Analysis of Helical Slow-Wave Structure in Tape-Helix Model

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You cannot teach a man anything; you can only help him to find it for himself

—Galileo Galilee

Pierce’s Sheath-helix model

Cylindrical sheath of infinitesimal thickness that has infinite and zero conductivity in the direction of helical winding and perpendicular to this direction, respectively, replacing the actual helix

At the sheath-helix surface

1. **Electric field intensity parallel to the helix winding direction in the region inside the sheath-helix helix is zero.**
2. **Electric field intensity parallel to the helix winding direction in the region outside the sheath-helix helix is zero**
3. **Magnetic field intensity parallel to the helix winding direction in the region inside the sheath-helix helix is continuous with that outside**
4. **Axial electric field intensity (or azimuthal electric field intensity) is continuous with that outside**

At the dielectric-dielectric interface

Axial and azimuthal electric field intensities as well as axial and azimuthal magnetic field intensities are each continuous

At the metal envelope

Axial and azimuthal electric field intensities are each equal to zero

J. R. Pierce: Traveling-Wave Tubes. D. Van Nostrand (Princeton, 1950)

Helix in Free-Space
(Sheath-Helix Model)

Dispersion relation of a helix in free-space in the sheath-helix model

$$\frac{k\cot\psi}{\gamma}=\left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\right)^{1/2}$$

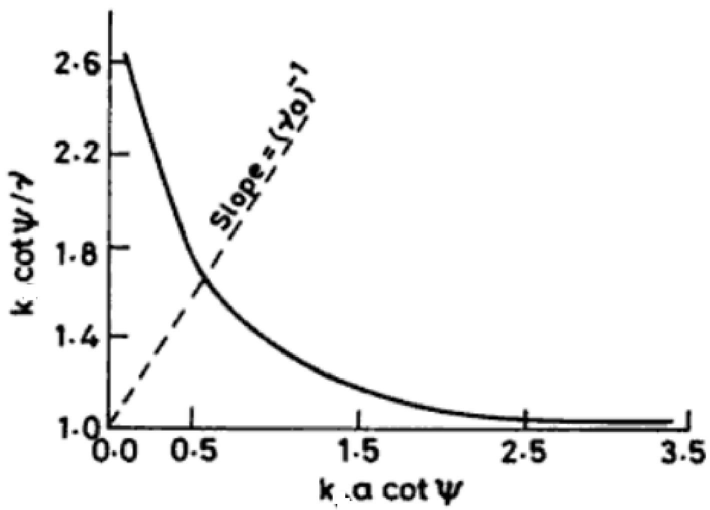
$$\left.\begin{array}{l}\beta >> k\\ v_p << c\end{array}\right\}$$

(slow-wave)

$$\left.\begin{array}{l}\beta=\dfrac{\omega}{v_p}\\ k=\omega\sqrt{\mu_0\varepsilon_0}=\dfrac{\omega}{c}\end{array}\right\}$$

$$\frac{k}{\gamma}=\frac{k}{\sqrt{\beta^2-k^2}}\approx\frac{k}{\beta}=\frac{v_p}{c}$$

$$\frac{k\cot\psi}{\gamma}\approx\frac{v_p}{c}\cot\psi=\left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\right)^{1/2}$$



$$\frac{k\cot\psi}{\gamma}\gamma a=ka\cot\psi$$

Sensiper's Tape-Helix Model

Axial harmonic effects due to the axial periodicity of the helix taken into account

Actual helix replaced by a tape of infinitesimal thickness that conducts in all directions

Zero tangential electric field intensity everywhere on the tape surface corresponding to the actual current distribution on the tape surface

Tape surface current density assumed to be predominantly along the helix winding direction, with a defined distribution, presumably caused by an electric field intensity parallel to the winding direction $E_{//}$, under narrow-tape approximation

Constant amplitude of surface current density over the tape width with its phase varying along the centreline of the tape winding as $\exp -(j\beta_0 p\theta/(2\pi))$, where β_0 is the fundamental axial phase propagation constant, and p is the helix pitch, according to a typically assumed current distribution

Expected satisfaction of the boundary condition $E_{//} = 0$ over the entire tape surface, for a narrow tape, if the boundary condition is satisfied along the centreline of the tape surface

Field expressions found using Floquet's theorem considering the helix periodicity

Relevant field constants and hence $E_{//}$ along the centreline of the tape surface found in terms of the assumed tape surface current density distribution

$E_{//}$ along the centreline of the tape surface set equal to zero $(E_{//} = 0)$ to obtain the dispersion relation in the tape-helix model

Excerpt from Samuel Sensiper: Electromagnetic wave propagation on helical conductors. Technical Report No. 194; May 16, 1951. Research Laboratory of Electronics, Institute of Technology, Cambridge, Massachusetts (Sc D Thesis); and *Proc. IRE* 42 (1955) 144-161.

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"If the tape is taken to be very narrow, that is, with δ small compared with a, p, and λ, it seems quite reasonable to assume that essentially all of the current flows only along the tape."

…….. "If the point of view is taken that the fields are produced by the currents which flow, with the tape narrow and current flowing primarily in the direction of the tape, the specific distribution of current across the tape will affect only to a small degree the fields in the near neighborhood of the wire and to a much less degree the fields on adjacent and faraway turns. Thus, if some reasonable assumptions are made concerning this current distribution, it is to be expected that only small errors will be made in the field expressions".

…….. "If an inexact current distribution on the tape is assumed, the tangential electric field can no longer be made zero everywhere on the tape, and this boundary conditions can be only approximately satisfied. This may be done in several ways. One could, for example, require the average value, or better the mean square value, of the tangential electric field on the tape to be a minimum, with the propagation constant, which gives this minimum as the solution. However, another procedure is used here which leads to a somewhat simpler determinantal equation for calculative purposes and which appears to be a quite adequate approximation. In this it is required that $E_{//}$ be zero along the centerline of the tape; in other words, one of the boundary conditions is matched exactly along a line. (As noted before), for a narrow tape the dominant current density is $K_{//}$, and, loosely speaking, it is $E_{//}$ which forces the current to flow along the tape. Thus, if the most important boundary condition is satisfied on a line, one may hope to obtain a reasonable good approximation to the exact case where the condition must be satisfied over a surface"…………….. "in the neighborhood of this line which is almost a narrow tape."

— Samuel Sensiper: Electromagnetic wave propagation on helical conductors. Technical Report No. 194; May 16, 1951. Research Laboratory of Electronics, Institute of Technology, Cambridge, Massachusetts (Sc D Thesis); and *Proc. IRE* 42 (1955) 144-161

Field expressions for a helix in free-space obtained in the sheath-helix model

$$\boxed{\begin{aligned}
E_z &= AI_0(\gamma r) + BK_0(\gamma r) \\
H_z &= CI_0(\gamma r) + DK_0(\gamma r) \\
E_\theta &= -\frac{j\omega\mu_0}{\gamma}[CI_1(\gamma r) - DK_1(\gamma r)] \\
H_\theta &= \frac{j\omega\varepsilon_0}{\gamma}[AI_1(\gamma r) - BK_1(\gamma r)] \\
E_r &= \frac{j\beta}{\gamma}[AI_1(\gamma r) - BK_1(\gamma r)] \\
H_r &= \frac{j\beta}{\gamma}[CI_1(\gamma r) - DK_1(\gamma r)]
\end{aligned}}$$

Field expressions

$$E_{z1} = A_1 I_0(\gamma r) + B_1 K_0(\gamma r)$$
$$H_{z1} = C_1 I_0(\gamma r) + D_1 K_0(\gamma r)$$
$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma}[C_1 I_1(\gamma r) - D_1 K_1(\gamma r)]$$
$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma}[A_1 I_1(\gamma r) - B_1 K_1(\gamma r)]$$

$$K_0(0) \to \infty$$
$$I_0(\infty) \to \infty$$
$$B_1 = 0$$
$$A_2 = 0$$
$$D_1 = 0$$
$$C_2 = 0$$

$$E_{z2} = A_2 I_0(\gamma r) + B_2 K_0(\gamma r)$$
$$H_{z2} = C_2 I_0(\gamma r) + D_2 K_0(\gamma r)$$
$$E_{\theta 2} = -\frac{j\omega\mu_0}{\gamma}[C_2 I_1(\gamma r) - D_2 K_1(\gamma r)]$$
$$H_{\theta 2} = \frac{j\omega\varepsilon_0}{\gamma}[A_2 I_1(\gamma r) - B_2 K_1(\gamma r)]$$

$$E_{z1} = A_1 I_0(\gamma r)$$
$$H_{z1} = C_1 I_0(\gamma r)$$
$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma r)$$
$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma r)$$

$$E_{z2} = B_2 K_0(\gamma r)$$
$$H_{z2} = D_2 K_0(\gamma r)$$
$$E_{\theta 2} = \frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma r)$$
$$H_{\theta 2} = \frac{-j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma r)$$

The subscript 1 refers to the quantities inside the helix.

The subscript 2 refers to the quantities outside the helix.

A_1, C_1, B_2, D_2 : **Four non-zero field constants**

Expressions for Fields Comprised of Space-Harmonics

$$\exp\, j(\omega t-\beta z)=\exp(j\omega t-\gamma z)=\exp(j\omega t-\Gamma z)$$

$$(\Gamma = j\gamma)$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}-\mu\varepsilon\frac{\partial^2}{\partial t^2}\right)(E_z,H_z)=0$$

RF quantities vary as

$$\Gamma^2=-\gamma^2=\beta^2$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)(E_z,H_z)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(E_z,H_z)+\Gamma^2(E_z,H_z)=0$$

Solution:

$$E_z=A_m I_m(\gamma r)\exp(-j\beta z)\exp(jm\theta)$$

$$H_z=C_m I_m(\gamma r)\exp(-j\beta z)\exp(jm\theta)$$

The subscript 1 refers to the quantities inside the helix.

The subscript 2 refers to the quantities outside the helix.

$$E_{z1} = [A_{1m} I_m(\gamma r)] \exp(-j\beta z) \exp(jm\theta)$$

$$H_{z1} = [C_{1m} I_m(\gamma r)] \exp(-j\beta z) \exp(jm\theta)$$

$$E_{\theta 1} = \left[\frac{-n\beta}{\gamma^2 r} A_{1m} I_m(\gamma r) - \frac{j\omega\mu_0}{\gamma} C_{1m} I'_m(\gamma r) \right] \exp(-j\beta z) \exp(jm\theta)$$

$$H_{\theta 1} = \left[\frac{j\omega\varepsilon_0}{\gamma} A_{1m} I'_m(\gamma r) - \frac{n\beta}{\gamma^2 r} C_{1m} I_m(\gamma r) \right] \exp(-j\beta z) \exp(jm\theta)$$

$$E_{z2} = [B_{2m} K_m(\gamma r)] \exp(-j\beta z) \exp(jm\theta)$$

$$E_{\theta 2} = \left[\frac{n\beta}{\gamma^2 r} B_{2m} K_m(\gamma r) - \frac{j\omega\mu_0}{\gamma} D_{2m} K'_m(\gamma r) \right] \exp(-j\beta z) \exp(jm\theta)$$

$$H_{z2} = [D_{2m} K_m(\gamma r)] \exp(-j\beta z) \exp(jm\theta)$$

$$H_{\theta 2} = \left[\frac{j\omega\varepsilon_0}{\gamma} B_{2m} K'_m(\gamma r) - \frac{n\beta}{\gamma^2 r} D_{2m} K_m(\gamma r) \right] \exp(-j\beta z) \exp(jm\theta)$$

Space periodicity of the helix

The helix coincides with itself if i) either it is translated through an axial period p (axial translation) or ii) it is translated through an arbitrary axial distance $z' < p$ and then rotated through an angle $2\pi z'/p$ (or, alternatively, it is rotated through an arbitrary angle $\theta' < 2\pi$ and then translated through an axial distance $p\theta'/2\pi$ (differential skew transformation).

Functional dependence

$$f(z) = \exp - j\beta_n z = \exp - (j\beta_0 + \frac{2\pi}{p}n)z$$

$$= (\exp - j\beta_0 z)\exp(-j\frac{2\pi n}{p})z \quad \longleftarrow \quad \beta_n = \beta_0 + \frac{2\pi}{p}n$$

$$f(z+p) = (\exp - j\beta_0(z+p)\exp(-j\frac{2\pi n}{p})(z+p) \qquad \exp - j2\pi = 1$$

$$= f(z)\exp - j\beta_0 p \exp - j2\pi = f(z)\exp - j\beta_0 p$$

$f(z+p)$ **differs from** $f(z)$ **by a constant phase factor and hence satisfies Floquet's theorem**

We can introduce this functional dependence in the field expressions to account for the axial periodicity of the structure.

$$E_{z1} = [A_{1m} I_m(\gamma r)]\exp(-j\beta z)\exp(jm\theta)$$

(recalled)

The subscript 1 refers to the quantities inside the helix.

Fields considering space harmonics

$$E_{z1}(r,\theta,z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} A_{1m,n} I_m(\gamma_n r)\exp(-j\beta_n z)\exp(jm\theta)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} A_{1m,n} I_m(\gamma_n r)\exp(-j2\pi nz/p)(\exp - j\beta_0 z)\exp(jm\theta)$$

$$\left.\begin{aligned} \gamma_n &= (\beta_n{}^2 - k^2)^{1/2} \\ \beta_n &= \beta_0 + 2\pi n/p \end{aligned}\right\}$$

$$E_{\theta 1}(r,\theta,z) =$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[\frac{-n\beta}{\gamma_n{}^2 r} A_{1m,n} I_m(\gamma_n r) - \frac{j\omega\mu_0}{\gamma_n} C_{1m,n} I'_m(\gamma_n r)\right]\exp(-j2\pi nz/p)(\exp - j\beta_0 z)\exp(jm\theta)$$

$$E_{z1}(r,\theta,z)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}A_{1m,n}I_m(\gamma_n r)\exp(-j2\pi nz/p)(\exp-j\beta_0 z)\exp(jm\theta)$$ **(rewritten)**

For the helix translated through an arbitrary axial distance $z' < p$ and then rotated through an angle $2\pi z'/p$

Skew transformation

$$E_{z1}(r,\theta+2\pi z'/p,z+z')=$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}A_{1m,n}I_m(\gamma_n r)\exp(-j2\pi nz/p)\exp(-j2\pi nz'/p)$$
$$\times(\exp-j\beta_0 z)(\exp-j\beta_0 z')\exp(jm\theta)\exp(jm2\pi z'/p)$$

$$=(\exp-j\beta_0 z')\exp-j2\pi(m-n)z'/p$$
$$\times\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}[A_{1m,n}I_m(\gamma_n r)\exp(-j2\pi nz/p)(\exp-j\beta_0 z)\exp(jm\theta)]$$

$$E_{z1}(r,\theta+2\pi z'/p,z+z')=(\exp-j\beta_0 z')$$
$$\times\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}[A_{1m,n}I_m(\gamma_n r)\exp(-j2\pi nz/p)(\exp-j\beta_0 z)\exp(jm\theta)]$$
$$=(\exp-j\beta_0 z')E_{z1}(r,\theta,z)$$

Thus, Floquet's theorem is obeyed through the invariance of skew transformation.

$$E_{z1}(r,\theta,z)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}A_{1m,n}I_m(\gamma_n r)\exp(-j2\pi nz/p)(\exp-j\beta_0 z)\exp(jm\theta)$$

(rewritten)

$$E_{\theta 1}(r,\theta,z)=$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}[\frac{-n\beta}{\gamma_n{}^2 r}A_{1m,n}I_m(\gamma_n r)-\frac{j\omega\mu_0}{\gamma_n}C_{1m,n}I'_m(\gamma_n r)]\exp(-j2\pi nz/p)(\exp-j\beta_0 z)\exp(jm\theta)$$

(rewritten)

Putting $m = n$**, to satisfy the invariance of skew transformation corresponding to** $f(z+p)$ **differing from** $f(z)$ **by a constant phase factor**

$$E_{z1}(r,\theta,z)=\sum_{n=-\infty}^{\infty}A_{1n}I_n(\gamma_n r)\exp(-j2\pi nz/p)(\exp-j\beta_0 z)\exp(jn\theta)$$

$$E_{\theta 1}(r,\theta,z)=$$

$$\sum_{n=-\infty}^{\infty}[\frac{-n\beta}{\gamma_n{}^2 r}A_{1n}I_n(\gamma_n r)-\frac{j\omega\mu_0}{\gamma_n}C_{1n}I'_n(\gamma_n r)]\exp(-j2\pi nz/p)(\exp-j\beta_0 z)\exp(jn\theta)$$

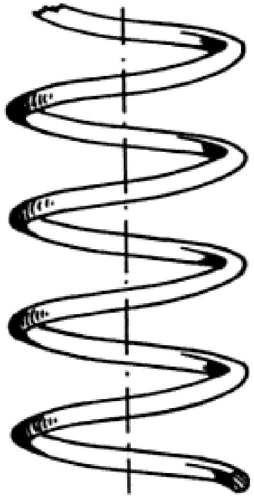
Electric field parallel to the winding direction

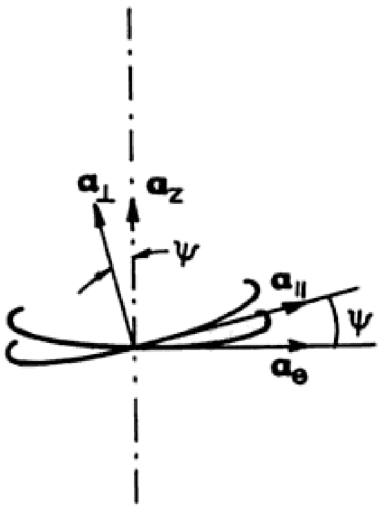
$$\vec{E} = E_r\vec{a}_r + E_\theta\vec{a}_\theta + E_z\vec{a}_z$$

$$E_{//} = \vec{E}\cdot\vec{a}_{//} = E_r\vec{a}_r\cdot\vec{a}_{//} + E_\theta\vec{a}_\theta\cdot\vec{a}_{//} + E_z\vec{a}_z\cdot\vec{a}_{//}$$

$$E_{//} = (E_r)(0) + (E_\theta)(\cos\psi) + (E_z)(\sin\psi)$$

$$E_{//} = E_\theta\cos\psi + E_z\sin\psi$$





$$E_{//} = E_{\theta} \cos\psi + E_{z} \sin\psi$$ **(rewritten)**

$$E_{\theta 1}(r,\theta,z) = \sum_{n=-\infty}^{\infty} \left[\frac{-n\beta}{\gamma_n^{2} r} A_{1n} I_n(\gamma_n r) - \frac{j\omega\mu_0}{\gamma_n} C_{1n} I_n'(\gamma_n r)\right] \exp(-j2\pi nz/p)(\exp - j\beta_0 z)\exp(jn\theta)$$

$$E_{z1}(r,\theta,z) = \sum_{n=-\infty}^{\infty} A_{1n} I_n(\gamma_n r) \exp(-j2\pi nz/p)(\exp - j\beta_0 z)\exp(jn\theta)$$

$$E_{1,//}(a) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{-n\beta_n}{\gamma_n^{2} a} I_n\{\gamma_n a\}\cos\psi - I_n\{\gamma_n a\}\sin\psi\right) A_{1n} - \left(\frac{j\omega\mu_0}{\gamma_n} I_n'\{\gamma_n a\}\cos\psi\right) C_{1n}\right] \times$$
$$\exp(-j\beta_0 z)\exp - jn((2\pi z/p) - \theta)$$

(to be recalled later)

Sensiper's Tape-Helix Model

Axial harmonic effects due to the axial periodicity of the helix taken into account

Actual helix replaced by a tape of infinitesimal thickness that conducts in all directions

Zero tangential electric field intensity everywhere on the tape surface corresponding to the actual current distribution on the tape surface

Tape surface current density assumed to be predominantly along the helix winding direction, with a defined distribution, presumably caused by an electric field intensity parallel to the winding direction $E_{//}$, under narrow-tape approximation

Constant amplitude of surface current density over the tape width with its phase varying along the centreline of the tape winding as $\exp - (j\beta_0 p\theta /(2\pi))$, where β_0 is the fundamental axial phase propagation constant, and p is the helix pitch, according to a typically assumed current distribution

Expected satisfaction of the boundary condition $E_{//} = 0$ over the entire tape surface, for a narrow tape, if the boundary condition is satisfied along the centreline of the tape surface

Field expressions found using Floquet's theorem considering the helix periodicity

Relevant field constants and hence $E_{//}$ along the centreline of the tape surface found in terms of the assumed tape surface current density distribution

$E_{//}$ along the centreline of the tape surface set equal to zero ($E_{//} = 0$) to obtain the dispersion relation in the tape-helix model

$$E_{1,//}(a)=\sum_{n=-\infty}^{\infty}\left[\left(\frac{-n\beta_n}{\gamma_n^2 a}I_n\{\gamma_n a\}\cos\psi - I_n\{\gamma_n a\}\sin\psi\right)A_{1n}-\left(\frac{j\omega\mu_0}{\gamma_n}I'_n\{\gamma_n a\}\cos\psi\right)C_{1n}\right]\times$$

$$\exp(-j\beta_0 z)\exp-jn((2\pi z/p)-\theta)$$

(recalled)

A_{1n}, C_{1n} : **Field constants**

We need to evaluate the field constants A_{1n} and C_{1n} in terms of helix currents with the help of the boundary conditions.

Boundary conditions at the mean helix radius $r = a$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

$$\downarrow$$

$$\vec{a}_r \times [(E_{r2} - E_{r1})\vec{a}_r + (E_{\theta 2} - E_{\theta 1})\vec{a}_\theta + (E_{z2} - E_{z1})\vec{a}_z] = 0$$

$$\downarrow$$

$$(E_{\theta 2} - E_{\theta 1})\vec{a}_z + (E_{z1} - E_{z2})\vec{a}_\theta = 0$$

$$\downarrow$$

$$\boxed{\begin{aligned} E_{z1} &= E_{z2} \\ E_{\theta 1} &= E_{\theta 2} \end{aligned}}$$

Subscript 1:
Inside the helix winding radius

Subscript 2:
outside the helix winding radius

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$$

$$\downarrow$$

$$\vec{a}_r \times [(H_{r2} - H_{r1})\vec{a}_r + (H_{\theta 2} - H_{\theta 1})\vec{a}_\theta + (H_{z2} - H_{z1})\vec{a}_z] = \vec{J}_s$$

$$\downarrow$$

$$\boxed{\begin{aligned} H_{\theta 2} - H_{\theta 1} &= J_{sz} \\ H_{z1} - H_{z2} &= J_{s\theta} \end{aligned}}$$

$$\boxed{\begin{aligned} E_{z1} &= E_{z2} \\ E_{\theta 1} &= E_{\theta 2} \end{aligned}}$$

Interpreting for the n^{th} harmonic component

$$\boxed{\begin{aligned} H_{\theta 2} - H_{\theta 1} &= J_{sz} \\ H_{z1} - H_{z2} &= J_{s\theta} \end{aligned}}$$

$$E_{z1,n} = E_{z2,n}$$
$$E_{\theta 1,n} = E_{\theta 2,n}$$

Boundary conditions at $r = a$

$$H_{\theta 2,n} - H_{\theta 1,n} = J_{sz,n}$$
$$H_{z1} - H_{z2} = J_{s\theta,n}$$

$$A_{1n} = \frac{\gamma_n K_n(\gamma_n a)}{j\omega\varepsilon_0} \frac{\hat{J}_{s//,n} - (n\beta_n / \gamma_n^2 a)\hat{J}_{s\theta,n}}{I_n(\gamma_n a)K_n'(\gamma_n a) - K_n(\gamma_n a)I_n'(\gamma_n a)}$$

The hat $\wedge$ represents the peak value.

$$C_{1n} = \frac{K_n'(\gamma_n a)I_n(\gamma_n a)\hat{J}_{s\theta,n}}{I_n(\gamma_n a)K_n'(\gamma_n a) - K_n(\gamma_n a)I_n'(\gamma_n a)}$$

Substituting field expressions into boundary conditions and solving the resulting equations in field constants

$$J_{szn} = J_{s//,n} \sin\psi$$
$$J_{\theta n} = J_{s//,n} \cos\psi$$
$$I_n(\gamma_n a)K_n'(\gamma_n a) - K_n(\gamma_n a)I_n'(\gamma_n a) = 1/\gamma_n a$$

$$A_{1,n} = -\frac{\gamma_n^2 a K_n(\gamma_n a)}{j\omega\varepsilon_0}[\hat{J}_{s//,n} \sin\psi - (n\beta / \gamma_n{}^2 a)\hat{J}_{s//,n} \cos\psi]$$

$$C_{1n} = -\gamma_n a K_n'(\gamma_n a)\hat{J}_{s//,n} \cos\psi$$

$$E_{1,//}(a)=\sum_{n=-\infty}^{\infty}\left[\left(\frac{-n\beta_n}{\gamma_n^2 a}I_n\{\gamma_n a\}\cos\psi+I_n\{\gamma_n a\}\sin\psi\right)A_{1n}-\left(\frac{j\omega\mu_0}{\gamma_n}I_n'\{\gamma_n a\}\cos\psi\right)C_{1n}\right]\times$$

$$\exp(-j\beta_0 z)\exp-jn((2\pi z/p)-\theta)$$

$$A_{1,n}=-\frac{\gamma_n^2 a K_n(\gamma_n a)}{j\omega\varepsilon_0}[\hat{J}_{s//,n}\sin\psi-(n\beta_n/\gamma_n{}^2 a)\hat{J}_{s//,n}\cos\psi]$$

$$C_{1n}=-\gamma_n a K_n'(\gamma_n a)\hat{J}_{s//,n}\cos\psi$$

We can find $\hat{J}_{s//,n}$ in terms of an assumed current distribution over the tape width.

Let us assume that the surface current density through the tape is parallel to the tape winding direction and that its amplitude is constant at a value $J_{//}$ and that its phase varies along the winding direction according to the phase factor $\exp(j\beta_0 z)$, where z corresponds to a point moving along the centre-line of the tape. The value of z, corresponding to a point on the centre line of the tape characterized by an azimuthal angle θ, is given $z = p\theta/2\pi$. Thus, over a period of the structure, that is, the helix pitch p, one may write:

$$J_{s//}=J\exp(-j\beta_0 z)$$

$$=J\exp(-j\beta_0 p\theta/2\pi)\quad[p\theta/2\pi-\delta/2<z<p\theta/2\pi+\delta/2]$$

$$=0\quad[p\theta/2\pi+\delta/2<z<p\theta/2\pi-\delta/2+p]$$

$$
\begin{aligned}
J_{s//} &= J\exp(-j\beta_0 z) \\
&= J\exp(-j\beta_0 p\theta/2\pi) \quad [p\theta/2\pi-\delta/2<z<p\theta/2\pi+\delta/2] \\
&= 0 \quad [p\theta/2\pi+\delta/2<z<p\theta/2\pi-\delta/2+p]
\end{aligned}
$$

(tape current distribution assumed)

$$J_{s//}=\sum_{n=-\infty}^{\infty} J_{s//,n}=\sum_{n=-\infty}^{\infty}\hat{J}_{s//,n}\exp(-j\beta_0 z)\exp(-jn(2\pi z/p)-\theta)$$

For the assumed current distribution, the following Fourier component is obtained:

$$\hat{J}_{s//,n}=J\left(\frac{\sin(\beta_n\delta/2)}{\beta_n\delta/2}\right)\left(\frac{\delta}{p}\right)$$

$$J_{s//} = \sum_{n=-\infty}^{\infty} \hat{J}_{s//,n} \exp(-j\beta_0 z)\exp(-jn(2\pi z/p)-\theta)$$

Multiplying both sides by $\exp(j\beta_0 z)\exp jm(2\pi z/p-\theta)dz$ **and integrating**

$$\int_{p\theta/2\pi-\delta/2}^{p\theta/2\pi+\delta/2} J_{s//}\exp(j\beta_0 z)\exp jm(2\pi z/p-\theta)dz + \int_{p\theta/2\pi+\delta/2}^{p\theta/2\pi-\delta/2+p} J_{s//}\exp(j\beta_0 z)\exp jm(2\pi z/p)-\theta)dz$$

$$= \int_{p\theta/2\pi-\delta/2}^{p\theta/2\pi-\delta/2+p} \hat{J}_{s//,n}\exp j(m-n)(2\pi z/p-\theta)dz$$

$$\begin{aligned} J_{s//} &= J\exp(-j\beta_0 z) \\ &= J\exp(-j\beta_0 p\theta/2\pi) \quad [p\theta/2\pi-\delta/2 < z < p\theta/2\pi+\delta/2] \\ &= 0 \quad [p\theta/2\pi+\delta/2 < z < p\theta/2\pi-\delta/2+p] \end{aligned}$$

(assumed tape current distribution)

$$J\exp(-j\beta_0 p\pi/2+m)\theta \int_{p\theta/2\pi-\delta/2}^{p\theta/2\pi+\delta/2} \exp j(\beta_0+2\pi m/p)zdz$$

$$= \int_{p\theta/2\pi-\delta/2\ (n=m)}^{p\theta/2\pi-\delta/2+p} \hat{J}_{s//,m}dz + \int_{p\theta/2\pi-\delta/2\ (n\neq m)}^{p\theta/2\pi-\delta/2+p} \sum_{n\neq m=-\infty}^{\infty} \hat{J}_{s//,m}\exp[-j(m-n)\theta]\exp j(m-n)(2\pi z/p)dz$$

$$\beta_n = \beta_0 + 2\pi n / p$$

$$\beta_m = \beta_0 + 2\pi mz / p$$

$$\beta_m p / 2\pi = \beta_0 p / 2\pi + mz$$

$$J \exp(-j\beta_0 p\pi / 2 + m)\theta \int_{p\theta/2\pi-\delta/2}^{p\theta/2\pi+\delta/2} \exp\ j(\beta_0 + 2\pi m / p) z dz$$

$$= \int_{p\theta/2\pi-\delta/2\ (n=m)}^{p\theta/2\pi-\delta/2+p} \hat{J}_{s//,m} dz + \sum_{n\neq m=-\infty}^{\infty} \hat{J}_{s//,m} \int_{z=p\theta/2\pi-\delta/2}^{p\theta/2\pi-\delta/2+p} \exp[-j(m-n)\theta] \exp\ j(m-n)(2\pi z / p) dz$$

(rewritten)

$$J \exp(-j\beta_m p\theta / 2\pi \int_{p\theta/2\pi-\delta/2}^{p\theta/2\pi+\delta/2} \exp\ j(\beta_m z dz$$

$$= \hat{J}_{s//,m} \int_{z=p\theta/2\pi-\delta/2}^{p\theta/2\pi-\delta/2+p} dz + \sum_{n\neq m=-\infty}^{\infty} \hat{J}_{s//,m} \int_{z=p\theta/2\pi-\delta/2}^{p\theta/2\pi-\delta/2+p} \exp[-j(m-n)\theta] \exp\ j(m-n)(2\pi z / p) dz$$

$$J\exp(-j\beta_m p\theta/2\pi \int_{p\theta/2\pi-\delta/2}^{p\theta/2\pi+\delta/2} \exp\ j(\beta_m z dz$$

$$= \hat{J}_{s//,m} \int_{z=p\theta/2\pi-\delta/2}^{p\theta/2\pi-\delta/2+p} dz + \sum_{n\neq m=-\infty}^{\infty} \hat{J}_{s//,m} \int_{z=p\theta/2\pi-\delta/2}^{p\theta/2\pi-\delta/2+p} \exp[-j(m-n)\theta]\exp\ j(m-n)(2\pi z/p)dz$$

(rewritten)

$\downarrow$

$$J\exp(-j\beta_m p\theta/2\pi \exp(j\beta_m p\theta/2\pi)[\frac{\exp(j\beta_m\delta/2)-\exp(j\beta_m\delta/2)}{j\beta_m}]$$

$$= \hat{J}_{s//,m} p + \sum_{n\neq m=-\infty}^{\infty} \hat{J}_{s//,m} \frac{\exp[-j(m-n)\theta}{j(m-n)\theta/p}[\exp\{j(m-n)(2\pi/p)(p\theta/2\pi-\delta/2)\}$$

$$\times[\exp\ j(m-n)2\pi-1]$$

$$J\exp(-j\beta_m p\theta/2\pi\exp(j\beta_m p\theta/2\pi)[\frac{\exp(j\beta_m\delta/2)-\exp(j\beta_m\delta/2)}{j\beta_m}]$$

$$=\hat{J}_{s//,m}p+\sum_{n\neq m=-\infty}^{\infty}\hat{J}_{s//,m}\frac{\exp[-j(m-n)\theta}{j(m-n)\theta/p}[\exp\{j(m-n)(2\pi/p)(p\theta/2\pi-\delta/2)\}$$

$$\times[\exp\ j(m-n)2\pi-1]$$

(rewritten)

$$\downarrow$$

$$J[\frac{2j\sin\beta_m\delta/2}{j\beta_m}]=\hat{J}_{s//,m}p$$

$$\exp(j\beta_m\delta/2)-\exp(j\beta_m\delta/2)=2j\sin(\beta_m\delta/2)$$

$$\downarrow$$

$$\hat{J}_{s//,m}=J[\frac{2\sin\beta_m\delta/2}{\beta_m p}]$$

$$\downarrow$$

$$\hat{J}_{s//,m}=J(\frac{\sin(\beta_m\delta/2)}{\beta_m\delta/2})(\frac{\delta}{p})$$

$$\downarrow$$

$$\hat{J}_{s//,n}=J(\frac{\sin(\beta_n\delta/2)}{\beta_n\delta/2})(\frac{\delta}{p})$$

$$E_{1,//}(a)=\sum_{n=-\infty}^{\infty}\left[\left(\frac{-n\beta_n}{\gamma_n^2 a}I_n\{\gamma_n a\}\cos\psi+I_n\{\gamma_n a\}\sin\psi\right)A_{1n}-\left(\frac{j\omega\mu_0}{\gamma_n}I_n'\{\gamma_n a\}\cos\psi\right)C_{1n}\right]\times$$

$$\exp(-j\beta_0 z)\exp-jn((2\pi z/p)-\theta)$$

$$\uparrow$$

$$\leftarrow$$

Set at the centre-line of the tape at $z=p\theta/2\pi$**:**

$$E_{1,//}(a)=0$$

$$\left.\begin{aligned}A_{1,n}&=-\frac{\gamma_n^2 aK_n(\gamma_n a)}{j\omega\varepsilon_0}[\hat{J}_{s//,n}\sin\psi-(n\beta_n/\gamma_n{}^2 a)\hat{J}_{s//,n}\cos\psi]\\C_{1n}&=-\gamma_n aK_n'(\gamma_n a)\hat{J}_{s//,n}\cos\psi\end{aligned}\right\}$$

$$\downarrow$$

$$E_{1,//}(a)=\sum_{n=-\infty}^{\infty}\frac{j\sin^2\psi\hat{J}_{s//,n}}{\omega\varepsilon_0 a}[(\frac{n\beta_n a}{\gamma_n a}\cot\psi-\gamma_n a)^2 I_n(\gamma_n a)K_n(\gamma_n a)$$

$$+k_0{}^2a^2\cot^2\psi I_n'(\gamma_n a)K_n'(\gamma_n a)]\exp(-j\beta_0 z)\exp-jn((2\pi z/p)-\theta)=0$$

$$E_{1,//}(a)=\sum_{n=-\infty}^{\infty}\frac{j\sin^{2}\psi\hat{J}_{s//,n}}{\omega\varepsilon_{0}a}[(\frac{n\beta_{n}a}{\gamma_{n}a}\cot\psi-\gamma_{n}a)^{2}I_{n}(\gamma_{n}a)K_{n}(\gamma_{n}a)$$
$$+k_{0}^{2}a^{2}\cot^{2}\psi I_{n}'(\gamma_{n}a)K_{n}'(\gamma_{n}a)]\exp(-j\beta_{0}z)\exp-jn((2\pi z/p)-\theta)=0$$

(rewritten)

$$z=\frac{p\theta}{2\pi}$$

$$\hat{J}_{s//,n}=J(\frac{\sin(\beta_{n}\delta/2)}{\beta_{n}\delta/2})(\frac{\delta}{p})$$

$$\frac{j\sin^{2}\psi}{\omega\varepsilon_{0}a}(J)(\frac{\delta}{p})\exp(-j\beta_{0}\frac{p\theta}{2\pi})\sum_{n=-\infty}^{\infty}[(\frac{n\beta_{n}a}{\gamma_{n}a}\cot\psi-\gamma_{n}a)^{2}I_{n}(\gamma_{n}a)K_{n}(\gamma_{n}a)$$
$$+k_{0}^{2}a^{2}\cot^{2}\psi I_{n}'(\gamma_{n}a)K_{n}'(\gamma_{n}a)][\frac{\sin(\beta_{n}\delta/2)}{\beta_{n}\delta/2}]=0$$

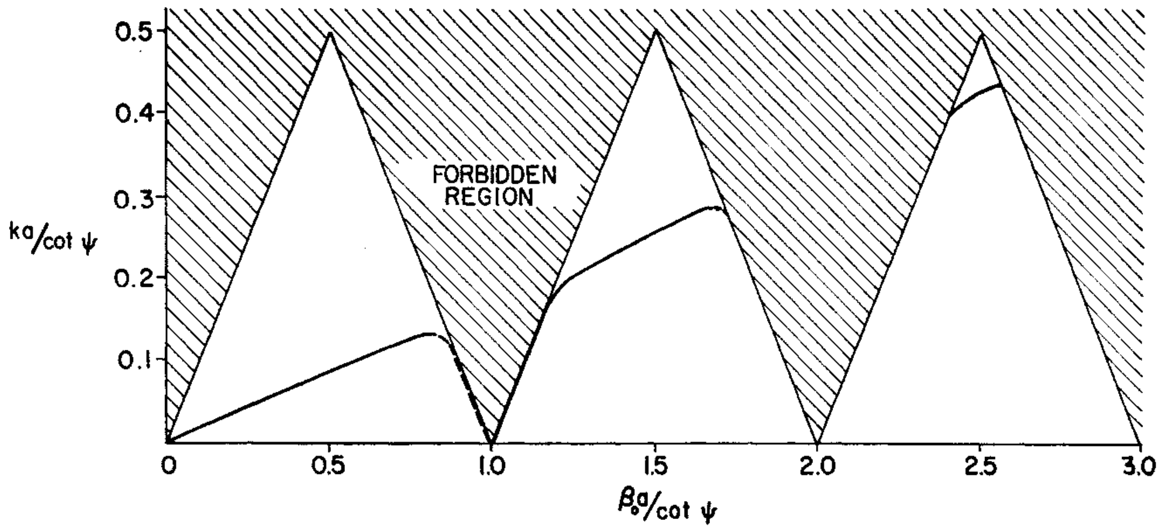
$$\sum_{n=-\infty}^{\infty}[(\frac{n\beta_{n}a}{\gamma_{n}a}\cot\psi-\gamma_{n}a)^{2}I_{n}(\gamma_{n}a)K_{n}(\gamma_{n}a)$$
$$+k_{0}^{2}a^{2}\cot^{2}\psi I_{n}'(\gamma_{n}a)K_{n}'(\gamma_{n}a)][\frac{\sin(\beta_{n}\delta/2)}{\beta_{n}\delta/2}]=0$$

$$\gamma_{n}=(\beta_{n}^{2}-k^{2})^{1/2}$$

$$\beta_{n}=\beta_{0}+2\pi n/p$$

(dispersion relation in the tape-helix model)

Dispersion plot for a helix in free-space obtained by tape-helix model



n = 0-**solution for** $\psi = 10^0$ **and** $\pi\delta/p = 0.1$

Forbidden regions

Permitted solution for real $\gamma_n a = [(\beta_n a)^2 - (ka)^2]^{1/2}$

$$\downarrow$$

$$(\beta_n a)^2 > (ka)^2$$

$$\downarrow$$

$$\beta_n a > \pm ka$$

$$\downarrow$$

$$\frac{\beta_0 a}{\cot\psi} + n > \pm \frac{ka}{\cot\psi}$$

$$\beta_n = \beta_0 + 2\pi n / p$$

$$\downarrow$$

$$\beta_n a = \beta_0 a + 2\pi a n / p = \beta_0 a + n(2\pi a / p)$$

$$\beta_n a = \beta_0 a + n \cot\psi$$

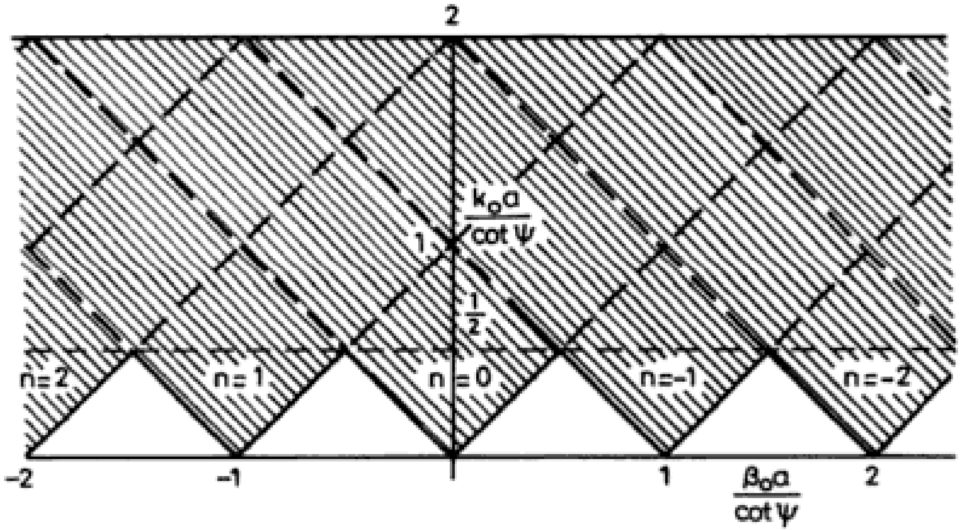
Boundaries between the allowed and forbidden regions are straight lines given by

$$\frac{ka}{\cot\psi} = \pm\left(\frac{\beta_0 a}{\cot\psi} + n\right)$$

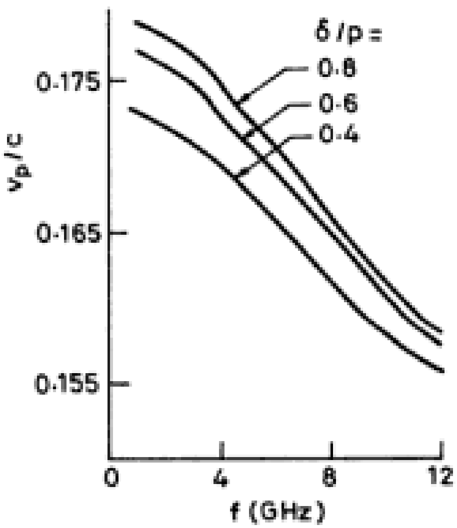
Boundaries between the allowed and forbidden regions are straight lines given by

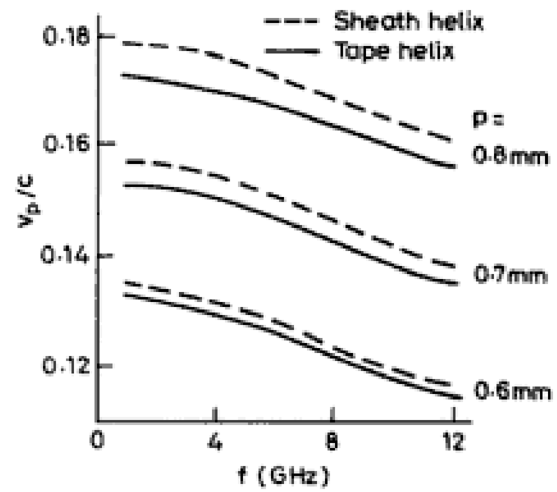
$$\frac{ka}{\cot\psi} = \pm\left(\frac{\beta_0 a}{\cot\psi} + n\right)$$

Hatched region shows the forbidden region, and the boundaries between the allowed (non-hatched) and forbidden regions are shown as straight lines of positive and negative unity slopes (± 1)**, for different values of** n (= 0, 1, -1, 2, -2, **etc.).**



Dispersion plot for a helix with dielectric wedge bar supports in a metal envelope





ϕ = 20^0, ψ = 10^0, $N = 3$, $a = 1.0$ mm, $b/a=2$, $\varepsilon_r=5.1$ (APBN)

ϕ = 20^0, $N = 3$, $a = 0.75$ mm, $b = 1.6$ mm, $\delta = 0.4$ mm, $\varepsilon_r = 5.1$ (APBN)

Heuristic tape-helix model

Makes the analysis leading to the dispersion relation of a loaded helical slow-wave structure rather simple!

Dispersion relation of an unloaded helix in free-space in the tape-helix model:

$$\boxed{\sum_{-\infty}^{\infty}(M_m(\gamma_m a)+N_m(\gamma_m a)\left(\frac{\sin(\beta_m\delta/2}{\beta_m\delta/2}\right)=0}$$

$$\delta=\text{tape width},\ \gamma_m=(\beta_m{}^2-k_0{}^2)^{1/2},\ k_0=\omega(\mu_0\varepsilon_0)^{1/2}$$

$$\boxed{M_m(\gamma_m a)=(\frac{m\beta_m}{\gamma_m}\cot\psi-\gamma_m a)^2 I_m(\gamma_m a)K_m(\gamma_m a)}$$

$$\boxed{N_m\{\gamma_m a\}=k_0{}^2 a^2\cot^2\psi\, I'_m\{\gamma_m a\}K'_m\{\gamma_m a\}}$$

$$\psi=\text{helix pitch angle}$$

$$a=\text{mean helix radius}$$

Prime indicates the derivative with respect to argument.

S. Sensiper: *Proc. IRE* 43 (1955) 149-161

Dispersion relation of a loaded helix in the sheath-helix model:

$$\frac{k_0 \cot\psi}{\gamma_0} = \left(-\frac{I_0(\gamma_0 a) K_0(\gamma_0 a)}{I_0'(\gamma_0 a) K_0'(\gamma_0 a)} \right)^{1/2} D_0(\gamma_0 a)$$

$$\Downarrow$$

$$M_0\{\gamma_0 a\} D_0^{\,2}\{\gamma_0 a\} + N_0\{\gamma_0 a\} = 0$$

Combinational approach

Combine the dispersion relation of an unloaded helix in free-space in the tape-helix model with the dispersion relation of a loaded helix in the sheath-helix model

A. K. Sinha and others: *IEE Proceedings-H: Microwave, Antenna & Propagation* 139 (1992), 347-350

Dispersion relation of an unloaded helix in free-space in the tape-helix model:

$\Downarrow$

$$\sum_{-\infty}^{\infty}\left(M_m(\gamma_m a)+N_m(\gamma_m a)\left(\frac{\sin(\beta_m\delta/2}{\beta_m\delta/2}\right)\right)=0$$

Dispersion relation of a loaded helix in the sheath-helix model:

$\Downarrow$

$$M_0(\gamma_0 a)D_0^{\,2}(\gamma_0 a)+N_0(\gamma_0 a)=0$$

Heuristic combinational approach:

Dispersion relation of a loaded helix in the tape-helix model

$\Downarrow$

$$\sum_{-\infty}^{\infty}\left(M_m(\gamma_m a)D_m^{\,2}(\gamma_m a)+N_m(\gamma_m a)\left(\frac{\sin(\beta_m\delta/2}{\beta_m\delta/2}\right)\right)=0$$

Dispersion relation of a loaded helix in the tape-helix model

$$\sum_{-\infty}^{\infty}(M_m(\gamma_m a)D_m{}^2(\gamma_m a)+N_m(\gamma_m a)\left(\frac{\sin(\beta_m\delta/2}{\beta_m\delta/2}\right)=0$$

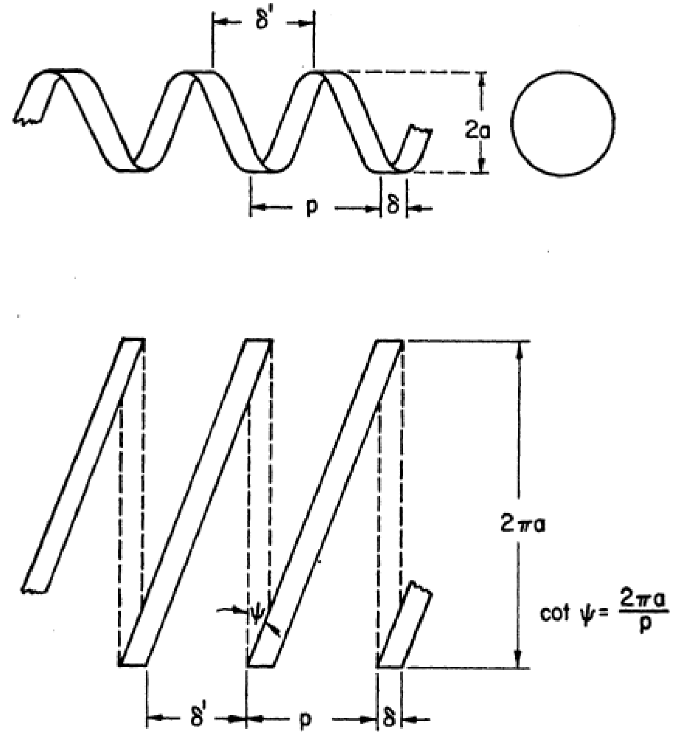
$D_m(\gamma_m a)$ is obtained by replacing $\gamma_0 a$ by $\gamma_m a$ in the expression for $D_0(\gamma_0 a)$

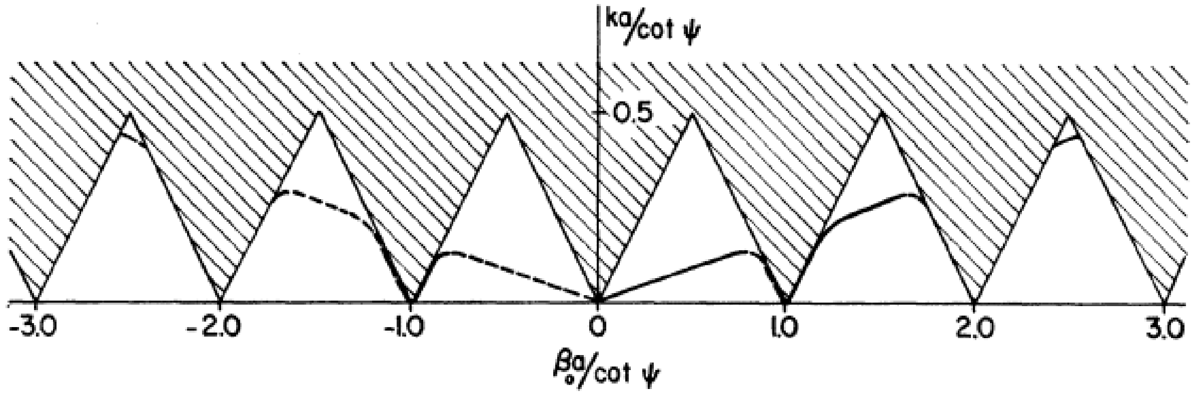
The above dispersion relation of a loaded helix obtained by the heuristic tape-helix model exactly agrees with the dispersion relation obtained by the analysis of the loaded helix by the rigorous tape-helix model!

A. K. Sinha and others: *IEE Proceedings-H: Microwave, Antenna & Propagation* 139 (1992), 347-350

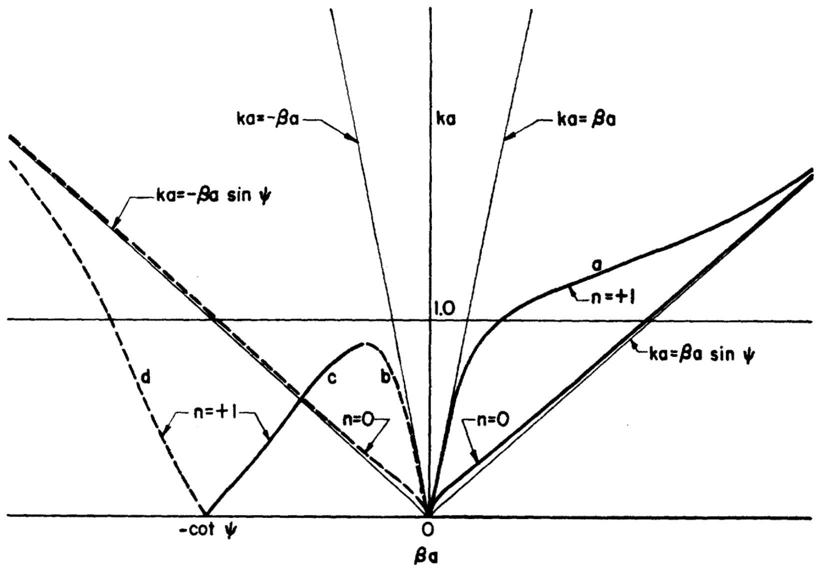
Thank you!

Appendix

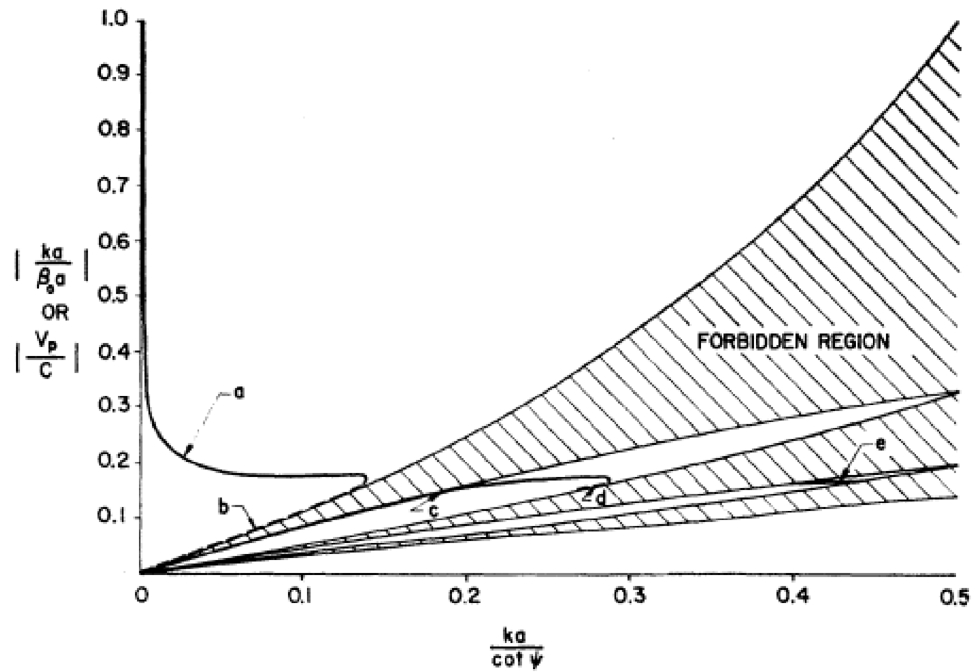

δ'
2a
p
δ
2πa
cot ψ = 2πa / p
ψ
δ'
p
δ



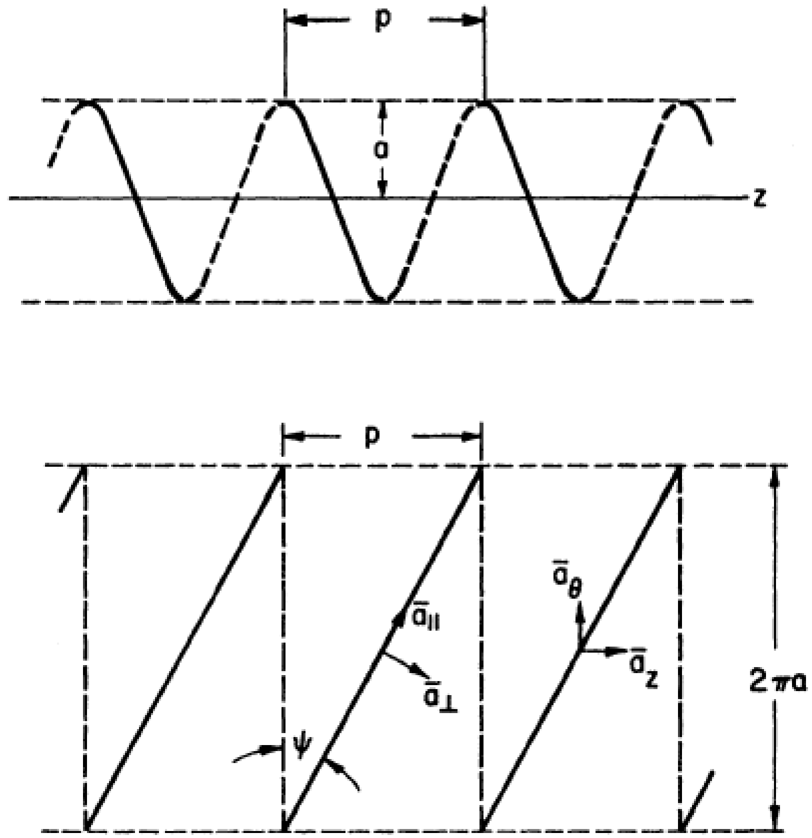
ka/cot ψ
0.5
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
βₒa/cot ψ



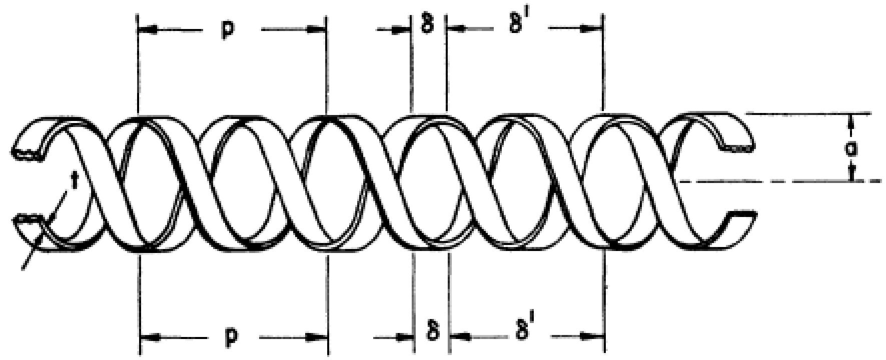
ka=-βa
ka
ka=βa
ka=-βa sin ψ
a
n=+1
1.0
ka=βa sin ψ
d
c
b
n=+1
n=0
n=0
-cot ψ
0
βa

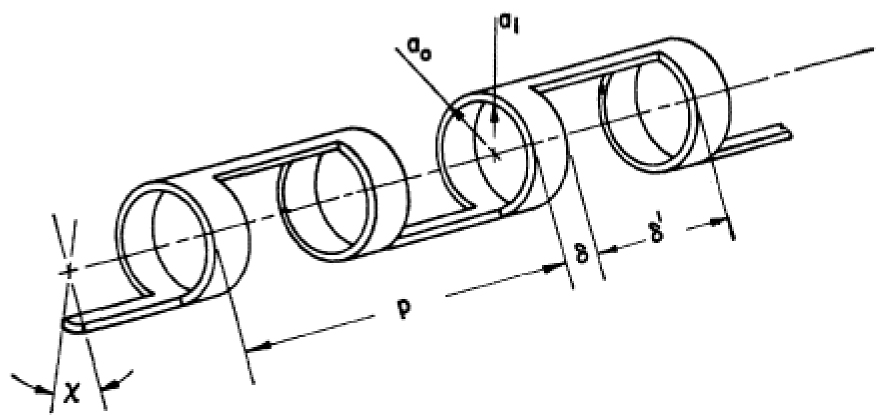


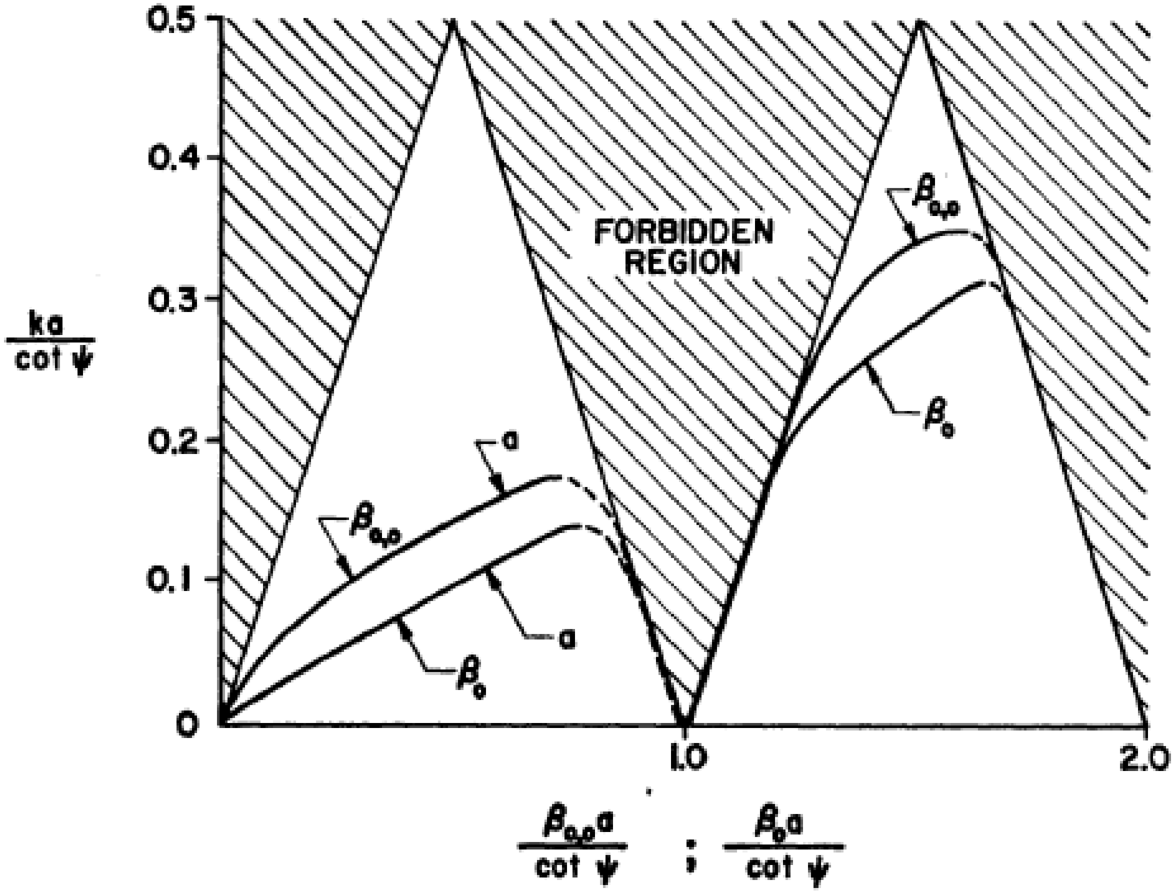
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
$\left|\frac{ka}{\beta_0 a}\right|$
OR
$\left|\frac{V_p}{C}\right|$
FORBIDDEN REGION
a
b
c
d
e
0
0.1
0.2
0.3
0.4
0.5
$\frac{ka}{\cot\psi}$

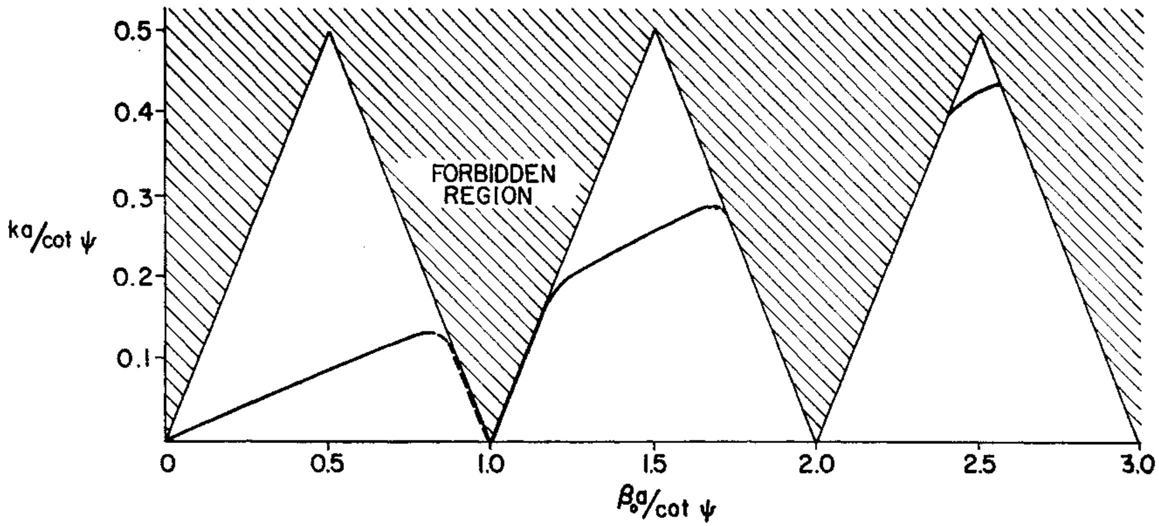


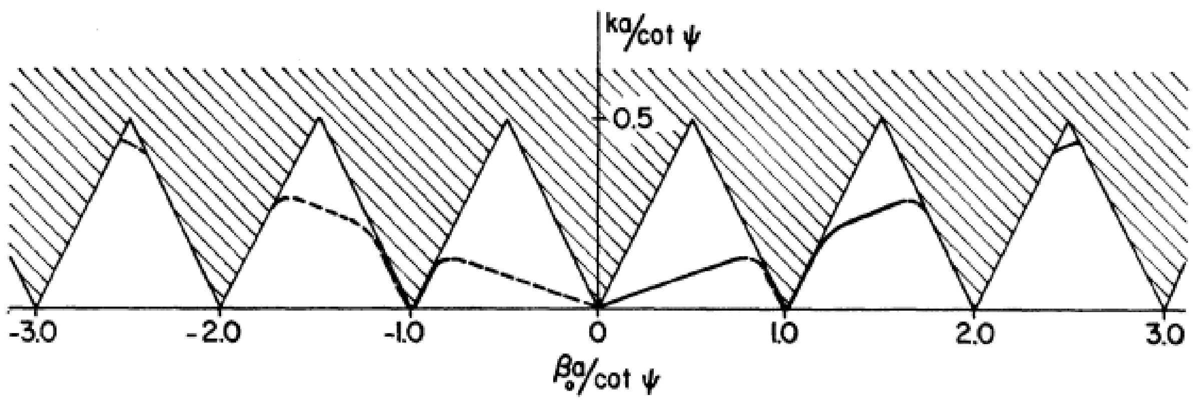
p
a
z
p
$\bar{a}_\theta$
$\bar{a}_{\parallel}$
$\bar{a}_z$
$\bar{a}_\perp$
$2\pi a$
ψ

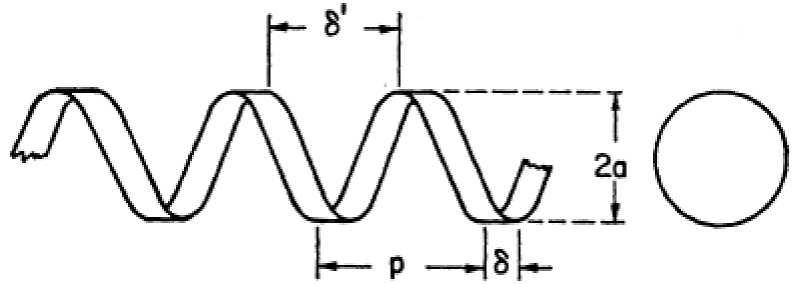

p
δ
δ'
a
t
p
δ
δ'

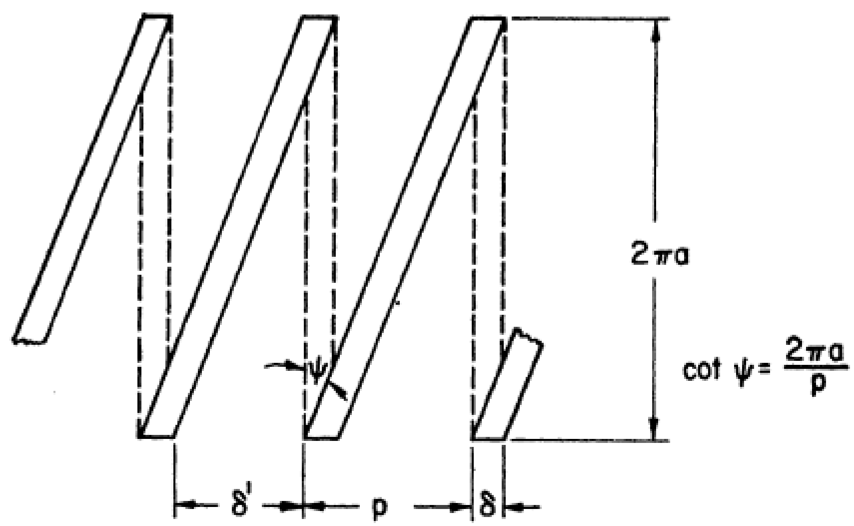

a₁
a₀
δ
δ'
p
χ

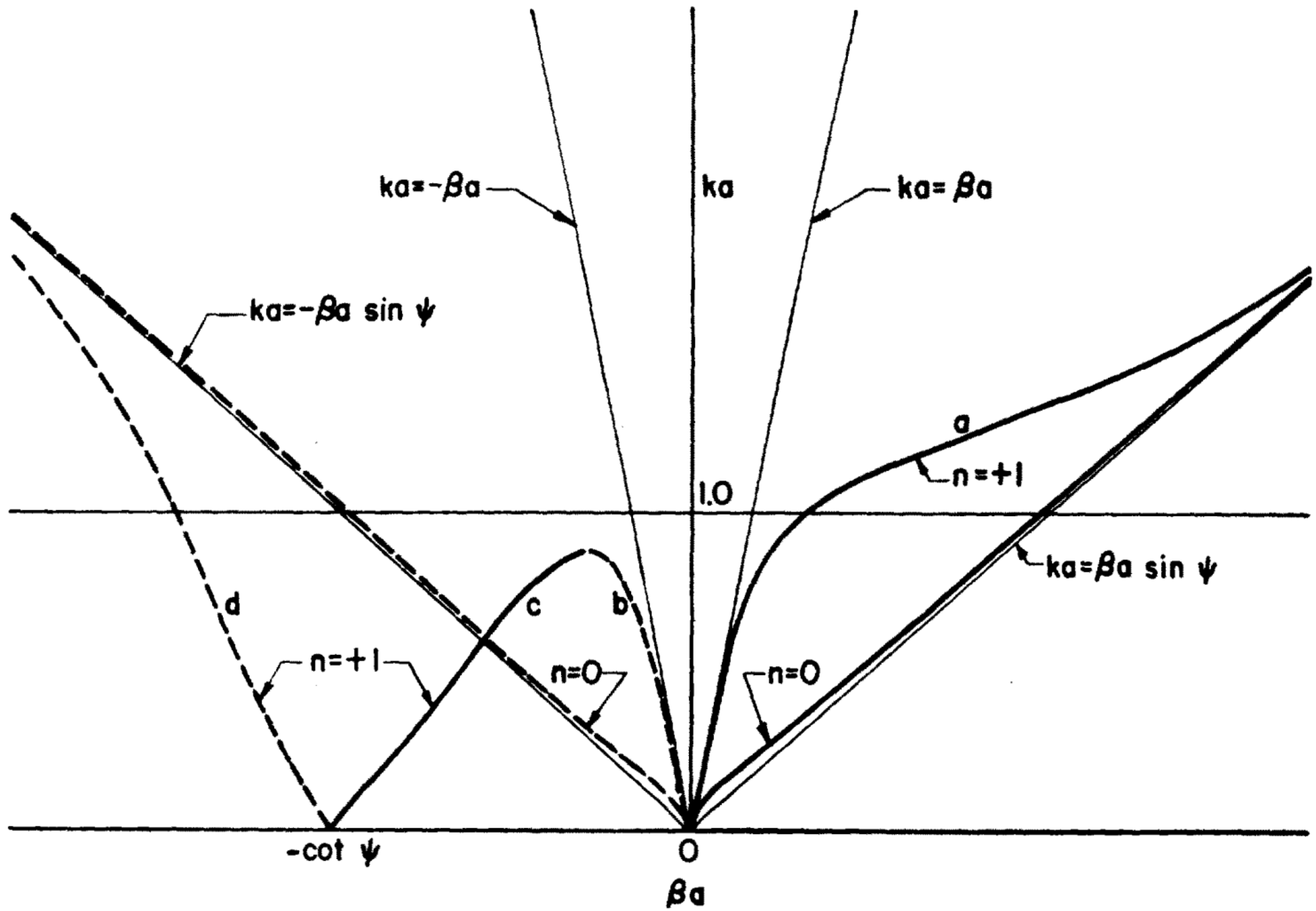

0.5
0.4
0.3
0.2
0.1
0
ka / cot ψ
FORBIDDEN REGION
β0,0
β0
a
1.0
2.0
β0,0 a / cot ψ ; β0 a / cot ψ

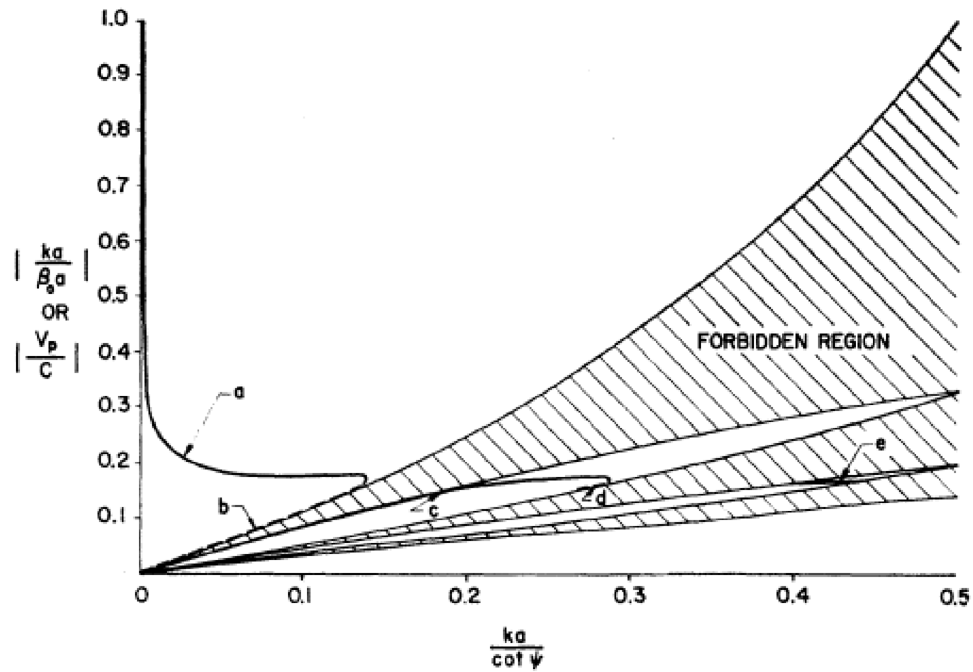

0.5
0.4
0.3
0.2
0.1
ka/cot ψ
FORBIDDEN
REGION
0
0.5
1.0
1.5
2.0
2.5
3.0
βoa/cot ψ

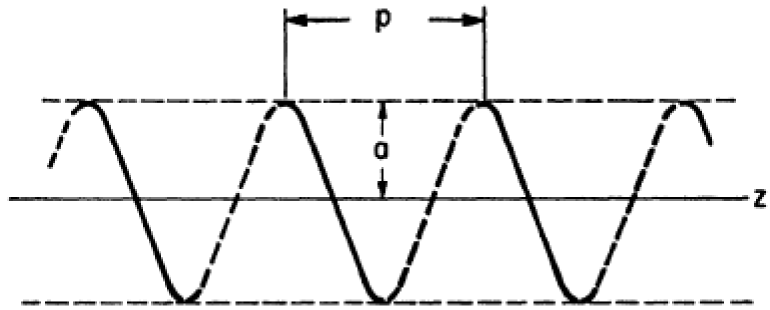

ka/cot ψ
0.5
-3.0
-2.0
-1.0
0
1.0
2.0
3.0
βoa/cot ψ

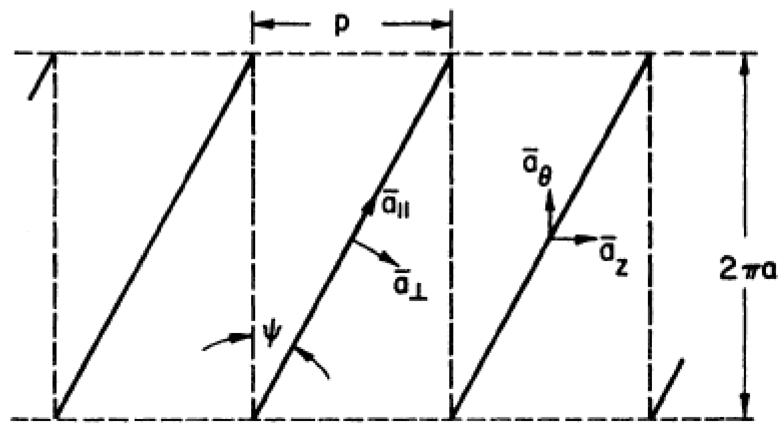

δ'
2a
p
δ

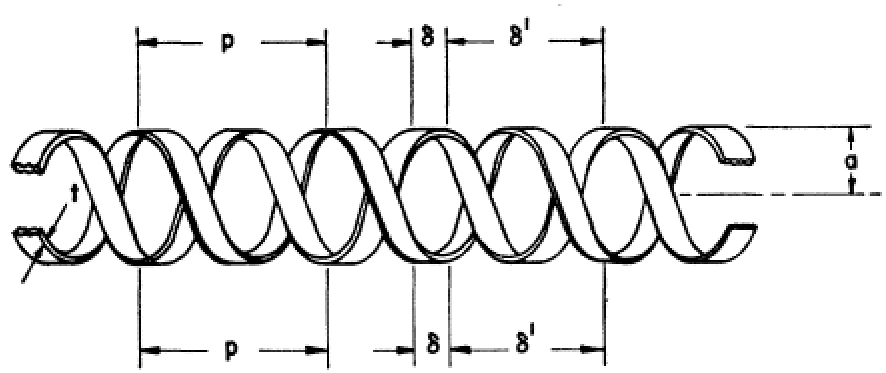

2πa
ψ
cot ψ = 2πa / p
δ'
p
δ

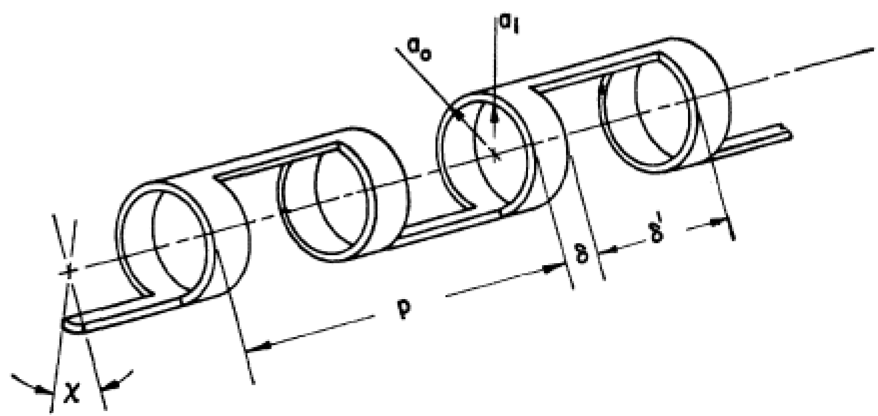

ka=-βa
ka
ka= βa
ka=-βa sin ψ
a
n=+1
1.0
ka=βa sin ψ
d
c
b
n=+1
n=0
n=0
-cot ψ
0
βa

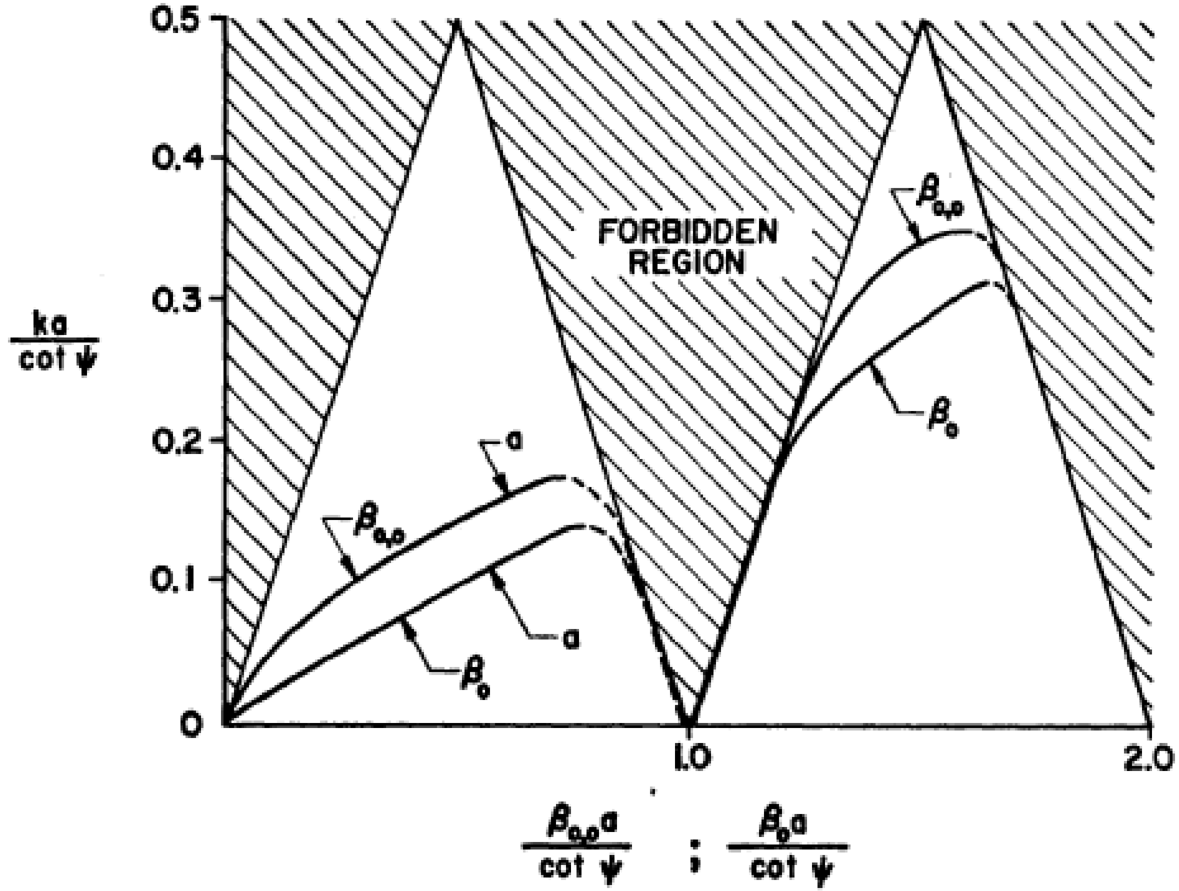

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
$\left|\frac{ka}{\beta_0 a}\right|$
OR
$\left|\frac{V_p}{C}\right|$
a
b
c
d
e
FORBIDDEN REGION
0
0.1
0.2
0.3
0.4
0.5
$\frac{ka}{\cot\psi}$


p
a
z


p
$\bar{a}_\theta$
$\bar{a}_\parallel$
$\bar{a}_\perp$
$\bar{a}_z$
$2\pi a$
ψ


p
δ
δ'
a
t
p
δ
δ'


a1
ao
δ
δ'
p
χ



0.5
0.4
0.3
0.2
0.1
0
$\frac{ka}{\cot\psi}$
FORBIDDEN REGION
$\beta_{o,o}$
β_o
a
1.0
2.0
$\frac{\beta_{o,o}a}{\cot\psi}$; $\frac{\beta_o a}{\cot\psi}$