Travelling-Wave Tube: Beam-Wave Interaction

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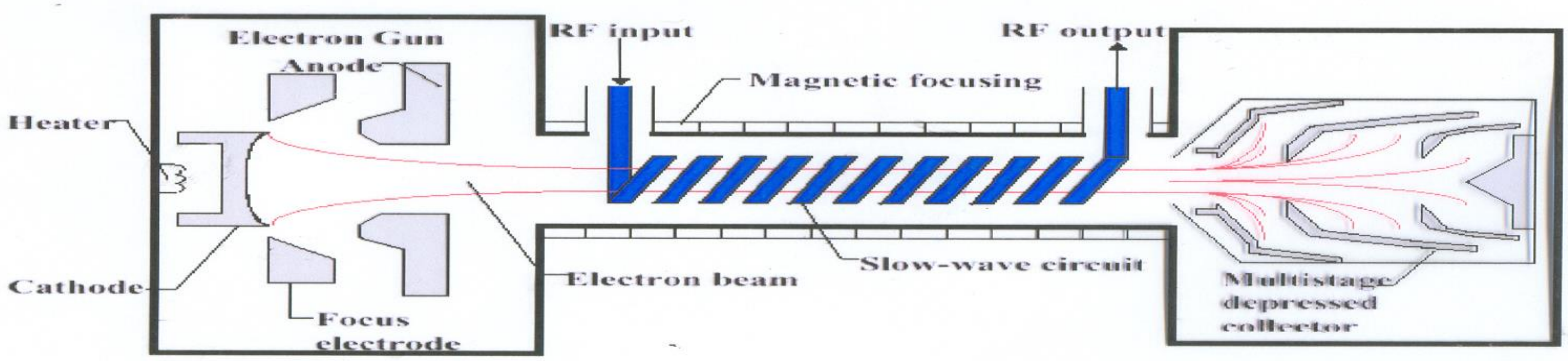
Principal parts of a TWT:

- Electron gun: electron beam formation
- Focusing structure: electron beam confinement
- Collector: collection of spent electron beam
- Slow-wave structure (SWS): excitation of slow RF wave for interaction with the electron beam
- Attenuator: suppression of oscillation
- RF input and output couplers



Scope of the present lecture

- Classification of the TWT in the family of microwave tubes
- Axial electron bunching and near-synchronism condition
- Space-charge waves
- Coupling between the circuit and space-charge waves
- Pierce’s theory for the growth parameter and gain of a TWT considering the coupling between the electron beam and the slow-wave structure
- Extension of Pierce's theory to estimate of hot attenuation
- Extension of Pierce's theory to arrive at Johnson’s start-oscillation condition

Identification of the TWT in the family of microwave tubes

- O type (TPO: tubes à propagation des ondes)
- Slow-wave type
- Axial bunching type
- Axial beam kinetic energy conversion type
- Distributed interaction type
- Growing-wave type
- Cerenkov radiation type

TWT is a Cerenkov radiation type of device

DC electron beam velocity is made close to but slightly greater than the phase velocity of the RF wave supported by the structure (near synchronization condition).

(1) This ensures the bunch of electrons in the beam to remain in the decelerating RF phase of the circuit (slow-wave structure) on the average transferring their kinetic energy to RF waves.

(2) This also makes the slow space-charge wave on the electron beam to couple to RF waves making on the average the beam kinetic power density to be negative and the electromagnetic power to be positive corresponding to the transfer of beam kinetic power to electromagnetic power of RF waves (Chu's kinetic power density concept).

Space-charge waves

Two forward (Hahn and Ramo) space-charge waves

Slow space-charge wave : $v_p < v_0$

Fast space-charge wave : $v_p > v_0$

One-dimensional small-signal analysis
A neutralizing presence of positive ions assumed

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DC beam velocity = constant beyond a narrow high frequency modulating gap at $z = 0$

Formulation of a differential equation in perturbed part of volume charge density and its solution for propagation constant of space-charge waves

∗ Current density equation ∗ Continuity equation

∗ Force equation ∗ Poisson's equation

Space-charge waves

$J = \rho v$ (Current density equation) $\longleftarrow$ $\vec{J} = \rho\vec{v}$ $\longleftarrow$

$\dfrac{\partial J_1}{\partial z} + \dfrac{\partial \rho_1}{\partial t} = 0$ (Continuity equation) $\longleftarrow$ $\nabla.\vec{J} + \dfrac{\partial \rho}{\partial t}$ $\longleftarrow$

$\dfrac{\partial E_s}{\partial z} = \dfrac{\rho_1}{\varepsilon_0}$ (Poisson's equation) $\longleftarrow$ $\nabla.\vec{E} = \dfrac{\rho}{\varepsilon}$ $\longleftarrow$

$\dfrac{dv_1}{dt} = \dfrac{e}{m}E_s = \eta E_s$ (Force/acceleration equation) $\longleftarrow$

$$\left.\begin{aligned} J &= J_0 + J_1 \\ \rho &= \rho_0 + \rho_1 \\ v &= v_0 + v_1 \end{aligned}\right\}$$

$J = \rho v$ (one-dimensional) $\longleftarrow$ $\vec{J} = \rho\vec{v}$ (current density equation)

$\downarrow \longleftarrow$ $J = J_0 + J_1, \rho = \rho_0 + \rho_1, v = v_0 + v_1$

$$J_0 + J_1 = (\rho_0 + \rho_1)(v_0 + v_1) = \rho_0 v_0 + \rho_0 v_1 + \rho_1 v_0 + \rho_1 v_1 \longleftarrow J_0 = \rho_0 v_0$$

$$\boxed{J_1 = \rho_0 v_1 + v_0 \rho_1 + \rho_1 v_1 = \rho_0 v_1 + v_0 \rho_1}$$ (small-signal approximation)

$$\frac{\partial J_1}{\partial z} = \rho_0 \frac{\partial v_1}{\partial z} + v_0 \frac{\partial \rho_1}{\partial z} \longleftarrow \frac{\partial J_1}{\partial z} + \frac{\partial \rho_1}{\partial t} = 0 \longleftarrow \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

(continuity equation)

$$-\frac{\partial \rho_1}{\partial t} = \rho_0 \frac{\partial v_1}{\partial z} + v_0 \frac{\partial \rho_1}{\partial z}$$

$$\frac{\partial \rho_1}{\partial t} + v_0 \frac{\partial \rho_1}{\partial z} = -\rho_0 \frac{\partial v_1}{\partial z} \longleftarrow D = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}$$

$$D\rho_1 = -\rho_0 \frac{\partial v_1}{\partial z}$$

Force equation

$$m\frac{dv_1}{dt} = eE_s \qquad v = v_0 + v_1$$

$$\frac{dv_1}{dt} = \frac{e}{m}E_s = \eta E_s$$

$$\frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + \frac{dz}{dt}\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + v\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + (v_0 + v_1)\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial z}$$

$$\frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial z} = \eta E_s \quad \longleftarrow \quad D = \frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}$$

$$Dv_1 = \eta E_s$$

$$D\rho_1 = -\rho_0 \frac{\partial v_1}{\partial z}$$ (obtained earlier) $$[\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z} = D]$$

$$Dv_1 = \eta E_s$$ (obtained earlier)

$$D^2\rho_1 = -\rho_0 D \frac{\partial v_1}{\partial z} = -\rho_0 \frac{\partial}{\partial z} Dv_1 = -\rho_0 \frac{\partial}{\partial z} \eta E_s = -\eta\rho_0 \frac{\partial E_s}{\partial z}$$

$$D^2\rho_1 = -\eta\rho_0 \frac{\partial E_s}{\partial z} \quad \longleftarrow \quad \frac{\partial E_s}{\partial z} = \frac{\rho_1}{\varepsilon_0}$$

$$\omega_p = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$

$$D^2\rho_1 = -\eta\rho_0 \frac{\rho_1}{\varepsilon_0} = \frac{-\eta\rho_0}{\varepsilon_0}\rho_1 = \frac{-|\eta||\rho_0|}{\varepsilon_0}\rho_1 = -\omega_p{}^2\rho_1$$

$$D^2 = -\omega_p{}^2$$

$$D = \pm j\omega_p$$

RF quantities vary as $\exp j(\omega t - \beta z)$

$$\frac{\partial}{\partial t} = j\omega, \quad \frac{\partial}{\partial z} = -j\beta$$

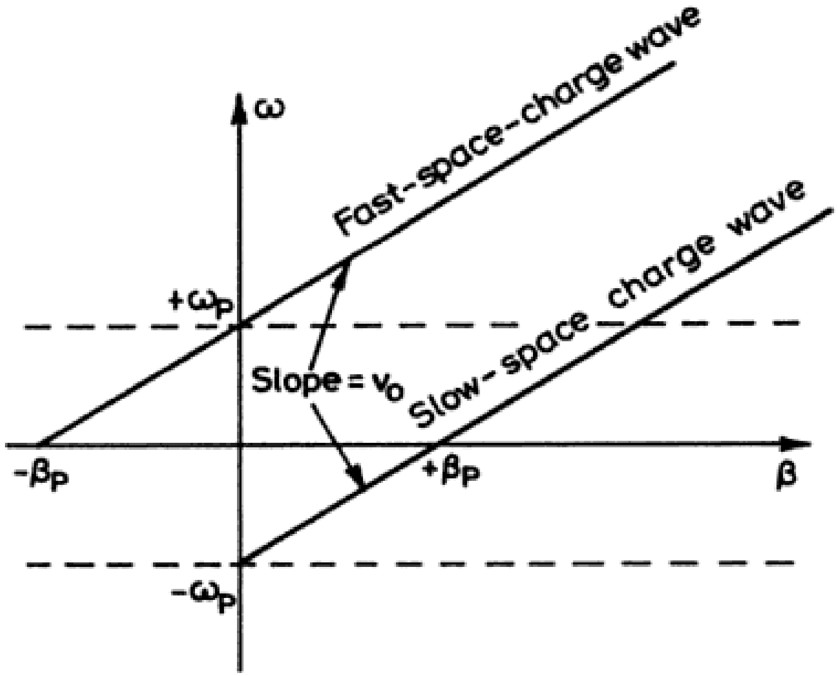
$$D = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z} = j\omega - j\beta v_0 = j(\omega - \beta v_0)$$

$D = \pm j\omega_p$ (recalled)

$$\pm j\omega_p = j(\omega - \beta v_0)$$

$$\boxed{\omega - \beta v_0 = \pm\omega_p}$$

Dispersion relation of Hahn and Ramo space-charge waves



$\omega - \beta v_0 = \pm\omega_p$ (Hahn and Ramo space-charge waves)

$$\frac{\omega}{v_0} = \beta_e, \ \frac{\omega_p}{v_0} = \beta_p$$

$$\boxed{\beta = \frac{\omega \mp \omega_p}{v_0} = \beta_e \mp \beta_p}$$

$$\boxed{v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \mp \omega_p} v_0 = \frac{\omega}{\beta_e \mp \beta_p}}$$ (phase velocity of space-charge waves)

The upper sign $\Rightarrow$ Fast space-charge wave

The lower sign $\Rightarrow$ Slow space-charge wave

$$\omega - \beta v_0 = \pm\omega_p$$ (space-charge-wave dispersion relation)

$$v_0 = 0 \Rightarrow \quad \omega = \pm\omega_p$$

In a frame of reference which moves with the dc beam velocity v_0, an observer 'sees' a Doppler-shifted frequency ω' given by

$$\omega' = \frac{v_p - v_0}{v_p}\omega = \left(1 - \frac{v_0}{v_p}\right)\omega \quad \leftarrow \quad v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \mp \omega_p} v_0$$

$$\omega' = \left(1 - \frac{\omega \mp \omega_p}{\omega}\right)\omega = \mp\omega_p$$

Dispersion relation of the beam-wave-coupled system (to be deduced later)

$$\frac{-\Gamma\Gamma_0 K}{\Gamma^2-\Gamma_0^2}=\frac{2V_0}{I_0}\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma}$$

$\downarrow$

$$(\Gamma^2-\Gamma_0^2)((j\beta_e-\Gamma)^2+\beta_p^2)=\Gamma^2\beta_e\beta_0\frac{KI_0}{2V_0}$$

$$(\Gamma^2-\Gamma_0^2)((j\beta_e-\Gamma)^2+\beta_p^2)\approx-\beta_e{}^4\frac{KI_0}{2V_0}$$

$\downarrow \quad \leftarrow \quad \Gamma\approx j\beta_0\approx j\beta_e$

$$(\Gamma^2-\Gamma_0^2)((j\beta_e-\Gamma)^2+\beta_p^2)\approx-\beta_e{}^4\frac{KI_0}{2V_0}$$

$V_0,\ I_0=$ beam voltage, beam current

$K=$ interaction impedance of the structure

RF quantities vary as

$$\exp(-\Gamma z)=\exp(-j\beta z)$$

$\Gamma_0=j\beta_0$

$\beta_0=$ cold (beam-absent) propagation constant of the circuit (SWS)

$\beta_e(=\omega/v_0)=$ cold (beam-absent) propagation constant of the circuit (SWS)

$v_0=$ Dc beam velocity

$$(\Gamma^2-\Gamma_0^2)((j\beta_e-\Gamma)^2+\beta_p^2)\approx-\beta_e{}^4\frac{KI_0}{2V_0}$$ (rewritten)

$I_0=0$ (for weak coupling)

$$\Gamma^2-\Gamma_0^2=0 \qquad (j\beta_e-\Gamma)^2+\beta_p^2=0$$

$$\Gamma=\pm\Gamma_0=\pm j\beta_0 \qquad j\beta_e-\Gamma=\pm j\beta_p$$

$\Gamma=j\beta$ $\qquad$ $\Gamma=j\beta$

$$\beta=\pm\beta_0$$

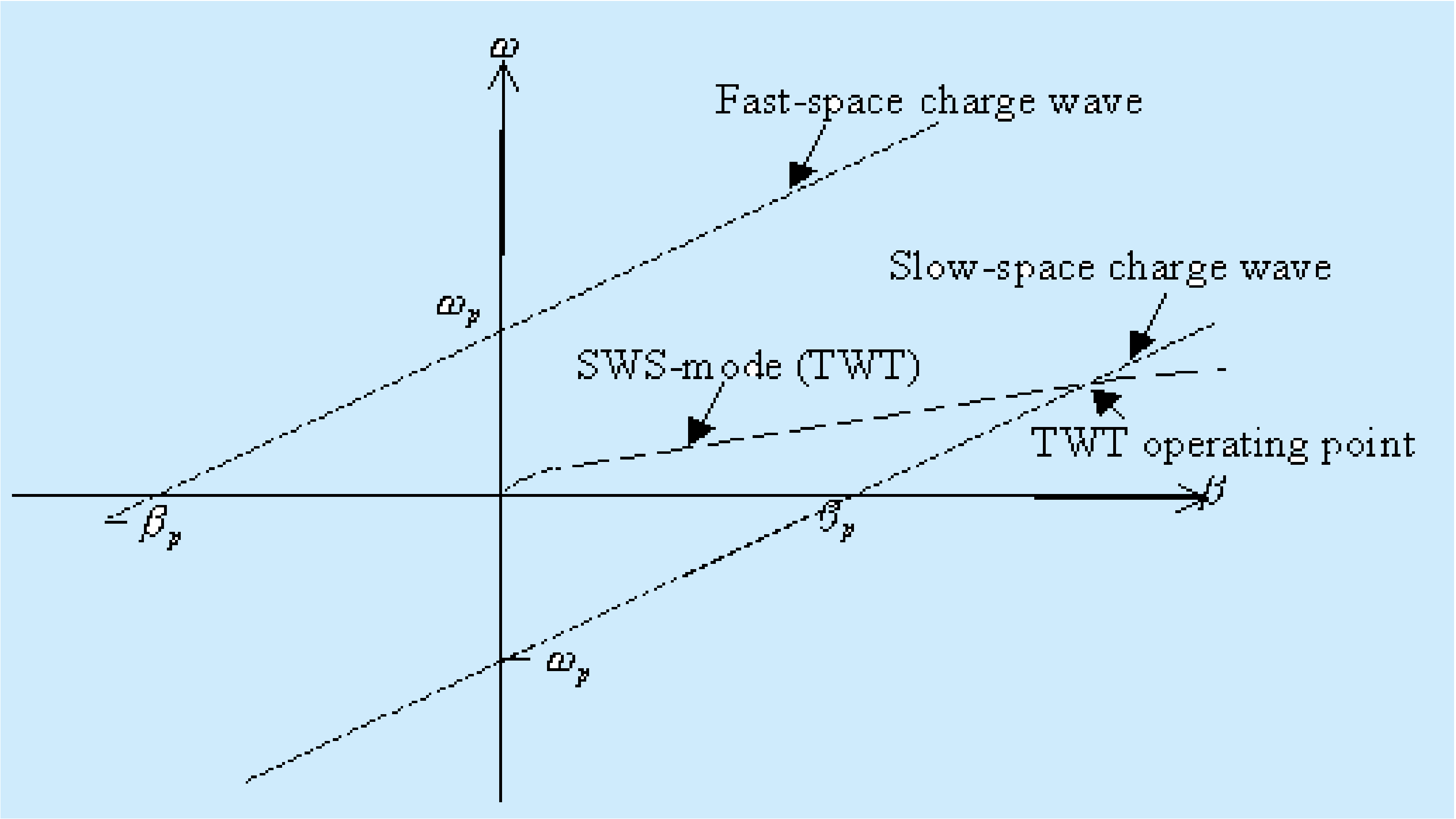
(Cold circuit wave decoupled)

$$\Gamma=j(\beta_e\mp\beta_p) \longrightarrow j\beta=j(\beta_e\mp\beta_p)$$

$$\beta=\beta_e\mp\beta_p$$

(Space-charge wave decoupled)

The operating point of the TWT corresponds to the intersection between the ω - β dispersion plot of the slow space-charge wave and that of a forward circuit wave.



The operating point of the TWT corresponds to the intersection between the ω - β dispersion plot of the slow space-charge wave and that of a forward circuit wave.

Pierce's theory for the beam-present dispersion relation of a TWT and the interpretation thereof for the TWT gain equation *G* = *A* + *BCN*

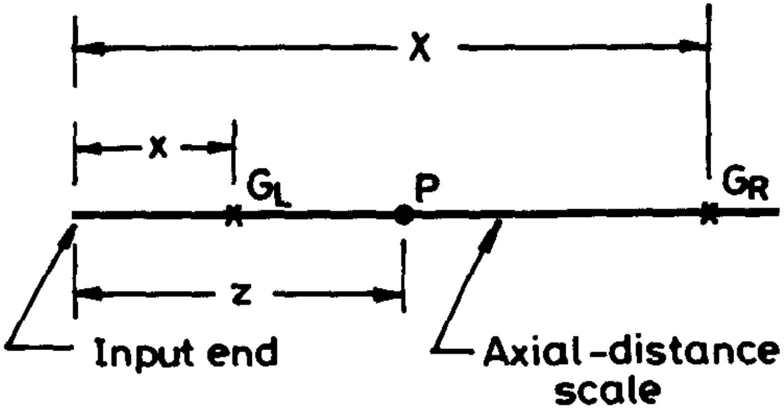
Two approaches are used to obtain the two expressions — one being the circuit equation and the other being the electronic equation, respectively, to obtain the same quantity, namely, the ratio of the circuit voltage *V* to the beam current *I*. These two expressions are then equated to obtain the dispersion relation of a TWT.

$$\frac{V}{I} = X$$ (circuit equation)

$$\frac{V}{I} = Y$$ (electronic equation)

$$X = Y$$ (TWT dispersion relation)

Circuit equation

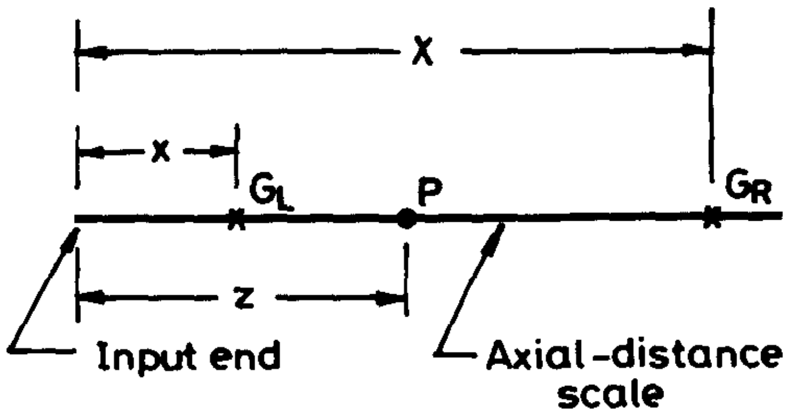


Effect of the element of a modulated beam at a point on the circuit (transmission line) is simulated by an infinitesimal current generator at that point.

The current generator 'sees' half the characteristic impedance of the transmission line, being equivalent to two such characteristic impedances in parallel, corresponding to two halves of the supposedly matched line. Such an infinitesimal generator sends two circuit waves in opposite directions, one to the left and one to the right such that the amplitudes of the circuit electric field intensity associated with these waves are equal.

z = distance from the input end of the line of a point P on the line where to find the electric field.

x = distance from the input end of the line of an infinitesimal current generator— G_L to the left of the point P sending the circuit wave to the right and G_R to the right of the point P sending the circuit wave to the left



$$E(z) = E_i \exp(-j\beta_0 z) + \int_{x=0}^{x=z} dE_R \ \exp - j\beta_0 (z-x) + \int_{x=z}^{x=\ell} dE_L \ \exp - j\beta_0 (x-z)$$

$dE_R =$ Circuit field amplitude of the wave traveling to the right due to an infinitesimal current generator G_L to the left of the point P

$dE_L =$ Circuit field amplitude of the wave traveling to the right due to an infinitesimal current generator G_R to the right of the point P

$E_i =$ Circuit field amplitude inputted to the line at $z = 0$

$$E(z) = E_i \exp(-j\beta_0 z) + \int_{x=0}^{x=z} dE_R \; \exp - j\beta_0 (z - x) + \int_{x=z}^{x=\ell} dE_L \; \exp - j\beta_0 (x - z)$$

$$\left.\begin{aligned} dE_L &= \zeta_L(x)\,dx \\ dE_R &= \zeta_R(x)\,dx \\ dE_L &= dE_R = dE = \zeta(x)\,dx \end{aligned}\right\}$$, say

$$E(z) = E_i \exp(-j\beta_0 z) + \int_0^z \zeta(x)\,dx \; \exp - j\beta_0 (z - x) + \int_{x=z}^{x=\ell} \zeta(x)\,dx \; \exp - j\beta_0 (x - z)$$

$$\boxed{E(z) = E_i \exp(-j\beta_0 z) + I_1 + I_2}$$

$$\left.\begin{aligned} I_1 &= \int_{x=0}^{x=z} \zeta(x)\,dx \; \exp - j\beta_0 (x - z)dx \\ I_2 &= \int_{x=z}^{x=\ell} \zeta(x)\,dx \; \exp - j\beta_0 (x - z)dx \end{aligned}\right\}$$

$$E(z) = E_i \exp(-j\beta_0 z) + I_1 + I_2$$ (rewritten)

$$I_1 = \int_{x=0}^{x=z} \zeta(x)\,dx \exp - j\beta_0(x-z)dx$$

$$I_2 = \int_{x=z}^{x=\ell} \zeta(x)\,dx \exp - j\beta_0(x-z)dx$$

↓

$$\frac{dE}{dz} = -j\beta_0 E_i \exp(-j\beta_0 z) + \frac{dI_1}{dz} + \frac{dI_2}{dz}$$

↓ ←

$$\frac{dI_1}{dz} = -j\beta_0 I_1 + \zeta(z)$$

$$\frac{dI_2}{dz} = j\beta_0 I_2 - \zeta(z)$$

(Leibnitz theorem)

$$\frac{dE}{dz} = -j\beta_0 E_i \exp(-j\beta_0 z) - j\beta_0 (I_1 - I_2)$$

↓

$$\frac{d^2E}{dz^2} = -\beta_0^2 E_i \exp(-j\beta_0 z) - j\beta_0 \left(\frac{dI_1}{dz} - \frac{dI_2}{dz}\right)$$

$$\frac{d^2E}{dz^2} = -\beta_0^2 E_i \exp(-j\beta_0 z) - j\beta_0(\frac{dI_1}{dz} - \frac{dI_2}{dz})$$ (rewritten)

$$\left.\begin{aligned} \frac{dI_1}{dz} &= -j\beta_0 I_1 + \zeta(z) \\ \frac{dI_2}{dz} &= j\beta_0 I_1 - \zeta\{z\} \end{aligned}\right\}$$

$$\frac{d^2E}{dz^2} = -\beta_0^2[E_i \exp(-j\beta_0 z) - (I_1 + I_2)] - 2j\beta_0\zeta(z)$$

$$E(z) = E_i \exp(-j\beta_0 z) + I_1 + I_2$$

$$\frac{d^2E}{dz^2} = -\beta_0^2 E(z) - 2j\beta_0\zeta(z)$$

$$dE = \zeta(x)\,dx = \zeta(z)\,dz$$

$$\boxed{\frac{d^2E}{dz^2} = -\beta_0^2 E \; - 2j\beta_0 \frac{dE}{dz}}$$

$$\frac{d^2E}{dz^2} = -\beta_0^2 E - 2j\beta_0 \frac{dE}{dz}$$

(rewritten)

Let us find dE/dz in terms of beam current i and circuit interaction impedance K

In terms of the axial voltage V the interaction impedance K is:

$$K = \frac{|V|^2}{2P} \longrightarrow K = \frac{E^2}{2\beta^2 P} \longrightarrow E^2 = 2\beta^2 KP \longrightarrow 2EdE = 2\beta^2 KdP$$

$$dP = \frac{EdE}{\beta^2 K} \xrightarrow{(\beta = \beta_0)} \left. \begin{aligned} dP_R &= \frac{E_R dE_R}{\beta_0{}^2 K} \\ dP_L &= \frac{E_L dE_L}{\beta_0{}^2 K} \end{aligned} \right\} \longrightarrow dP = dP_R + dP_L = \frac{E_R\, dE_R}{\beta_0{}^2 K} + \frac{E_L\, dE_L}{\beta_0{}^2 K}$$

(propagation constants of beam-wave coupled and cold systems are equal)

$$dE_R = dE_L = dE$$

$$dP = \frac{E_R + E_L}{\beta_0{}^2 K} dE$$

dP = Increment of circuit power at a point due to a modulated beam element of length dz

dP_{R} = Increments of circuit power due to the wave sent by the infinitesimal current generators to the left of the point

dP_{L} = Increments of circuit power due to th wave sent by the infinitesimal current generators to the left and to the right of the point

$$dP = \frac{E_R + E_L}{\beta_0{}^2 K} dE$$ (rewritten)

$dP =$ Increment of circuit power at a point due to a modulated beam element of length dz

From another angle of view:

$dP =$ Power lost by the beam element of length dz

$=$ Power lost by half beam element of length $dz/2$ experiencing E_L +
Power lost by half beam element of length $dz/2$ experiencing E_R

$$= (-eE_L v_1)(n\alpha\, dz/2) + (-eE_R v_1)(n\alpha\, dz/2)$$

$$= -e(E_R + E_L) v_1 (n\alpha\, dz/2)$$

n = electron concentration

α = beam cross-sectional area

e = electron charge

$$\left. \begin{aligned} J_1 &= nev_1 \\ i &= J_1 \alpha \end{aligned} \right\}$$

$$dP = -i(E_R + E_L)\frac{dz}{2}$$

$$dP = \frac{E_R + E_L}{\beta_0{}^2 K} dE$$ (recalled)

$$dP = -i(E_R + E_L)\frac{dz}{2}$$ (recalled)

↘ ↙

$$\frac{E_R + E_L}{\beta_0{}^2 K} dE = -i(E_R + E_L)\frac{dz}{2}$$

↓

$$\frac{dE}{dz} = -\frac{\beta_0{}^2 K\, i}{2}$$

⟶

$$\frac{d^2E}{dz^2} = -\beta_0^2 E - 2j\beta_0 \frac{dE}{dz}$$ (recalled)

↓

$$\boxed{\frac{d^2E}{dz^2} = -\beta_0^2 E + j\beta_0^3 K i}$$

$$\frac{d^2E}{dz^2} = -\beta_0^2 E + j\beta_0^3 Ki$$ (rewritten)

$\downarrow \longleftarrow$ $d^2/dz^2 = \Gamma^2 \longleftarrow$ RF quantities vary as $\exp(-\Gamma z)$

$$\Gamma^2 E = -\beta_0^2 E + j\beta_0^3 Ki$$ $\longleftarrow$ RF quantities vary as $\exp(-\Gamma z)$

$\downarrow$

$$(\Gamma^2 + \beta_0^2)E = j\beta_0^3 Ki$$

$\downarrow \longleftarrow$ $\boxed{E = -\partial V/\partial z = \Gamma V}$ $\longleftarrow$ $d/dz = -\Gamma \longleftarrow$ RF quantities vary as $\exp(-\Gamma z)$

$$\frac{V}{i} = \frac{j\beta_0^3 K}{(\Gamma^2 + \beta_0^2)\Gamma}$$

$\downarrow \longleftarrow$ $\Gamma_0 = j\beta_0, \Gamma \approx \Gamma_0$

$$\boxed{\frac{V}{i} = -\frac{\Gamma\Gamma_0 K}{(\Gamma^2 - \Gamma_0^2)}}$$ (Circuit equation)

Electronic equation

$$J_1 = \rho_0 v_1 + v_0 \rho_1 \text{ (recalled)} \quad \longleftarrow \quad v_1?, \rho_1?$$

Electronic motion in the presence of the circuit electric field intensity E plus the space-charge electric field intensity E_s

$$\frac{\partial v_1}{\partial t} + \frac{dz}{dt}\frac{\partial v_1}{\partial z} = \eta(E + E_s) \quad \longrightarrow \quad \frac{\partial v_1}{\partial t} + v\frac{\partial v_1}{\partial z} = \eta(E + E_s) \longleftarrow v = v_0 + v_1$$

$$\downarrow$$

$$\frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial z} + v_1\frac{\partial v_1}{\partial z} = \eta(E + E_s) \longleftarrow \frac{\partial v_1}{\partial t} + (v_0 + v_1)\frac{\partial v_1}{\partial z} = \eta(E + E_s)$$

$\downarrow \longleftarrow$ ignoring $v_1 \partial v_1 / \partial z$

RF quantities vary as $\exp(j\omega t - \Gamma z)$

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)v_1 = \eta(E + E_s) \longleftarrow \frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z} = j\omega - v_0\Gamma$$

$$\downarrow$$

$$(j\omega - v_0\Gamma)v_1 = \eta(E + E_s) \longrightarrow \boxed{v_1 = \frac{\eta(E + E_s)}{j\omega - v_0\Gamma}}$$

$J_1 = \rho_0 v_1 + v_0 \rho_1$ (recalled) $\longleftarrow$ $v_1 = \dfrac{\eta(E + E_s)}{j\omega - v_0\Gamma}, \rho_1 ?$ (rewritten)

RF quantities vary as $\exp(j\omega t - \Gamma z)$

$$\frac{\partial E_s}{\partial z} = \frac{\rho_1}{\varepsilon_0}$$ (Poisson's equation) $\longleftarrow$ RF quantities vary as $\exp(j\omega t - \Gamma z)$

$$-\Gamma E_s = \frac{\rho_1}{\varepsilon_0} \longrightarrow E_s = -\frac{\rho_1}{\Gamma \varepsilon_0}$$

$$\frac{\partial J_1}{\partial z} = -\frac{\partial \rho_1}{\partial t}$$ (continuity equation)

$$\rho_1 = \frac{\Gamma J_1}{j\omega}$$

$$E_s = -\frac{J_1}{j\omega\varepsilon_0}$$

$$v_1 = \frac{\eta(E + E_s)}{j\omega - v_0\Gamma} \longrightarrow v_1 = \frac{\eta\left(E - \dfrac{J_1}{j\omega\varepsilon_0}\right)}{j\omega - v_0\Gamma} \longrightarrow J_1 = \rho_0 v_1 + v_0 \rho_1 \text{ (recalled)}$$

$$J_1 = \frac{\rho_0 \eta\left(E - \dfrac{J_1}{j\omega\varepsilon_0}\right)}{j\omega - v_0\Gamma} + v_0 \frac{\Gamma J_1}{j\omega}$$

$$J_1 = \frac{\rho_0 \eta (E - \frac{J_1}{j\omega\varepsilon_0})}{j\omega - v_0 \Gamma} + v_0 \frac{\Gamma J_1}{j\omega}$$ (recalled)

$$\omega_p^2 = \eta\rho_0 / \varepsilon_0 = |\eta||\rho_0| / \varepsilon_0$$
$$J_0 = \rho_0 v_0$$
$$\beta_e = \omega / v_0$$
$$E = \Gamma V$$

↓

$$\boxed{J_1((j\omega - v_0\Gamma)^2 + \omega_p^2) = j\beta_e \eta J_0 \Gamma V}$$

$$J_1 \alpha = i$$
$$J_0 \alpha = i_0$$

↓

$$\frac{V}{i} = \frac{(j\omega - v_0\Gamma)^2 + \omega_p^2}{j\beta_e \eta i_0 \Gamma}$$

$$\frac{V}{i}=\frac{(j\omega-v_0\Gamma)^2+\omega_p^2}{j\beta_e\eta i_0\Gamma} \quad \text{(rewritten)}$$

$$\beta_e=\omega/v_0$$

$$\beta_p=\omega_p/v_0$$

$$i_0=-|i_0|=-I_0$$

$$v_0^{\,2}=2|\eta|V_0$$

$$\boxed{\frac{V}{i}=\left(\frac{2V_0}{I_0}\right)\left(\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma}\right)}$$

(electronic equation)

$$\frac{V}{i} = -\frac{\Gamma\Gamma_0 K}{(\Gamma^2 - \Gamma_0^2)}$$

(Circuit equation)

$$\frac{V}{i} = \left(\frac{2V_0}{I_0}\right)\left(\frac{(j\beta_e - \Gamma)^2 + \beta_p^2}{j\beta_e\Gamma}\right)$$

(Electronic equation)

$$-\frac{\Gamma\Gamma_0 K}{(\Gamma^2 - \Gamma_0^2)} = \left(\frac{2V_0}{I_0}\right)\left(\frac{(j\beta_e - \Gamma)^2 + \beta_p^2}{j\beta_e\Gamma}\right)$$

$$\boxed{\frac{-\Gamma\Gamma_0 K}{(\Gamma + \Gamma_0)(\Gamma - \Gamma_0)} = \frac{2V_0}{I_0}\frac{(j\beta_e - \Gamma)^2 + \beta_p^2}{j\beta_e\Gamma}}$$

(Beam-wave coupled dispersion relation of a TWT)

Gain equation from the beam-wave coupled dispersion relation of a TWT

$$\frac{-\Gamma\Gamma_0 K}{(\Gamma+\Gamma_0)(\Gamma-\Gamma_0)} = \frac{2V_0}{I_0}\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma} \quad \text{(rewritten)}$$

(beam-wave coupled dispersion relation of a TWT)

Fourth-degree equation in the propagation constant Γ

RF quantities vary as $\exp(j\omega t - \Gamma z)$

In an isolated circuit (slow-wave structure), there is one forward wave and one backward wave (1 forward + 1 backward).

On an isolated electron beam, there is one forward space-charge wave and one forward space-charge wave (1 forward + 1 forward).

In a beam-wave coupled system, it can be intuitively guessed that there would be
1 forward wave +1 backward wave + 1 forward wave + 1 forward wave
= 3 forward wave + 1 backward wave.

$$\frac{-\Gamma\Gamma_0 K}{(\Gamma+\Gamma_0)(\Gamma-\Gamma_0)} = \frac{2V_0}{I_0}\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma}$$ (rewritten)

(beam-wave coupled dispersion relation of a TWT)

$$(\Gamma^2-\Gamma_0^2)[(j\beta_e-\Gamma)^2+\beta_p^2]=0$$

$I_0 = 0$ (weak coupling)

$$\Gamma = \pm\Gamma_0 = \pm j\beta_0$$

(1 forward + 1 backward circuit wave)

$$\Gamma = j(\beta_e \mp \beta_p)$$

(1 forward + 1 forward space-charge wave)

(Circuit and space-charge waves are decoupled)

In a beam-wave coupled system, it has been intuitively guessed that there would be 3 forward-wave + 1 backward-wave solutions to the following dispersion relation:

$$\frac{-\Gamma\Gamma_0 K}{(\Gamma+\Gamma_0)(\Gamma-\Gamma_0)} = \frac{2V_0}{I_0}\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma} \quad \text{(rewritten)}$$

We look forward to the solution for the propagation constant of the beam-wave coupled system close to that of the beam $(\Gamma \approx j\beta_e)$

RF quantities vary as $\exp(j\omega t - \Gamma z)$

Three forward wave solutions:

$$-\Gamma = -j\beta_e + \beta_e C\delta \quad C\delta << 1$$ (C, δ are dimensionless quantities)

$$\beta_0 = \beta_e(1+bC) \quad (\beta_e = \omega / v_0)$$

$$b = \frac{\beta_0 - \beta_e}{\beta_e C} = \frac{v_0 - v_p}{v_p C}$$

(b = velocity synchronization parameter)

$$(\Gamma \approx j\beta_e)$$

$$\Gamma \approx j\beta_e$$

$$-\Gamma = -j\beta_e + \beta_e C\delta \; (C\delta << 1)$$

$$\Gamma_0 = \beta_e C d + j\beta_0 \; (Cd << 1)$$

d = Loss parameter

$$\beta_e = \omega / v_0$$

$$\beta_0 = \beta_e(1 + bC) \qquad (b = \frac{\beta_0 - \beta_e}{\beta_e C} = \frac{v_0 - v_p}{v_p C})$$

(Velocity synchronization parameter)

$$\Gamma_0 = \beta_e C d + j\beta_e(1 + bC) \; (bC << 1)$$

$$j\beta_e - \Gamma = \beta_e C\delta$$

$$\Gamma + \Gamma_0 = j\beta_e + j\beta_e(1 + bC) + \beta_e(Cd - C\delta) \approx 2j\beta_e$$

$$\Gamma - \Gamma_0 = -\beta_e C(\delta + d + jb)$$

$$\frac{-\Gamma\Gamma_0 K}{(\Gamma + \Gamma_0)(\Gamma - \Gamma_0)} = \frac{2V_0}{I_0} \frac{(j\beta_e - \Gamma)^2 + \beta_p^2}{j\beta_e \Gamma}$$

$$\frac{-\Gamma\Gamma_0 K}{(\Gamma+\Gamma_0)(\Gamma-\Gamma_0)} = \frac{2V_0}{I_0}\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma} \longleftarrow \left.\begin{aligned} C^3 &= \frac{KI_0}{4V_0} \\ QC &= \frac{\beta_p{}^2}{4\beta_e{}^2 C^2} \end{aligned}\right\}$$

$QC =$ Space-charge parameter

$$\downarrow$$

$$\boxed{(\delta^2+4QC)(j\delta+jd-b)=1}$$

(Cubic dispersion relation)

$$\downarrow \longleftarrow \boxed{b=d=QC=0}$$

(Special case)

$$\boxed{\delta^3=-j} \longrightarrow -\Gamma=-j\beta_e+\beta_e C\delta$$

$$\left.\begin{aligned} -\Gamma_1 &= -j\beta_e+\beta_e C\delta_1 \\ -\Gamma_2 &= -j\beta_e+\beta_e C\delta_2 \\ -\Gamma_3 &= -j\beta_e+\beta_e C\delta_3 \end{aligned}\right\}$$

Special case: $b = d = QC = 0$

$$\left.\begin{aligned} \delta_1 &= \sqrt{3}/2 - j(1/2) \\ \delta_2 &= -\sqrt{3}/2 - j(1/2) \\ \delta_3 &= j \end{aligned}\right\}$$

RF quantities vary as $\exp(j\omega t - \Gamma z)$

$$\downarrow \quad \longleftarrow \quad -\Gamma = -j\beta_e + \beta_e C\delta$$

$$-\Gamma_1 = \beta_e C\sqrt{3}/2 - j\beta_e(1 + C/2) \longrightarrow$$ Growing√ √ √

$$-\Gamma_2 = -\beta_e C\sqrt{3}/2 - j\beta_e(1 + C/2) \longrightarrow$$ Decaying $(b = d = QC = 0)$

$$-\Gamma_3 = -j\beta_e(1 - C) \longrightarrow$$ Neither growing nor decaying

Growing-wave component:

$$-\Gamma_1 = \beta_e C\sqrt{3}/2 - j\beta_e(1+C/2) \qquad \beta_e = \omega / v_0$$

$$(b = d = QC = 0)$$

Phase velocity $= \omega / \beta_e(1+C/2) = v_0 /(1+C/2) < v_0$

Decaying-wave component:

$$-\Gamma_2 = -\beta_e C\sqrt{3}/2 - j\beta_e(1+C/2) \qquad \beta_e = \omega / v_0 \qquad (b = d = QC = 0)$$

Phase velocity $= \omega / \beta_e(1+C/2) = v_0 /(1+C/2) < v_0$

Neither growing- nor decaying-wave component:

$$-\Gamma_3 = -j\beta_e(1-C) \qquad (b = d = QC = 0)$$

Phase velocity $= \omega / \beta_e(1-C) = v_0 /(1-C) > v_0$

Solution of the dispersion relation for the fourth backward-wave component

$$\frac{-\Gamma\Gamma_0 K}{(\Gamma+\Gamma_0)(\Gamma-\Gamma_0)} = \frac{2V_0}{I_0}\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma}$$

$$-\Gamma = -j\beta_e + \beta_e C\delta$$

Forward-wave component

$$-\Gamma = j\beta_e + \beta_e C\delta$$

Backward-wave component
(4th propagation constant)

$$b = d = QC = 0 \longrightarrow \delta_4 = -\frac{jC^2}{4} \longrightarrow -\Gamma_4 = j\beta_e + \beta_e C\delta_4 = j\beta_e(1-C^3/4)$$

Phase velocity $=\omega/\beta_e(1-C^3/4) = v_0/(1-C^3/4) > v_0$

$$\beta_e = \omega/v_0$$

Fourth backward-wave component neither grows nor decays

If the structure is perfectly matched, the fourth wave is not excited to a significant extent.

Gain equation of a TWT

$$J_1((j\omega - v_0\Gamma)^2 + \omega_p^2) = j\beta_e \eta J_0 \Gamma V$$ (recalled)

$\downarrow$

$$J_1 = \frac{j\beta_e \eta J_0 \Gamma}{(j\omega - v_0\Gamma)^2 + \omega_p^2} V$$

$\downarrow$ $\longleftarrow$

$$\left.\begin{aligned} E &= \Gamma V \\ \omega_p{}^2 &= \eta\,\rho_0 / \varepsilon_0 \\ \omega / v_0 &= \beta_e \end{aligned}\right\}$$

$$v_1 = \frac{\eta\left(E - \dfrac{J_1}{j\omega\varepsilon_0}\right)}{j\omega - v_0\Gamma}$$ (recalled) $\longrightarrow$ $$v_1 = \frac{\eta\,\Gamma(j\beta_e - \Gamma)}{v_0((j\beta_e - \Gamma)^2 + \beta_p{}^2)} V$$

$$v_1 = \frac{\eta\,\Gamma(j\beta_e - \Gamma)}{v_0((j\beta_e - \Gamma)^2 + \beta_p{}^2)} V$$ (rewritten)

$$-\Gamma = -j\beta_e + \beta_e C\delta$$

$$QC = \frac{\beta_p{}^2}{4\beta_e{}^2 C^2}$$

$$\Gamma \approx j\beta_e$$

$$v_1 = \frac{j\eta}{v_0 C\delta\left(1 + \frac{4QC}{\delta^2}\right)} V$$

$$V' = \frac{V}{1 + \frac{4QC}{\delta^2}}$$

$$v_1 = \frac{j\eta}{v_0 C\delta} V'$$

$$\longrightarrow$$

$$v_1 \propto \frac{V'}{\delta}$$

,

$$J_1 = \frac{j\beta_e \eta J_0 \Gamma}{(j\omega - v_0\Gamma)^2 + \omega_p^2} V$$ (recalled)

$$\left.\begin{aligned} \beta_e &= \omega / v_0 \\ \beta_p &= \omega_p / v_0 \end{aligned}\right\} \rightarrow$$

$$\left.\begin{aligned} -\Gamma &= -j\beta_e + \beta_e C\delta \\ QC &= \frac{\beta_p{}^2}{4\beta_e{}^2 C^2} \\ \Gamma &\approx j\beta_e \end{aligned}\right\}$$

$$\boxed{V' = \frac{V}{1 + \frac{4QC}{\delta^2}}}$$

$$J_1 = \frac{j\beta_e \eta J_0 \Gamma}{v_0{}^2((j\beta_e - \Gamma)^2 + \beta_p^2))} V \longrightarrow J_1 = -\frac{\eta J_0}{v_0{}^2 C^2 \delta^2 (1 + \frac{4QC}{\delta^2})} V$$

,

$$\boxed{J_1 = -\frac{\eta J_0}{v_0{}^2 C^2 \delta^2} V'}$$

$$\boxed{J_1 \propto \frac{V'}{\delta^2}}$$

Circuit voltage

At a distance z from the input end:

$$V(z) = V_1(0)\exp(-\Gamma_1 z) + V_2(0)\exp(-\Gamma_2 z) + V_3(0)\exp(-\Gamma_3 z)$$

$$-\Gamma = -j\beta_e + \beta_e C\delta \longleftarrow \delta = x + jy$$

$$\left.\begin{aligned} -\Gamma_1 &= -j\beta_e + \beta_e C(x_1 + jy_1) \\ -\Gamma_2 &= -j\beta_e + \beta_e C(x_2 + jy_2) \\ -\Gamma_3 &= -j\beta_e + \beta_e C(x_3 + jy_3) \end{aligned}\right\} \longleftarrow \left.\begin{aligned} \delta_1 &= x_1 + jy_1 \\ \delta_2 &= x_2 + jy_2 \\ \delta_3 &= x_3 + jy_3 \end{aligned}\right\}$$

$$\begin{aligned} V(z) = {} & V_1(0)\exp(\beta_e C x_1 z)\exp - j\beta_e(1 - Cy_1)z \\ & + V_2(0)\exp(\beta_e C x_2 z)\exp - j\beta_e(1 - Cy_2)z \\ & + V_3(0)\exp(\beta_e C x_3 z)\exp - j\beta_e(1 - Cy_3)z \end{aligned}$$

Input conditions $(z=0)$

$$\left.\begin{aligned} &V_{\text{input}} = V_1(0) + V_2(0) + V_3(0) \\ &v_{1,\text{ input}} = 0 \\ &J_{1,\text{input}} = 0 \end{aligned}\right\} (z=0)$$

At a distance z from the input end:

$$\begin{aligned} V(z) = {} & V_1(0)\exp(\beta_e C x_1 z)\exp - j\beta_e(1 - C y_1)z \\ & + V_2(0)\exp(\beta_e C x_2 z)\exp - j\beta_e(1 - C y_2)z \quad \text{(recalled)} \\ & + V_3(0)\exp(\beta_e C x_3 z)\exp - j\beta_e(1 - C y_3)z \end{aligned}$$

$$V_{\text{input}} = V_1(0) + V_2(0) + V_3(0) \quad \longleftarrow \quad V' = \frac{V}{1 + \dfrac{4QC}{\delta^2}}$$

$$\downarrow$$

$$\boxed{V_{\text{input}} = (V_1'(0) + V_2'(0) + V_3'(0)) + 4QC\left(\frac{V_1'(0)}{\delta_1{}^2} + \frac{V_2'(0)}{\delta_2{}^2} + \frac{V_3'(0)}{\delta_3{}^2}\right)}$$

$$v_{1,\,\text{input}} = 0 \quad (z = 0) \quad \longleftarrow \quad v_1 \propto \frac{V'}{\delta}$$

$$\downarrow$$

$$\boxed{\frac{V_1'(0)}{\delta_1} + \frac{V_2'(0)}{\delta_2} + \frac{V_3'(0)}{\delta_3} = 0}$$

$$J_{1,\text{input}} = 0 \quad (z=0) \quad \longleftarrow \quad J_1 \propto \frac{V'}{\delta^2}$$

$$V_{\text{input}} = (V_1'(0) + V_2'(0) + V_3'(0)) + 4QC\left(\frac{V_1'(0)}{\delta_1^2} + \frac{V_2'(0)}{\delta_2^2} + \frac{V_3'(0)}{\delta_3^2}\right)$$

(recalled)

$$\boxed{\frac{V_1'(0)}{\delta_1^2} + \frac{V_2'(0)}{\delta_2^2} + \frac{V_3'(0)}{\delta_3^2} = 0}$$

$$V_{\text{input}} = V_1'(0) + V_2'(0) + V_3'(0)$$

$$\left.\begin{aligned} V_{\text{input}} &= V_1(0) + V_2(0) + V_3(0) \\ V_{\text{input}} &== V_1'(0) + V_2'(0) + V_3'(0) \\ &\frac{V_1'(0)}{\delta_1} + \frac{V_2'(0)}{\delta_2} + \frac{V_3'(0)}{\delta_3} = 0 \\ &\frac{V_1'(0)}{\delta_1^2} + \frac{V_2'(0)}{\delta_2^2} + \frac{V_3'(0)}{\delta_3^2} = \end{aligned}\right\}$$

(input conditions)

$$V_1'(0)=\frac{V_{\text{input}}}{\left(1-\frac{\delta_2}{\delta_1}\right)\left(1-\frac{\delta_3}{\delta_1}\right)}$$

$$V_2'(0)=\frac{V_{\text{input}}}{\left(1-\frac{\delta_3}{\delta_2}\right)\left(1-\frac{\delta_1}{\delta_2}\right)}$$

$$V_3'(0)=\frac{V_{\text{input}}}{\left(1-\frac{\delta_1}{\delta_3}\right)\left(1-\frac{\delta_2}{\delta_3}\right)}$$

$$\left.\begin{aligned} &V_{\text{input}}=V_1(0)+V_2(0)+V_3(0)\\ &V_{\text{input}}==V_1'(0)+V_2'(0)+V_3'(0)\\ &\frac{V_1'(0)}{\delta_1}+\frac{V_2'(0)}{\delta_2}+\frac{V_3'(0)}{\delta_3}=0\\ &\frac{V_1'(0)}{\delta_1^{\ 2}}+\frac{V_2'(0)}{\delta_2^{\ 2}}+\frac{V_3'(0)}{\delta_3^{\ 2}}= \end{aligned}\right\}\text{(recalled)}$$

Three equations can be solved for $V_{1,2,3}'(0)$

$$V'=\frac{V}{1+\frac{4QC}{\delta^2}}$$

$$V_1'(0)==\frac{V_{\text{input}}}{\left(1-\frac{\delta_2}{\delta_1}\right)\left(1-\frac{\delta_3}{\delta_1}\right)}$$

$$\boxed{V_1(0)=\left(1+\frac{4QC}{\delta_1^2}\right)\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)V_{in}}$$

$l =$ interaction length

$$V_{\text{out}} = V(l) = V_1(0)\exp(-\Gamma_1 l)$$

$\delta = x + jy$

$$-\Gamma_1 = -j\beta_e + \beta_e C\delta_1 = -j\beta_e + \beta_e C(x_1 + jy_1)$$

Considering the growing-wave component

$$V_{\text{out}} = V(l) = V_1(0)\exp(\beta_e C x_1 l)\exp - j\beta_e(1 - Cy_1)l$$

$$V_1(0) = \left(1 + \frac{4QC}{\delta_1^2}\right)\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)V_{in} \quad \text{(recalled)}$$

$$\boxed{V_{\text{out}} = V_{\text{in}}(1 + 4QC/\delta_1^{\,2})\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)\times\exp(\beta_e C x_1 l)\exp - j\beta_e(1 - Cy_1)l}$$

$$V_{\text{out}} = V_{\text{in}}(1 + 4QC/\delta_1^{\;2})\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right) \times \exp(\beta_e C x_1 l)\exp - j\beta_e(1 - Cy_1)l$$

(rewritten)

$$\downarrow$$

$$G = 20\log_{10}\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = A + 20\log_{10}(\exp(\beta_e C x_1 l))$$ (Gain in dB)

$$A = 20\log_{10}\left|(1 + 4QC/\delta_1^2)\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)\right|$$

$$\downarrow$$

$$G = A + 20\,\log_e(\exp(\beta_e C x_1 \ell))\log_{10} e = A + 20Cx_1(\log_{10} e)(\beta_e l)$$

$$G = A + 20\log_e(\exp(\beta_e C x_1 \ell))\log_{10} e = A + 20Cx_1(\log_{10} e)(\beta_e l)$$ (rewritten)

$$A = 20\log_{10}\left|(1+4QC/\delta_1^2)\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)\right|$$

$$\beta_e l = 2\pi N$$

$$N(2\pi/\beta_e) = l$$

$$N\lambda_e = l$$

$$G = A + 20Cx_1(\log_{10} e)(2\pi N) = A + 40\pi(\log_{10} e)x_1 CN$$

$$B = 40\pi(\log_{10} e)x_1 \approx 54.6x_1$$

N = number of beam wavelengths in the interaction length l

$$G = A + BCN$$ (Gain in dB)

$$\delta_1 = x_1 + jy_1$$

$$\delta_1, \delta_2, \delta_3$$

$$(\delta^2 + 4QC)(j\delta + jd - b) = 1$$

Special case: $b = d = QC = 0 \longrightarrow V_1(0) = ???$

$$V_1(0) = \left(1 + \frac{4QC}{\delta_1^2}\right)\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)V_{in} \quad \text{(recalled)}$$

$$\downarrow \longleftarrow QC = 0$$

$$V_1(0) = \left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)V_{in} \longleftarrow \left.\begin{aligned} \delta_1 &= \sqrt{3}/2 - j(1/2) \\ \delta_2 &= -\sqrt{3}/2 - j(1/2) \\ \delta_3 &= j \end{aligned}\right\} (b = d = QC = 0)$$

$$\downarrow$$

$$\boxed{V_1(0) = \frac{V_{\text{in}}}{3}} \quad (b = d = QC = 0)$$

Special case: $b = d = QC = 0 \longrightarrow V_2(0) = ???$

$$V_2(0) = \left(1 + \frac{4QC}{\delta_2^2}\right)\left(\frac{1}{(1-\delta_3/\delta_2)(1-\delta_1/\delta_2)}\right)V_{in}$$

Three equations can be solved for $V'_{1,2,3}(0)$

$$\downarrow \longleftarrow QC = 0$$

$$V_2(0) = \left(\frac{1}{(1-\delta_3/\delta_2)(1-\delta_1/\delta_2)}\right)V_{in} \longleftarrow \left.\begin{aligned}\delta_1 &= \sqrt{3}/2 - j(1/2)\\ \delta_2 &= -\sqrt{3}/2 - j(1/2)\\ \delta_3 &= j\end{aligned}\right\} (b = d = QC = 0)$$

$$\downarrow$$

$$\boxed{V_2(0) = \frac{V_{\text{in}}}{3}} \quad (b = d = QC = 0)$$

Special case: $b = d = QC = 0 \longrightarrow V_3(0) = ???$

Three equations can be solved for $V'_{1,2,3}(0)$

$$V_3(0) = \left(1 + \frac{4QC}{\delta_3^2}\right)\left(\frac{1}{(1-\delta_1/\delta_3)(1-\delta_2/\delta_3)}\right)V_{in}$$

$$\downarrow \longleftarrow QC = 0$$

$$V_3(0) = \left(\frac{1}{(1-\delta_1/\delta_3)(1-\delta_2/\delta_3)}\right)V_{in} \longleftarrow \left.\begin{aligned} \delta_1 &= \sqrt{3}/2 - j(1/2) \\ \delta_2 &= -\sqrt{3}/2 - j(1/2) \\ \delta_3 &= j \end{aligned}\right\} (b = d = QC = 0)$$

$$\downarrow$$

$$\boxed{V_3(0) = \frac{V_{\text{in}}}{3}} \quad (b = d = QC = 0) \quad \boxed{V_1(0) = V_2(0) = V_3(0) = \frac{V_{\text{in}}}{3}}$$

Voltage is equally divided among three forward wave components at the input for the special case: $b = d = QC = 0$

Gain for the special case: $b = d = QC = 0$

$$G = A + BCN$$

$$A = 20\log_{10}\left|(1+4QC/\delta_1^2)\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)\right|$$

$$A = 20\log_{10}(1/3) \cong -9.54 \longleftarrow A = 20\log_{10}\left|\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right|$$

$$B = 40\pi(\log_{10} e)x_1 \approx 54.6x_1 \cong 47.3 \longleftarrow x_1 = \sqrt{3}/2$$

$$\left.\begin{aligned}\delta_1 &= \sqrt{3}/2 - j(1/2)\\ \delta_2 &= -\sqrt{3}/2 - j(1/2)\\ \delta_3 &= j\end{aligned}\right\}$$

$$G = -954 + 47.3CN \ (b = d = QC = 0)$$

(Gain in dB)

Extension of Pierce's theory to estimate hot attenuation

Lossy section is provided with the slow-wave structure to prevent the device from oscillation due to imperfect matching

One attenuator section per about 20 dB gain of the device

Estimate of 'hot' attenuation for infinite 'cold' attenuation
Beyond the attenuator

Circuit voltage = 0 ('Cold' attenuation = ∞)
RF modulation on the beam, however, remains.

We assume that the circuit voltage becomes null following the attenuator of negligibly small length, though the RF velocity and the RF current density sweep through the attenuator without any change:

$$V_1^{\mathrm{b}} + V_2^{\mathrm{b}} + V_3^{\mathrm{b}} = 0$$

$$v_1^{\mathrm{b}} + v_2^{\mathrm{b}} + v_3^{\mathrm{b}} = v_1^{\mathrm{a}} + v_2^{\mathrm{a}} + v_3^{\mathrm{a}}$$

$$J_1^{\mathrm{b}} + J_2^{\mathrm{b}} + J_3^{\mathrm{b}} = J_1^{\mathrm{a}} + J_2^{\mathrm{a}} + J_3^{\mathrm{a}}$$

Superscripts 'a' and '*b*' represent the quantities immediately preceding and beyond the attenuator, respectively.

Superscripts 'a' and 'b' represent

Quantities immediately preceding and beyond the attenuator, respectively.

Subscripts 1, 2, 3 refer to

Three forward waves, respectively.

Attenuator length is negligibly small:

$$v_1^{\mathrm{b}} + v_2^{\mathrm{b}} + v_3^{\mathrm{b}} = v_1^{\mathrm{a}} + v_2^{\mathrm{a}} + v_3^{\mathrm{a}} \longleftarrow v_1 \propto \frac{V}{\delta} \longleftarrow v_1 \propto \frac{V'}{\delta} \longleftarrow QC = 0$$

$$\downarrow$$

$$\boxed{\frac{V_1^{\mathrm{b}}}{\delta_1} + \frac{V_2^{\mathrm{b}}}{\delta_2} + \frac{V_3^{\mathrm{b}}}{\delta_3} = \frac{V_1^{\mathrm{a}}}{\delta_1} + \frac{V_2^{\mathrm{a}}}{\delta_2} + \frac{V_3^{\mathrm{a}}}{\delta_3}}$$

$$J_1^{\mathrm{b}} + J_2^{\mathrm{b}} + J_3^{\mathrm{b}} = J_1^{\mathrm{a}} + J_2^{\mathrm{a}} + J_3^{\mathrm{a}} \longleftarrow J_1 \propto \frac{V}{\delta^2} \longleftarrow J_1 \propto \frac{V'}{\delta^2} \longleftarrow QC = 0$$

$$\downarrow$$

$$\boxed{\frac{V_1^{\mathrm{b}}}{\delta_1^{\ 2}} + \frac{V_2^{\mathrm{b}}}{\delta_2^{\ 2}} + \frac{V_3^{\mathrm{b}}}{\delta_3^{\ 2}} = \frac{V_1^{\mathrm{a}}}{\delta_1^{\ 2}} + \frac{V_2^{\mathrm{a}}}{\delta_2^{\ 2}} + \frac{V_3^{\mathrm{a}}}{\delta_3^{\ 2}}}$$

Considering one of the forward-wave components (one that axially grows:

$$V_{\text{out}} = V_{\text{in}} (1 + 4QC/\delta_1^2)(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}) \times \exp(\beta_e C x_1 l) \exp - j\beta_e (1 - C y_1) l$$

(recalled)

$$\downarrow \quad \longleftarrow \quad QC = 0$$

$$V_{\text{out}} = V_{\text{in}} (\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}) \times \exp(\beta_e C x_1 l) \exp - j\beta_e (1 - C y_1) l$$

For the contributions from all the three forward wave components, and taking l_1 as the distance of the attenuator from the input of the interaction length

$$V_1^{\text{a}} = V_{\text{in}} (\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}) \exp(\beta_e C x_1 l_1) \exp - j\beta_e (1 - C y_1) l_1$$

$$V_2^{\text{a}} = V_{\text{in}} (\frac{1}{(1-\delta_3/\delta_2)(1-\delta_1/\delta_2)}) \exp(\beta_e C x_2 l_1) \exp - j\beta_e (1 - C y_2) l_1$$

$$V_3^{\text{a}} = V_{\text{in}} (\frac{1}{(1-\delta_1/\delta_3)(1-\delta_2/\delta_3)}) \exp(\beta_e C x_3 l_1) \exp - j\beta_e (1 - C y_3) l_1$$

$$\frac{V_1^{\mathrm{b}}}{\delta_1}+\frac{V_2^{\mathrm{b}}}{\delta_2}+\frac{V_3^{\mathrm{b}}}{\delta_3}=\frac{V_1^{\mathrm{a}}}{\delta_1}+\frac{V_2^{\mathrm{a}}}{\delta_2}+\frac{V_3^{\mathrm{a}}}{\delta_3}$$

$$\frac{V_1^{\mathrm{b}}}{\delta_1^{\,2}}+\frac{V_2^{\mathrm{b}}}{\delta_2^{\,2}}+\frac{V_3^{\mathrm{b}}}{\delta_3^{\,2}}=\frac{V_1^{\mathrm{a}}}{\delta_1^{\,2}}+\frac{V_2^{\mathrm{a}}}{\delta_2^{\,2}}+\frac{V_3^{\mathrm{a}}}{\delta_3^{\,2}}$$

$$V_1^{\mathrm{b}}+V_2^{\mathrm{b}}+V_3^{\mathrm{b}}=0$$

(recalled)

Can be solved for V_1^{b} in terms of $V_1^{\mathrm{a}}, V_2^{\mathrm{a}}, V_3^{\mathrm{a}}$ given by

$$V_1^{\mathrm{a}}=V_{\mathrm{in}}\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right)\exp(\beta_e C x_1 l_1)\exp-j\beta_e(1-Cy_1)l_1$$

$$V_2^{\mathrm{a}}=V_{\mathrm{in}}\left(\frac{1}{(1-\delta_3/\delta_2)(1-\delta_1/\delta_2)}\right)\exp(\beta_e C x_2 l_1)\exp-j\beta_e(1-Cy_2)l_1$$

$$V_3^{\mathrm{a}}=V_{\mathrm{in}}\left(\frac{1}{(1-\delta_1/\delta_3)(1-\delta_2/\delta_3)}\right)\exp(\beta_e C x_3 l_1)\exp-j\beta_e(1-Cy_3)l_1$$

(recalled)

$\beta_e l_1 = 2\pi N_1$ (recalled)

Hence, we obtain:

$$V_1^b = \frac{V_{in}}{3}\exp-(j2\pi N_1)[\frac{2}{3}\exp(2\pi CN_1(x_1+jy_1))-\frac{1}{3}\exp(2\pi CN_1(x_2+jy_2))$$
$$-\frac{1}{3}\exp(2\pi CN_1(x_3+jy_3))]$$

$$b = d = QC = 0:$$

$$\left.\begin{aligned} x_1 &= \sqrt{3}/2, y_1 = -1/2 \\ x_2 &= -\sqrt{3}/2, y_2 = -1/2 \\ x_3 &= 0, y_3 = 1 \end{aligned}\right\} \leftarrow \left\{\begin{aligned} \delta_1 &= \sqrt{3}/2 - j(1/2) \\ \delta_2 &= -\sqrt{3}/2 - j(1/2) \\ \delta_3 &= j \end{aligned}\right.$$

$$\left|\frac{V_1^b}{V_1^a}\right| = \left|\frac{2}{3}+\frac{1}{3}\exp-(2\pi CN_1\sqrt{3})+\frac{1}{3}\exp-2\pi CN_1(\frac{\sqrt{3}}{2}-j\frac{3}{2})\right|$$

Taking $CN_1 >$ 0.2 (practical values)

$$\left|\frac{V_1^b}{V_1^a}\right| \cong \frac{2}{3} \longrightarrow$$

'Hot' attenuation ~ 20 $\log_{10}3/2$ = 3.52 dB, though
'Cold' attenuation = ∞ !

Extension of Pierce's theory for Johnson's start-oscillation condition

(H. R. Johnson, "Backward-wave oscillators, " *Proc. IRE*, June 1955, pp. 684-694)

Backward-wave mode: v_{ph} is positive and v_{g} is negative

$\Gamma_0 = j\beta_0$ (cold circuit propagation constant)

$\beta_0 = \beta_e(1+bC)$ $\Gamma_0 = \beta_e Cd + j\beta_e(1+bC)$ (d = circuit loss parameter)

Forward-wave mode

$$\frac{-\Gamma\Gamma_0 K}{(\Gamma+\Gamma_0)(\Gamma-\Gamma_0)} = \frac{2V_0}{I_0}\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma}$$

↓

$$(\delta^2+4QC)(j\delta+jd-b)=1$$

Backward-wave mode

$$\frac{+\Gamma\Gamma_0 K}{(\Gamma+\Gamma_0)(\Gamma-\Gamma_0)} = \frac{2V_0}{I_0}\frac{(j\beta_e-\Gamma)^2+\beta_p^2}{j\beta_e\Gamma}$$

↓ K is interpreted as negative (power propagating in the negative direction) causing a change in the sign in the expression (with – K replaced by + K

$$(\delta^2+4QC)(j\delta+jd-b)=-1$$

That circuit voltage in the presence of loss would have to be less at the input (gun) end than in the absence of loss has to be interpreted with a change in the sign of d

Forward-wave mode

$$(\delta^2 + 4QC)(j\delta + jd - b) = -1 \text{ (recalled)}$$

Backward-wave mode

$$(\delta^2 + 4QC)(j\delta - jd - b) = -1$$

$$V_{\text{out}} = V_{\text{in}}(1 + 4QC/\delta_1^{\,2})\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right) \times \exp(\beta_e C x_1 l)\exp - j\beta_e(1 - Cy_1)l$$

(recalled)

$$V_{\text{out}} = V_{\text{in}}(1 + 4QC/\delta_1^{\,2})\left(\frac{1}{(1-\delta_2/\delta_1)(1-\delta_3/\delta_1)}\right) \times \exp(\beta_e C x_1 l)\exp - j\beta_e(1 - Cy_1)l$$

$$+V_{\text{in}}(1 + 4QC/\delta_2^{\,2})\left(\frac{1}{(1-\delta_3/\delta_2)(1-\delta_1/\delta_2)}\right) \times \exp(\beta_e C x_1 l)\exp - j\beta_e(1 - Cy_1)l$$

$$+V_{\text{in}}(1 + 4QC/\delta_3^{\,2})\left(\frac{1}{(1-\delta_1/\delta_2)(1-\delta_2/\delta_3)}\right) \times \exp(\beta_e C x_1 l)\exp - j\beta_e(1 - Cy_1)l$$

$$e^{j2\pi N}\frac{V_{out}}{V_{in}}=\left(\frac{\delta_1{}^2+4QC}{(\delta_1-\delta_2)(\delta_1-\delta_3)}\right)(\exp 2\pi CN\delta_1)$$

$$+\left(\frac{\delta_2{}^2+4QC}{(\delta_2-\delta_3)(\delta_2-\delta_1)}\right)(\exp 2\pi CN\delta_2)+\left(\frac{\delta_3{}^2+4QC}{(\delta_3-\delta_1)(\delta_3-\delta_2)}\right)(\exp 2\pi CN\delta_3)$$

$$\frac{V_{out}}{V_{in}}=0$$

Oscillation condition

$\delta_1,\delta_2,\delta_3$ are the solutions of

$$(\delta^2+4QC)(j\delta-jd-b)=-1$$

$$\left(\frac{\delta_1{}^2+4QC}{(\delta_1-\delta_2)(\delta_1-\delta_3)}\right)(\exp 2\pi CN\delta_1)$$

$$+\left(\frac{\delta_2{}^2+4QC}{(\delta_2-\delta_3)(\delta_2-\delta_1)}\right)(\exp 2\pi CN\delta_2)+\left(\frac{\delta_3{}^2+4QC}{(\delta_3-\delta_1)(\delta_3-\delta_2)}\right)(\exp 2\pi CN\delta_3)=0$$

The parameter QC may be interpreted as

$$\frac{Q}{N} = \frac{QC}{CN} = \frac{2|\eta|\,V_0}{\varepsilon_0\, r_b{}^2 \omega^3\, K\, l} \longleftarrow \left.\begin{aligned} & QC = \frac{1}{4}\left(\frac{\beta_p}{\beta_e C}\right)^2 = \frac{1}{4}\left(\frac{\omega_p / v_0}{(\omega / v_0)C}\right)^2 \\ & C^3 = \frac{KI_0}{4V_0};\ \omega_p{}^2 = \frac{|\eta||\rho_0|}{\varepsilon_0} \\ & \beta_e l = 2\pi\, N;\ J_0 = \rho_0 v_0 = \frac{I_0}{\pi r_b{}^2} \end{aligned}\right\}$$

(a parameter independent of beam current I_0 and of relevance to TWT beam-wave interaction)

ω has to be interpreted as the frequency where the phase velocity of the forward-wave mode of the SWS becomes equal to that of the backward-wave mode. K has to be taken as the interaction impedance at this frequency.

One can find the solution for *CN* with the help of the following two equations :

(i)

$$(\delta^2 + 4QC)(j\delta - jd - b) = -1$$

Parameters: $d, b, QC = (\frac{Q}{N})(CN)$

(ii)

$$\frac{Q}{N} = \frac{2|\eta|\, V_0}{\varepsilon_0\, r_b^{\,2} \omega^3 K\, l}$$

$$\left(\frac{\delta_1^{\,2} + 4QC}{(\delta_1 - \delta_2)(\delta_1 - \delta_3)}\right)(\exp 2\pi CN\delta_1)$$

$$+\left(\frac{\delta_2^{\,2} + 4QC}{(\delta_2 - \delta_3)(\delta_2 - \delta_1)}\right)(\exp 2\pi CN\delta_2) + \left(\frac{\delta_3^{\,2} + 4QC}{(\delta_3 - \delta_1)(\delta_3 - \delta_2)}\right)(\exp 2\pi CN\delta_3) = 0$$

The solution for CN thus obtained may be interpreted as the start-oscillation current I_0 in view of the relations:

$$C^3 = \frac{KI_0}{4V_0}, \beta_e l = 2\pi N$$

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Thank you