\cos\psi + H_{z1}\sin\psi = H_{\theta 2}\cos\psi + H_{z2}\sin\psi$$

$$E_{\theta 1}\cos\psi + E_{z1}\sin\psi = 0$$

$$E_{z1} = E_{z2}$$

$$\Downarrow$$

$$\frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma a)\cos\psi + B_2 K_0(\gamma a)\sin\psi = 0$$

$$\frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma a)\cos\psi + C_1 I_0(\gamma a)\sin\psi +$$

$$\frac{j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma a)\cos\psi - D_2 K_0(\gamma a)\sin\psi = 0$$

$$-\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma a)\cos\psi + A_1 I_0(\gamma a)\sin\psi = 0$$

$$A_1 I_0(\gamma a) E_{z2} - B_2 K_0(\gamma a) = 0$$

$$E_{z1} = A_1 I_0(\gamma r)$$

$$H_{z1} = C_1 I_0(\gamma r)$$

$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma r)$$

$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma r)$$

$$E_{z2} = B_2 K_0(\gamma r)$$

$$H_{z2} = D_2 K_0(\gamma r)$$

$$E_{\theta 2} = \frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma r)$$

$$H_{\theta 2} = \frac{-j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma r)$$

$$a_{11}A_1 + a_{12}C_1 + a_{13}B_2 + a_{14}D_2 = 0$$
$$a_{21}A_1 + a_{22}C_1 + a_{23}B_2 + a_{24}D_2 = 0$$
$$a_{31}A_1 + a_{32}C_1 + a_{33}B_2 + a_{34}D_2 = 0$$
$$a_{41}A_1 + a_{42}C_1 + a_{43}B_2 + a_{44}D_2 = 0$$

$$a_{11} = 0, a_{12} = 0, a_{13} = K_0(\gamma a)\sin\psi$$
$$a_{14} = \frac{j\omega\mu_0}{\gamma} K_1(\gamma a)\cos\psi$$
$$a_{21} = \frac{j\omega\varepsilon_0}{\gamma} I_1(\gamma a)\cos\psi,\ a_{22} = I_0(\gamma a)\sin\psi$$
$$a_{23} = \frac{j\omega\varepsilon_0}{\gamma} K_1(\gamma a)\cos\psi,\ a_{24} = -K_0(\gamma a)\sin\psi$$
$$a_{31} = I_0(\gamma a)\sin\psi,\ a_{32} = -\frac{j\omega\mu_0}{\gamma} I_1(\gamma a)\cos\psi$$
$$a_{33} = 0,\ a_{34} = 0$$
$$a_{41} = I_0(\gamma a),\ a_{42} = 0,\ a_{43} = -K_0(\gamma a),\ a_{44} = 0$$

$$\frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma a)\cos\psi + B_2 K_0(\gamma a)\sin\psi = 0$$

$$\frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma a)\cos\psi + C_1 I_0(\gamma a)\sin\psi +$$
$$\frac{j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma a)\cos\psi - D_2 K_0(\gamma a)\sin\psi = 0$$

$$-\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma a)\cos\psi + A_1 I_0(\gamma a)\sin\psi = 0$$

$$A_1 I_0(\gamma a) - B_2 K_0(\gamma a) = 0$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0$$

(condition for non-trivial solution)

$$a_{11} = 0, a_{12} = 0, a_{13} = K_0(\gamma a)\sin\psi,$$

$$a_{14} = \frac{j\omega\mu_0}{\gamma} K_1(\gamma a)\cos\psi,$$

$$a_{21} = \frac{j\omega\varepsilon_0}{\gamma} I_1(\gamma a)\cos\psi,\ a_{22} = I_0(\gamma a)\sin\psi,$$

$$a_{23} = \frac{j\omega\varepsilon_0}{\gamma} K_1(\gamma a)\cos\psi,\ a_{24} = -K_0(\gamma a)\sin\psi,$$

$$a_{31} = I_0(\gamma a)\sin\psi,\ a_{32} = -\frac{j\omega\mu_0}{\gamma} I_1(\gamma a)\cos\psi,$$

$$a_{33} = 0,\ a_{34} = 0,$$

$$a_{41} = I_0(\gamma a),\ a_{42} = 0,\ a_{43} = -K_0(\gamma a),\ a_{44} = 0.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0$$

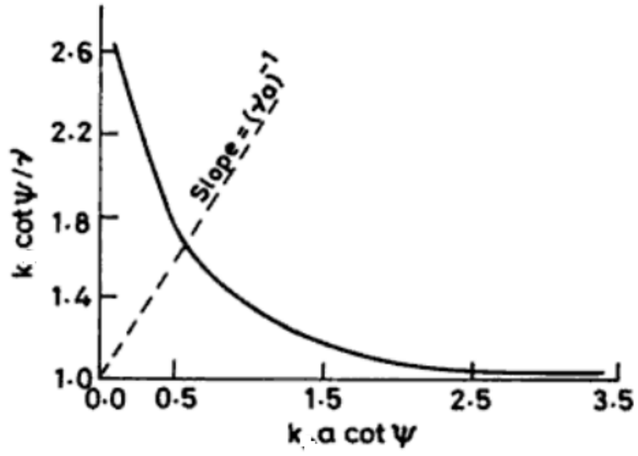
$$I_0(x)K_1(x) + K_0(x)I_1(x) = \frac{1}{x} \qquad (x = \gamma a)$$

$$\gamma^2 = \beta^2 - \omega^2 \mu_0 \varepsilon_0 = \beta^2 - \frac{\omega^2}{c^2} = \beta^2 - k^2$$

$$\gamma = \sqrt{\beta^2 - k^2} \qquad k = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} \qquad \beta = \frac{\omega}{v_p}$$

$$\frac{k \cot\psi}{\gamma} = \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)} \right)^{1/2}$$

(electromagnetic field analysis)



$$\frac{k \cot\psi}{\gamma} \gamma a = k a \cot\psi$$

$$\frac{v_p}{c} \cot\psi \approx \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)} \right)^{1/2}$$

Equivalent circuit analysis of a helix

Treating the helix as an equivalent transmission line with distributed line parameters *L* and *C*

Helix treated as a lossless transmission line

Capacitance per unit length *C*

Boundary conditions at the sheath-helix radius *r* = *a*:

$$E_{z1} = E_{z2}$$

$$A_1 I_0(\gamma a) = B_2 K_0(\gamma a)$$

$$H_{\theta 2} - H_{\theta 1} = \frac{I_z}{2\pi a}$$

$$-\frac{j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma a) - \frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma a) = \frac{I_{za}}{2\pi a}$$

Eliminating B_2

$$A_1 = -\left(\frac{\gamma}{j\omega\varepsilon_0}\right)\gamma a K_0(\gamma a)\frac{I_{za}}{2\pi a}$$

$$E_{z1} = A_1 I_0(\gamma r)$$

$$E_{za} = A_1 I_0(\gamma a)$$

$$E_{za} = -\left(\frac{\gamma}{j\omega\varepsilon_0}\right)\gamma a I_0(\gamma a) K_0(\gamma a)\frac{I_{za}}{2\pi a}$$

(to be recalled)

$$E_{za} = j\beta V - j\omega A_z \longleftarrow E_{za} = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t} \longleftarrow E_{za} = -(\nabla V)_z - \frac{\partial A_z}{\partial t} \longleftarrow \vec{E} = -\nabla V - \frac{\partial A}{\partial t}$$

$$\partial / \partial t = j\omega \quad \partial / \partial z = -j\beta \longleftarrow$$ **RF quantities vary as** $\exp j(\omega t - \beta z)$

$$\longleftarrow -j\beta A_z + j\omega\mu_0\varepsilon_0 V = 0 \longleftarrow \frac{\partial A_z}{\partial z} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = 0 \longleftarrow \nabla.\vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = 0$$

$$E_{za} = j(\frac{\beta^2 - \omega^2\mu_0\varepsilon_0}{\beta})V = j(\frac{\beta^2 - k^2}{\beta})V \longleftarrow \gamma^2 = \beta^2 - k^2$$

$$E_{za} = \frac{j\gamma^2}{\beta}V$$ **(to be recalled)**

RF quantities vary as $\exp j(\omega t - \beta z)$

$$\frac{\partial I_{za}}{\partial z} + C\frac{\partial V}{\partial t} = 0$$

$$-j\beta I_{za} + j\omega C V = 0$$

$$\boxed{C = \frac{2\pi\varepsilon_0}{I_0(\gamma a)K_0(\gamma a)}}$$

$$E_{za} = \frac{j\gamma^2}{\beta}V$$

$$E_{za} = -\left(\frac{\gamma}{j\omega\varepsilon_0}\right)\gamma a I_0(\gamma a)K_0(\gamma a)\frac{I_{za}}{2\pi a}$$

$$I_0(x)K_1(x) + K_0(x)I_1(x) = \frac{1}{x}$$

$$(x = \gamma a)$$

Inductance per unit length L

$$-\frac{j\omega\mu_0}{\gamma}C_1 I_1(\gamma a) = \frac{j\omega\mu_0}{\gamma}D_2 K_1(\gamma a)$$

$$C_1 I_0(\gamma a) - D_2 K_0(\gamma a) = \frac{I_{\theta a}}{2\pi a}$$

$$C_1 = \gamma a K_1(\gamma a)\frac{I_{\theta a}}{2\pi a}$$

$$E_{\theta a} = -\frac{j\omega\mu_0}{\gamma}C_1 I_1(\gamma a)$$

$$E_{\theta a} = -\frac{j\omega\mu_0}{\gamma}\gamma a I_1(\gamma a) K_1(\gamma a)\frac{I_{\theta a}}{2\pi a}$$

Boundary conditions at the sheath-helix radius $r = a$:

$$E_{\theta 1} = E_{\theta 2}$$

$$H_{z1} - H_{z2} = \frac{I_\theta}{2\pi a}$$

$$\frac{\partial V}{\partial z} + L\frac{\partial I_{za}}{\partial t} = 0$$

$$\downarrow$$

$$-j\beta V + j\omega L I_{za} = 0$$

$$\downarrow$$

$$\boxed{L = \frac{\mu_0}{2\pi}\left(\frac{\beta}{\gamma}\right)^2 \cot^2\psi\, I_1(\gamma a)K_1(\gamma a)}$$

$$E_{\theta a} = -\frac{j\omega\mu_0}{\gamma}\gamma a I_1(\gamma a)K_1(\gamma a)\frac{I_{\theta a}}{2\pi a}$$

$$E_{za} = \frac{j\gamma^2}{\beta}V$$

$$E_{\theta a}\cos\psi + E_{za}\sin\psi = 0$$

$$-I_{\theta a}\sin\psi + I_{za}\cos\psi = 0$$

$$I_0(x)K_1(x) + K_0(x)I_1(x) = \frac{1}{x} \quad (x = \gamma a)$$

$$C = \frac{2\pi\varepsilon_0}{I_0(\gamma a)K_0(\gamma a)}$$

$$L = \frac{\mu_0}{2\pi}\left(\frac{\beta}{\gamma}\right)^2 \cot^2\psi \, I_1(\gamma a)K_1(\gamma a)$$

$$\beta^2 = \omega^2 LC$$

$$\frac{k\cot\psi}{\gamma} = \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\right)^{1/2}$$

(dispersion relation derived by equivalent circuit analysis)

$$\frac{k\cot\psi}{\gamma} = \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\right)^{1/2}$$

(dispersion relation derived by electromagnetic field analysis)

(derived earlier)

The two approaches of analysis yield one and the same dispersion relation of a helical structure!

Electromagnetic field analysis	Equivalent circuit analysis
Ohm's electromagnetic law $\vec{J} = \sigma\vec{E}$	**Ohm's circuital law** $V_A - V_B = IR$
Poynting vector $\vec{S} = \vec{E} \times \vec{H}$	**Joule's circuit loss** $I^2 R$
Electromagnetic boundary condition at the interface between two conducting media $\vec{a}_n \times (E_2 - E_1) = 0$	**Law of parallel resistances** $\frac{1}{R_{\text{equivalent}}} = \frac{1}{R_1} + \frac{1}{R_2}$
Waveguide dispersion relation $\omega^2 - \beta^2 c^2 - \omega_c^2 = 0$	**Waveguide dispersion relation** $\omega^2 - \beta^2 c^2 - \omega_c^2 = 0$
Dispersion relation of a helix in free space $\frac{k\cot\psi}{\gamma} = \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\right)^{1/2}$	**Dispersion relation of a helix in free space** $\frac{k\cot\psi}{\gamma} = \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\right)^{1/2}$

Concluding Remarks

The circuit concepts can be developed from the corresponding field concepts. Thus, Ohm's law in circuit theory expressed in terms of voltage, current and resistance can be derived from Ohm's law in electromagnetic theory expressed in terms of current density, electric field and material conductivity. Joule's power loss expression in circuit theory in terms of circuit current and circuit resistance can be derived from Poynting theorem of electromagnetic theory. Further, one of the electromagnetic boundary conditions at the interface between two media can be used to easily appreciate the law of parallel resistances of circuit theory.

On the same note, the behavior of interaction structures of vacuum electron devices can be described by either of the field and circuit analytical concepts. These two theoretical concepts yield one and the same dispersion relation of a hollow-pipe waveguide which can be made to support either a TE or a TM fast waveguide-mode. Similarly, these two concepts yield one and the same dispersion relation of helix which supports hybrid TE and TM slow waveguide-modes. The circuit analysis enjoys the simplicity in that it handles at a time half the number of boundary conditions than the field analysis.

We can find, with the help of circuit analysis, the characteristic impedance of an interaction structure, which is a parameter of relevance from the standpoint of the impedance matching of the structure with the RF couplers which couple power in and out of the structure. On the other hand, We can find, with the help of electromagnetic field analysis, the interaction impedance of an interaction structure, which is a parameter of relevance from the standpoint of the device gain and efficiency.

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Thank you!