Analysis of Helical Slow-Wave Structure in the Sheath-Helix Model

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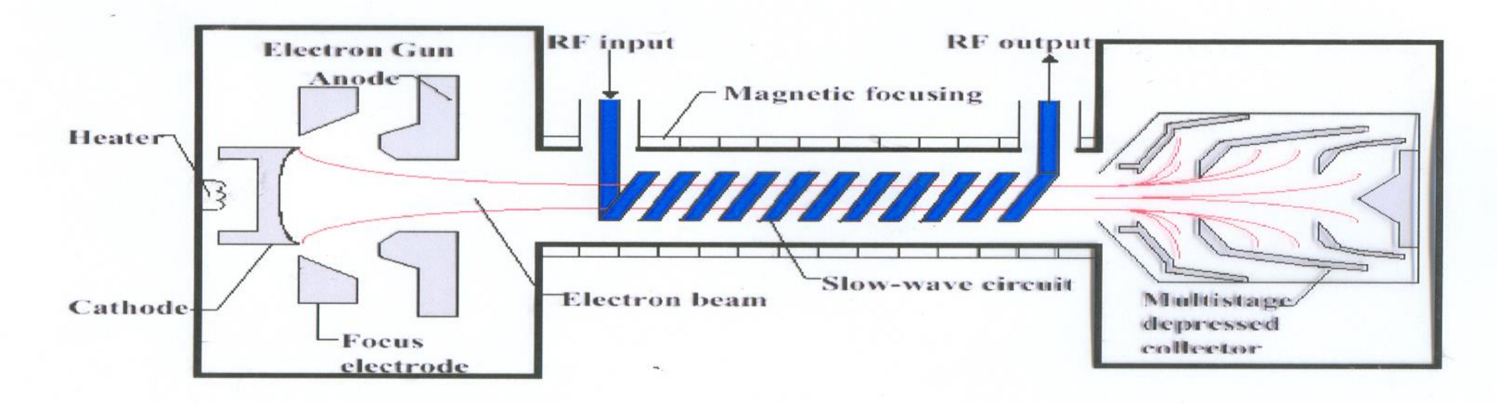
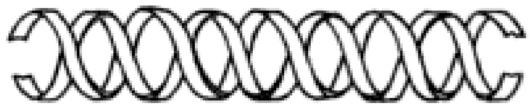
Helical slow-wave structure of a travelling-wave tube

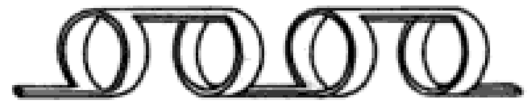


Travelling-wave tube is also known as Kompfner tube

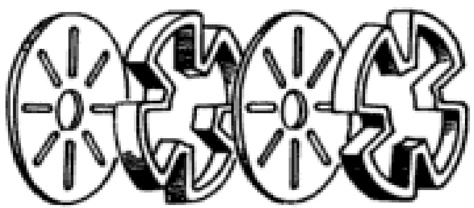
Slow-wave structures other than a single helix



BIFILAR CONTRA-WOUND HELICES

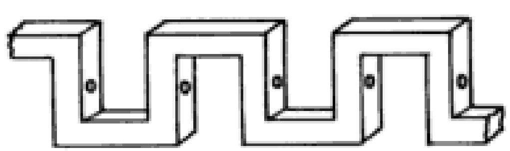


RING AND BAR STRUCTURE

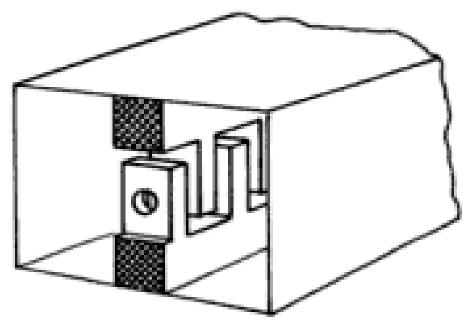


CLOVER-LEAF STRUCTURE

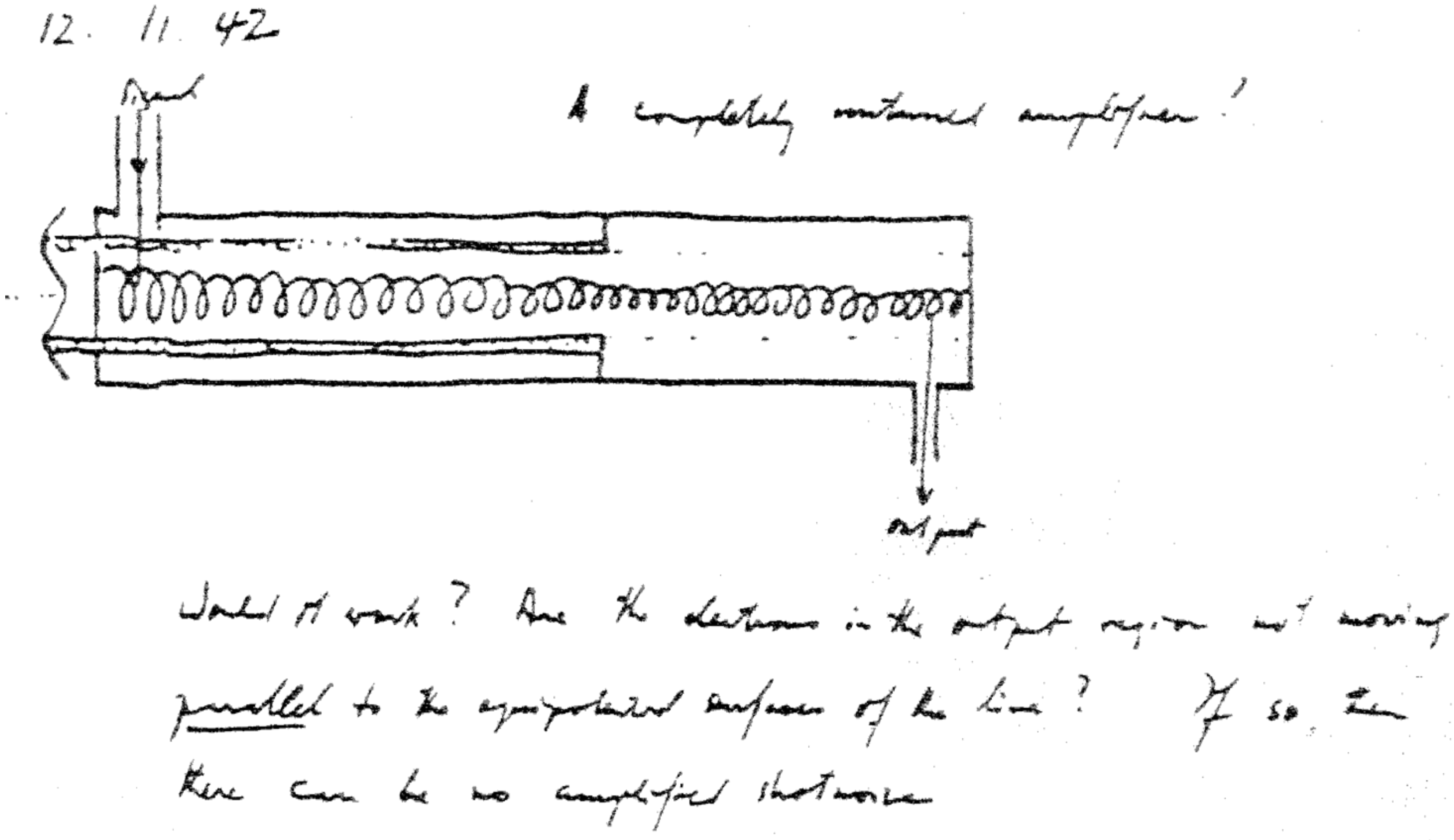
STAGGERED-SLOT COUPLED-CAVITY STRUCTURE



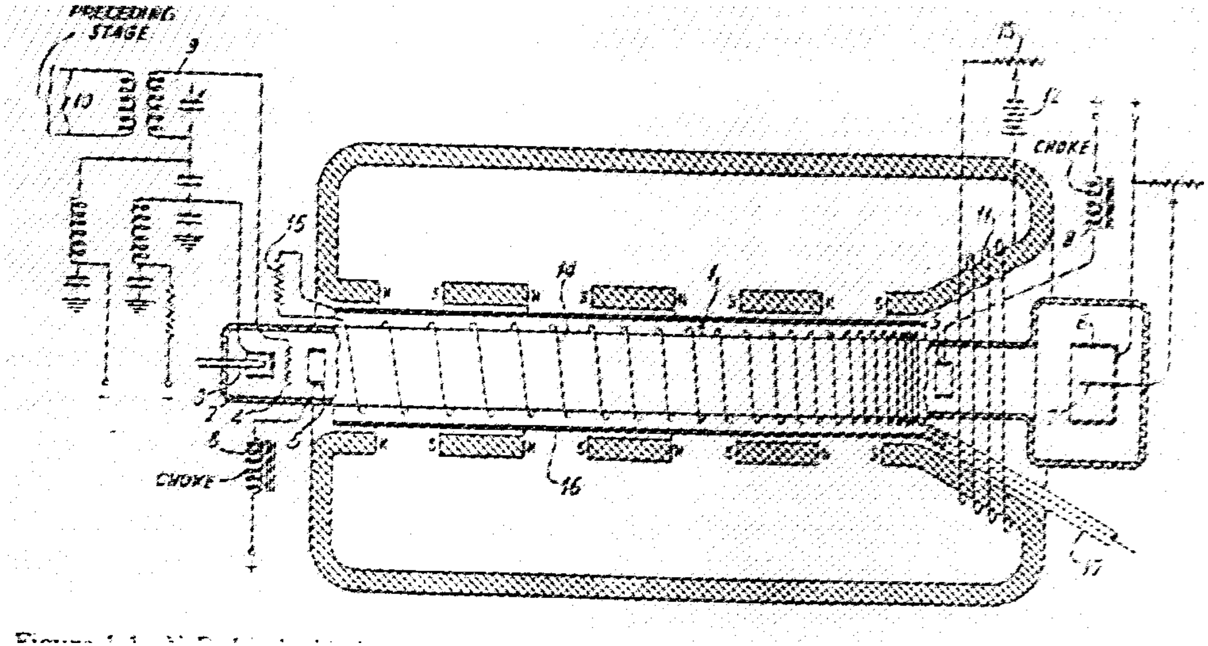
FOLDED-WAVEGUIDE STRUCTURE



INTERDIGITAL STRUCTURE



Sketch of the travelling-wave tube from R. Kompfner’s note book



N. E. Lindenblad's travelling-wave tube amplification at 390 MHz over a 30 MHz band (U. S. Patent 2,300,052, filed on May 4, 1940 issued on October 27, 1942)

Helix wound around the outside the glass envelope. Signal applied to the grid of the electron gun (also applied to the helix in other experiments). Series of permanent magnets (non-periodic). Pitch tapered for velocity re-synchronization

Sheath-helix model

In the sheath-helix model, the actual helix is replaced by a circular cylindrical sheath that has

- **Infinitesimal thickness**
- **Radius equal to the mean radius of the actual helix of a finite thickness**
- **Anisotropic conductivity: infinite conductivity and zero conductivity in directions parallel and perpendicular to the helix winding direction, respectively**

The sheath-helix model is valid for large number of turns per guide wave length

$$\frac{v_p}{c} = \frac{p}{2\pi a}$$ **(phase velocity reduction factor)**

$$v_p = f\lambda_g \qquad c = f\lambda$$

p = **helix pitch**

a = **sheath-helix radius**

λ_g = **guide wavelength in the helix**

λ = **free-space wavelength**

$$\frac{\lambda_g}{\lambda} = \frac{p}{2\pi a} \longrightarrow \frac{\lambda_g}{p} = \frac{\lambda}{2\pi a} \longrightarrow \frac{\lambda_g}{p} >> 1 \longrightarrow \frac{\lambda}{2\pi a} >> 1$$

Valid region of the sheath-helix model

Wave equation and its solution

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}-\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)(E_z,H_z) = 0 \longleftarrow \left.\begin{aligned}\nabla^2 E_z-\mu_0\varepsilon_0\frac{\partial^2 E_z}{\partial t^2}=0\\ \nabla^2 H_z-\mu_0\varepsilon_0\frac{\partial^2 H_z}{\partial t^2}=0\end{aligned}\right\}$$

$\longleftarrow$ $\partial/\partial\theta=0$ **(non-azimuthally varying mode)**

$\longleftarrow$ **RF quantities vary as** $\exp j(\omega t-\beta z)$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)(E_z,H_z)-(\beta^2-\omega^2\mu_0\varepsilon_0)(E_z,H_z)=0$$

$$c=1/\sqrt{\mu_0\varepsilon_0}$$

$$\omega^2\mu_0\varepsilon_0=\omega^2/c^2=k^2$$

$$\gamma^2=\beta^2-\omega^2\mu_0\varepsilon_0=\beta^2-\omega^2/c^2=\beta^2-k^2$$

$$k=\omega\sqrt{\mu_0\varepsilon_0}=\omega/c$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)(E_z,H_z)-\gamma^2(E_z,H_z)=0$$ **(wave equation) (non-azimuthally symmetric mode)**

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)(E_z,H_z)-\gamma^2(E_z,H_z)=0$$ **(wave equation) (non-azimuthally symmetric mode)**

$\gamma^2=\beta^2-k^2$ is positive $\leftarrow$ $\beta>k$ $\leftarrow$ $\omega/v_p>\omega/c$ $\leftarrow$ $v_p<c$

(slow waves)

Field solutions involve modified Bessel functions

$$E_z=AI_0(\gamma r)+BK_0(\gamma r)$$
$$H_z=CI_0(\gamma r)+DK_0(\gamma r)$$

With the help of Maxwell's equations

$$E_\theta=-\frac{j\omega\mu_0}{\gamma}[CI_1(\gamma r)-DK_1(\gamma r)]$$

$$H_\theta=\frac{j\omega\varepsilon_0}{\gamma}[AI_1(\gamma r)-BK_1(\gamma r)]$$

$$E_r=\frac{j\beta}{\gamma}[AI_1(\gamma r)-BK_1(\gamma r)]$$

$$H_r=\frac{j\beta}{\gamma}[CI_1(\gamma r)-DK_1(\gamma r)]$$

$I_0(x)$: **zeroth order modified Bessel function of the first kind**

$K_0(x)$: **zeroth order modified Bessel function of the second kind**

$I_1(x)$: **first order modified Bessel function of the first kind**

$K_1(x)$: **first order modified Bessel function of the second kind**

$I_0'(x)=I_1(x)$ $\leftarrow$ $x=\gamma r$

$K_0'(x)=-K_1(x)$

r and θ components of Maxwell's curl equations

$$E_z = AI_0(\gamma r) + BK_0(\gamma r)$$
$$H_z = CI_0(\gamma r) + DK_0(\gamma r)$$

$$\exp j(\omega t - \beta z) = \exp(j\omega t - \gamma z)$$

RF quantities vary as

$$\partial / \partial\theta = 0$$

$$\frac{1}{r}\frac{\partial E_z}{\partial\theta} - \frac{\partial E_\theta}{\partial z} = -\mu\frac{\partial H_r}{\partial t}$$ **(*r*-component)** $(\nabla\times\vec{E})_r$

$$\boxed{j\beta E_\theta = -j\omega\mu_0 H_r}$$

$$\frac{1}{r}\frac{\partial H_z}{\partial\theta} - \frac{\partial H_\theta}{\partial z} = \varepsilon_0\frac{\partial E_r}{\partial t}$$ **(*r*-component)** $(\nabla\times\vec{H})_r$

$$\boxed{j\beta H_\theta = j\omega\varepsilon_0 E_r}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\mu_0\frac{\partial H_\theta}{\partial t}$$ **(θ-component)** $(\nabla\times\vec{E})_\theta$

$$\boxed{-j\beta E_r - [AI_0'(\gamma r) + BK_0'(\gamma r)]\gamma = -j\omega\mu_0 H_\theta}$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \varepsilon_0\frac{\partial E_\theta}{\partial t}$$ **(θ-component)** $(\nabla\times\vec{H})_\theta$

$$\boxed{-j\beta H_r - [CI_0'(\gamma r) + DK_0'(\gamma r)]\gamma = j\omega\varepsilon_0 E_\theta}$$

$$\left.\begin{aligned} &j\beta E_\theta = -j\omega\mu_0 H_r \\ &j\beta H_\theta = j\omega\varepsilon_0 E_r \\ &-j\beta E_r - [AI_0'(\gamma r) + BK_0'(\gamma r)]\gamma = -j\omega\mu_0 H_\theta \\ &-j\beta H_r - [CI_0'(\gamma r) + DK_0'(\gamma r)]\gamma = j\omega\varepsilon_0 E_\theta \end{aligned}\right\}$$

(rewritten)

$$\downarrow$$

$$\left.\begin{aligned} &H_\theta = \left(\frac{j\omega\varepsilon_0}{\gamma}\right)[AI_0'(\gamma r) + BK_0'(\gamma r)] \\ &E_\theta = \left(-\frac{j\omega\mu_0}{\gamma}\right)[CI_0'(\gamma r) + DK_0'(\gamma r)] \end{aligned}\right\}$$

$$I_0'(\gamma r) = I_1(\gamma r)$$

$$K_0'(\gamma r) = -K_1(\gamma r) \longrightarrow$$

$$\downarrow$$

$$\boxed{\left.\begin{aligned} &H_\theta = \left(\frac{j\omega\varepsilon_0}{\gamma}\right)[AI_1(\gamma r) - BK_1(\gamma r)] \\ &E_\theta = -\frac{j\omega\mu_0}{\gamma}[CI_1(\gamma r) - DK_1(\gamma r)] \end{aligned}\right\}}$$

Similarly,

$$\left.\begin{aligned} j\beta E_\theta &= -j\omega\mu_0 H_r \\ j\beta H_\theta &= j\omega\varepsilon_0 E_r \\ -j\beta E_r - [AI_0'(\gamma r) + BK_0'(\gamma r)]\gamma &= -j\omega\mu_0 H_\theta \\ -j\beta H_r - [CI_0'(\gamma r) + DK_0'(\gamma r)]\gamma &= j\omega\varepsilon_0 E_\theta \end{aligned}\right\}$$ **(rewritten)**

$$\downarrow$$

$$\left.\begin{aligned} E_r &= \frac{j\beta}{\gamma}[AI_0'(\gamma r) + BK_0'(\gamma r)] \\ H_r &= \frac{j\beta}{\gamma}[CI_1(\gamma r) + DK_0'(\gamma r)]] \end{aligned}\right\}$$

$$I_0'(\gamma r) = I_1(\gamma r)$$
$$K_0'(\gamma r) = -K_1(\gamma r)$$
$$\longrightarrow \quad \downarrow$$

$$\boxed{\left.\begin{aligned} E_r &= \frac{j\beta}{\gamma}[AI_1(\gamma r) - BK_1(\gamma r)] \\ H_r &= \frac{j\beta}{\gamma}[CI_1(\gamma r) - DK_1(\gamma r)] \end{aligned}\right\}}$$

Thus, we obtained:

$$E_z = AI_0(\gamma r) + BK_0(\gamma r)$$

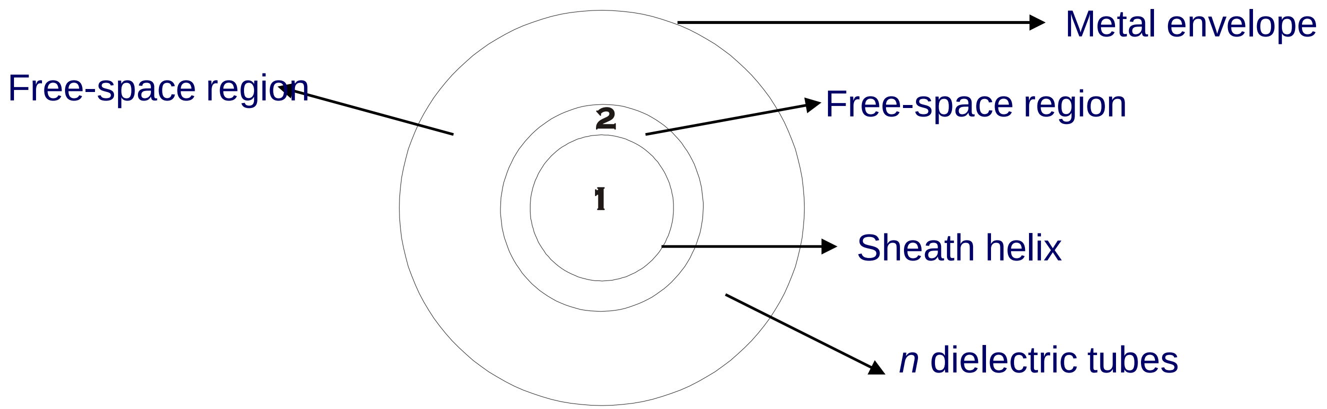
$$H_z = CI_0(\gamma r) + DK_0(\gamma r)$$

$$E_\theta = -\frac{j\omega\mu_0}{\gamma}[CI_1(\gamma r) - DK_1(\gamma r)]$$

$$H_\theta = \frac{j\omega\varepsilon_0}{\gamma}[AI_1(\gamma r) - BK_1(\gamma r)]$$

$$E_r = \frac{j\beta}{\gamma}[AI_1(\gamma r) - BK_1(\gamma r)]$$

$$H_r = \frac{j\beta}{\gamma}[CI_1(\gamma r) - DK_1(\gamma r)]$$



Structure model

$$E_{z1} = A_1 I_0(\gamma r) + B_1 K_0(\gamma r) \qquad E_{z2} = A_2 I_0(\gamma r) + B_2 K_0(\gamma r)$$

$$H_{z1} = C_1 I_0(\gamma r) + D_1 K_0(\gamma r) \qquad H_{z2} = C_2 I_0(\gamma r) + D_2 K_0(\gamma r)$$

$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma}[C_1 I_1(\gamma r) - D_1 K_1(\gamma r)] \qquad E_{\theta 2} = -\frac{j\omega\mu_0}{\gamma}[C_2 I_1(\gamma r) - D_2 K_1(\gamma r)]$$

$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma}[A_1 I_1(\gamma r) - B_1 K_1(\gamma r)] \qquad H_{\theta 2} = \frac{j\omega\varepsilon_0}{\gamma}[A_2 I_1(\gamma r) - B_2 K_1(\gamma r)]$$

$$E_{r1} = \frac{j\beta}{\gamma}[A_1 I_1(\gamma r) - B_1 K_1(\gamma r)] \qquad E_{r2} = \frac{j\beta}{\gamma}[A_2 I_1(\gamma r) - B_2 K_1(\gamma r)]$$

$$H_{r1} = \frac{j\beta}{\gamma}[C_1 I_1(\gamma r) - D_1 K_1(\gamma r)] \qquad H_{r2} = \frac{j\beta}{\gamma}[C_2 I_1(\gamma r) - D_2 K_1(\gamma r)]$$

$$\vec{E} = E_r \vec{a}_r + E_\theta \vec{a}_\theta + E_z \vec{a}_z$$

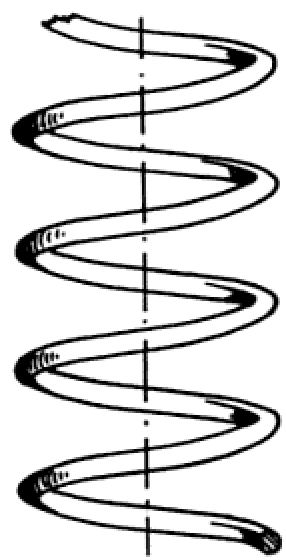
$$\downarrow$$

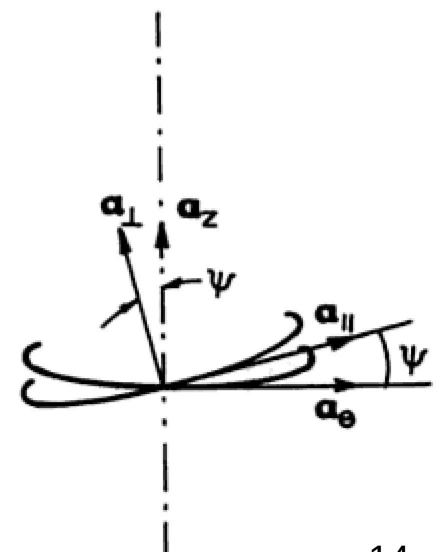
$$E_{//} = \vec{E} \cdot \vec{a}_{//} = E_r \vec{a}_r \cdot \vec{a}_{//} + E_\theta \vec{a}_\theta \cdot \vec{a}_{//} + E_z \vec{a}_z \cdot \vec{a}_{//}$$

$$\downarrow$$

$$E_{//} = (E_r)(0) + (E_\theta)(\cos\psi) + (E_z)(\sin\psi)$$

$$\boxed{E_{//} = E_\theta \cos\psi + E_z \sin\psi}$$





Boundary conditions at the mean helix radius *r* = *a*

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0 \rightarrow \vec{a}_r \times [(E_{r2} - E_{r1})\vec{a}_r + (E_{\theta 2} - E_{\theta 1})\vec{a}_\theta + (E_{z2} - E_{z1})\vec{a}_z] = 0$$

$$\downarrow$$

$$(E_{\theta 2} - E_{\theta 1})\vec{a}_z + (E_{z1} - E_{z2})\vec{a}_\theta = 0 \rightarrow \begin{array}{l} E_{z1} = E_{z2} \\ E_{\theta 1} = E_{\theta 2} \end{array}$$

$\downarrow$ ← **can be interpreted as**

$$E_{z1} = E_{zi} \qquad E_{zi} = E_{z2}$$

$$E_{\theta 1} = E_{\theta i} \qquad E_{\theta i} = E_{\theta 2}$$

$$\boxed{\begin{array}{l} E_{z1} = E_{z2} \\ E_{\theta 1} = E_{\theta 2} \end{array}}$$

Subscript *i*:
inside the helix thickness

Subscript 1:
Inside the helix winding radius $(0 \le r \le a)$

Subscript 2:
outside the helix winding radius $(a \le r \le \infty)$

Boundary conditions at the mean helix radius *r* = *a*

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0 \longleftarrow \quad \vec{E} = E_{//}\vec{a}_{//} + E_{\perp}\vec{a}_{\perp} + E_{\perp}\vec{a}_{\perp} + E_r\vec{a}_r, \quad \vec{a}_n = \vec{a}_r$$

$$\downarrow$$

$$\vec{a}_r \times [(E_{//2} - E_{//1})\vec{a}_{//} + (E_{\perp 2} - E_{\perp 1})\vec{a}_{\perp} + (E_{r2} - E_{r1})\vec{a}_r] = 0$$

$$\downarrow$$

$$(E_{//2} - E_{//1})\vec{a}_{\perp} + (E_{\perp 1} - E_{\perp 2})\vec{a}_{//} = 0 \longrightarrow \quad E_{//1} = E_{//2}, \quad E_{\perp 1} = E_{\perp 2}$$

$$E_{//1} = E_{//i}, \quad E_{//i} = E_{//2} \qquad\qquad E_{\perp 1} = E_{\perp i}, \quad E_{\perp i} = E_{\perp 2}$$

$$E_{//1} = E_{//2} \longleftarrow \quad E_{//1} = E_{//i}, \quad E_{//i} = E_{//2}$$

$$\vec{E} = E_{//}\vec{a}_{//} + E_{\perp}\vec{a}_{\perp} + E_{\perp}\vec{a}_{\perp} \longrightarrow \downarrow$$

$$E_{//1} = E_{//2} = 0$$ **(infinite conductivity parallel to the winding direction)**

$$\downarrow$$

$$\boxed{E_{\theta 1}\cos\psi + E_{z1}\sin\psi = E_{\theta 2}\cos\psi + E_{z2}\sin\psi = 0} \longleftarrow \quad E_{//} = E_{\theta}\cos\psi + E_z\sin\psi$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s \longrightarrow \vec{a}_r \times [(H_{r2} - H_{r1})\vec{a}_r + (H_{\theta 2} - H_{\theta 1})\vec{a}_\theta + (H_{z2} - H_{z1})\vec{a}_z] = \vec{J}_s$$

$$\downarrow$$

$$(H_{\theta 2} - H_{\theta 1})\vec{a}_z + (H_{z1} - H_{z2})\vec{a}_\theta = \vec{J}_s = J_{sz}\vec{a}_z + J_{s\theta}\vec{a}_\theta$$

$$H_{\theta 2} - H_{\theta 1} = J_{sz}$$

$$\downarrow$$

$$H_{\theta 2} - H_{\theta i} = J_{szo}$$

$$H_{\theta i} - H_{\theta 1} = J_{szi}$$

$$\downarrow$$

$$H_{\theta 2} - H_{\theta 1} = J_{szi} + J_{szo} = J_{sz}$$

$$\downarrow$$

$$\boxed{H_{\theta 2} - H_{\theta 1} = \frac{I_z}{2\pi a}}$$

$$\vec{J}_s = Lt\,\vec{J}dh$$

$$dh \to 0$$

$$J_{s\theta} = Lt\,\frac{I_\theta}{2\pi a dh}dh = \frac{I_\theta}{2\pi a}$$

$$dh \to 0$$

$$J_{sz} = Lt\,\frac{I_z}{2\pi a dh}dh = \frac{I_z}{2\pi a}$$

$$dh \to 0$$

$$H_{z1} - H_{z2} = J_{s\theta}$$

$$\downarrow$$

$$H_{z1} - H_{zi} = J_{s\theta i}$$

$$H_{zi} - H_{z2} = J_{s\theta 0}$$

$$\downarrow$$

$$H_{z1} - H_{z2} = J_{s\theta i} + J_{s\theta o} = J_{s\theta}$$

$$\downarrow$$

$$\boxed{H_{z1} - H_{z2} = \frac{I_\theta}{2\pi a}}$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s \longrightarrow \vec{a}_r \times [(H_{//2} - H_{//1})\vec{a}_{//} + (H_{\perp 2} - H_{\perp 1})\vec{a}_{\perp} + (H_{r2} - H_{r1})\vec{a}_r] = \vec{J}_s$$

$$\downarrow$$

$$(H_{//2} - H_{//1})\vec{a}_{\perp} + (H_{\perp 1} - H_{\perp 2})\vec{a}_{//} = J_{s\perp}\vec{a}_{\perp} + J_{//}\vec{a}_{//}$$

$$\downarrow$$

$$J_{s\perp} = 0$$

(no current perpendicular to the winding direction)

$$H_{//2} - H_{//1} = 0$$

$$\downarrow$$

$$H_{//1} = H_{//2}$$

$$\vec{H} = H_r\vec{a}_r + H_\theta\vec{a}_\theta + H_z\vec{a}_z$$

$$H_{//} = \vec{H} \cdot \vec{a}_{//} = H_r\vec{a}_r \cdot \vec{a}_{//} + H_\theta\vec{a}_\theta \cdot \vec{a}_{//} + H_z\vec{a}_z \cdot \vec{a}_{//}$$

$$H_{//} = (H_r)(0) + (H_\theta)(\cos\psi) + (H_z)(\sin\psi)$$

$$H_{//} = H_\theta \cos\psi + H_z \sin\psi$$

$$\downarrow$$

$$\boxed{H_{\theta 1}\cos\psi + H_{z1}\sin\psi = H_{\theta 2}\cos\psi + H_{z2}\sin\psi}$$

The following four boundary conditions at the mean helix radius = sheath-helix radius $r = a$ are used in deriving the dispersion relation of a helix using the electromagnetic field analysis:

$$
\begin{aligned}
&E_{z1} = E_{z2} \\
&E_{\theta 1}\cos\psi + E_{z1}\sin\psi = 0 \\
&E_{\theta 2}\cos\psi + E_{z2}\sin\psi = 0 \\
&H_{\theta 1}\cos\psi + H_{z1}\sin\psi = H_{\theta 2}\cos\psi + H_{z2}\sin\psi
\end{aligned}
$$

Field expressions

$$E_{z1} = A_1 I_0(\gamma r) + B_1 K_0(\gamma r)$$
$$H_{z1} = C_1 I_0(\gamma r) + D_1 K_0(\gamma r)$$
$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma}[C_1 I_1(\gamma r) - D_1 K_1(\gamma r)]$$
$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma}[A_1 I_1(\gamma r) - B_1 K_1(\gamma r)]$$

$$K_0(0) \to \infty$$
$$I_0(\infty) \to \infty$$
$$B_1 = 0$$
$$A_2 = 0$$
$$D_1 = 0$$
$$C_2 = 0$$

$$E_{z2} = A_2 I_0(\gamma r) + B_2 K_0(\gamma r)$$
$$H_{z2} = C_2 I_0(\gamma r) + D_2 K_0(\gamma r)$$
$$E_{\theta 2} = -\frac{j\omega\mu_0}{\gamma}[C_2 I_1(\gamma r) - D_2 K_1(\gamma r)]$$
$$H_{\theta 2} = \frac{j\omega\varepsilon_0}{\gamma}[A_2 I_1(\gamma r) - B_2 K_1(\gamma r)]$$

$$E_{z1} = A_1 I_0(\gamma r)$$
$$H_{z1} = C_1 I_0(\gamma r)$$
$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma r)$$
$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma r)$$

$$E_{z2} = B_2 K_0(\gamma r)$$
$$H_{z2} = D_2 K_0(\gamma r)$$
$$E_{\theta 2} = \frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma r)$$
$$H_{\theta 2} = \frac{-j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma r)$$

A_1, C_1, B_2, D_2 : **four non-zero field constants**

At the sheath-helix radius *r* = *a*:

$$E_{\theta 2}\cos\psi + E_{z2}\sin\psi = 0$$

$$H_{\theta 1}\cos\psi + H_{z1}\sin\psi = H_{\theta 2}\cos\psi + E_{z1} = E_{z2}$$

$$E_{\theta 1}\cos\psi + E_{z1}\sin\psi = 0$$

$$E_{z1} = E_{z2}$$

$$E_{z1} = A_1 I_0(\gamma r)$$

$$H_{z1} = C_1 I_0(\gamma r)$$

$$E_{\theta 1} = -\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma r)$$

$$H_{\theta 1} = \frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma r)$$

$$E_{z2} = B_2 K_0(\gamma r)$$

$$H_{z2} = D_2 K_0(\gamma r)$$

$$E_{\theta 2} = \frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma r)$$

$$H_{\theta 2} = \frac{-j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma r)$$

↓

$$\frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma a)\cos\psi + B_2 K_0(\gamma a)\sin\psi = 0$$

$$\frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma a)\cos\psi + C_1 I_0(\gamma a)\sin\psi +$$

$$\frac{j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma a)\cos\psi - D_2 K_0(\gamma a)\sin\psi = 0$$

$$-\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma a)\cos\psi + A_1 I_0(\gamma a)\sin\psi = 0$$

$$A_1 I_0(\gamma a) E_{z2} - B_2 K_0(\gamma a) = 0$$

$$\frac{j\omega\mu_0}{\gamma} D_2 K_1(\gamma a)\cos\psi + B_2 K_0(\gamma a)\sin\psi = 0$$

$$\frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma a)\cos\psi + C_1 I_0(\gamma a)\sin\psi +$$

$$\frac{j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma a)\cos\psi - D_2 K_0(\gamma a)\sin\psi = 0$$

$$-\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma a)\cos\psi + A_1 I_0(\gamma a)\sin\psi = 0$$

$$A_1 I_0(\gamma a) E_{z2} - B_2 K_0(\gamma a) = 0$$

$$\longrightarrow \begin{aligned} a_{11}A_1 + a_{12}C_1 + a_{13}B_2 + a_{14}D_2 &= 0 \\ a_{21}A_1 + a_{22}C_1 + a_{23}B_2 + a_{24}D_2 &= 0 \\ a_{31}A_1 + a_{32}C_1 + a_{33}B_2 + a_{34}D_2 &= 0 \\ a_{41}A_1 + a_{42}C_1 + a_{43}B_2 + a_{44}D_2 &= 0 \end{aligned}$$

where

$$a_{11} = 0, a_{12} = 0, a_{13} = K_0(\gamma a)\sin\psi,$$

$$a_{14} = \frac{j\omega\mu_0}{\gamma} K_1(\gamma a)\cos\psi$$

$$a_{21} = \frac{j\omega\varepsilon_0}{\gamma} I_1(\gamma a)\cos\psi,\ a_{22} = I_0(\gamma a)\sin\psi$$

$$a_{23} = \frac{j\omega\varepsilon_0}{\gamma} K_1(\gamma a)\cos\psi,\ a_{24} = -K_0(\gamma a)\sin\psi$$

$$a_{31} = I_0(\gamma a)\sin\psi,\ a_{32} = -\frac{j\omega\mu_0}{\gamma} I_1(\gamma a)\cos\psi$$

$$a_{33} = 0,\ a_{34} = 0$$

$$a_{41} = I_0(\gamma a),\ a_{42} = 0,\ a_{43} = -K_0(\gamma a),\ a_{44} = 0$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0$$

(condition for non-trivial solution)

$$a_{11} = 0, a_{12} = 0, a_{13} = K_0(\gamma a)\sin\psi,$$

$$a_{14} = \frac{j\omega\mu_0}{\gamma} K_1(\gamma a)\cos\psi$$

$$a_{21} = \frac{j\omega\varepsilon_0}{\gamma} I_1(\gamma a)\cos\psi,\ a_{22} = I_0(\gamma a)\sin\psi$$

$$a_{23} = \frac{j\omega\varepsilon_0}{\gamma} K_1(\gamma a)\cos\psi,\ a_{24} = -K_0(\gamma a)\sin\psi$$

$$a_{31} = I_0(\gamma a)\sin\psi,\ a_{32} = -\frac{j\omega\mu_0}{\gamma} I_1(\gamma a)\cos\psi$$

$$a_{33} = 0,\ a_{34} = 0$$

$$a_{41} = I_0(\gamma a),\ a_{42} = 0,\ a_{43} = -K_0(\gamma a),\ a_{44} = 0$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0$$

$$\boxed{\frac{k\cot\psi}{\gamma} = \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)} \right)^{1/2}}$$

(Dispersion relation of a helix in free-space)

$$\frac{v_p}{c}\cot\psi \approx \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)} \right)^{1/2}$$

$$I_0(x)K_1(x) + K_0(x)I_1(x) = \frac{1}{x}$$

$$\downarrow$$

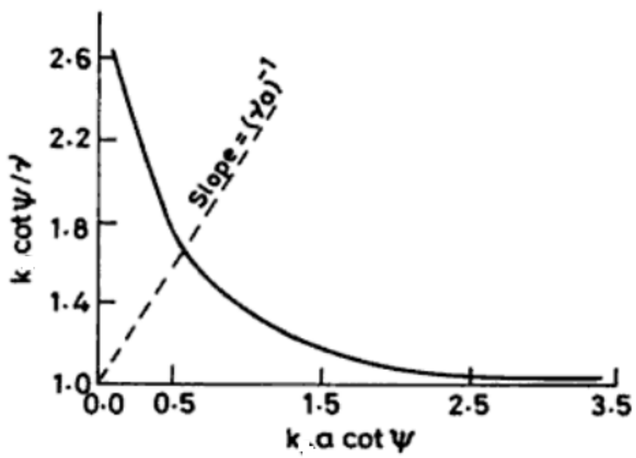
$$I_0(\gamma a)K_1(\gamma a) + K_0(\gamma a)I_1(\gamma a) = \frac{1}{\gamma a}$$

$$\gamma^2 = \beta^2 - \omega^2\mu_0\varepsilon_0 = \beta^2 - \frac{\omega^2}{c^2} = \beta^2 - k^2$$

$$\gamma = \sqrt{\beta^2 - k^2}$$

$$k = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c}$$

$$\beta = \frac{\omega}{v_p}$$



Equivalent Circuit Analysis of a Helix

The following four boundary conditions at the mean helix radius = sheath-helix radius r = a are used in deriving the dispersion relation of a helix using the equivalent circuit analysis:

$$E_{z1} = E_{z2}$$

$$E_{\theta 1} = E_{\theta 2}$$

$$H_{\theta 2} - H_{\theta 1} = \frac{I_z}{2\pi a}$$

$$H_{z1} - H_{z2} = \frac{I_\theta}{2\pi a}$$

Capacitance per unit length

$$A_1 I_0(\gamma a) = B_2 K_0(\gamma a) \longleftarrow E_{z1} = E_{z2}$$

$$-\frac{j\omega\varepsilon_0}{\gamma} B_2 K_1(\gamma a) - \frac{j\omega\varepsilon_0}{\gamma} A_1 I_1(\gamma a) = \frac{I_{za}}{2\pi a} \longleftarrow H_{\theta 2} - H_{\theta 1} = \frac{I_z}{2\pi a}$$

$$\downarrow$$

$$\boxed{A_1 = -\left(\frac{\gamma}{j\omega\varepsilon_0}\right)\gamma a K_0(\gamma a)\frac{I_{za}}{2\pi a}}$$

$$\boxed{E_{za} = A_1 I_0(\gamma a)} \longleftarrow E_{z1} = A_1 I_0(\gamma r)$$

$$E_{za} = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t} \longleftarrow E_{za} = -(\nabla V)_z - \frac{\partial A_z}{\partial t} \longleftarrow \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\nwarrow \frac{\partial A_z}{\partial z} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = 0 \longleftarrow \nabla \cdot \vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = 0$$

(Lorentz condition recalled)

$$\downarrow$$

$$E_{za} = j\frac{\beta^2 - \omega^2\mu_0\varepsilon_0}{\beta} V$$

$$\nwarrow \frac{\partial}{\partial t} = j\omega, \quad \frac{\partial}{\partial z} = -j\beta \longleftarrow$$

RF quantities vary as $\exp j(\omega t - \beta)z$

$$\downarrow \longleftarrow \beta^2 - \omega^2\mu_0\varepsilon_0 = \beta^2 - k^2 = \gamma^2$$

$$\boxed{E_{za} = \frac{j\gamma^2}{\beta} V}$$

$$\frac{\partial I_{za}}{\partial z} + C\frac{\partial V}{\partial t} = 0$$ **(Telegrapher's equation)**

$$\frac{\partial}{\partial t} = j\omega, \ \frac{\partial}{\partial z} = -j\beta$$

$$E_{za} = A_1 I_0(\gamma a)$$

$$A_1 = -\left(\frac{\gamma}{j\omega\varepsilon_0}\right)\gamma a K_0(\gamma a)\frac{I_{za}}{2\pi a}$$

$$E_{za} = \frac{j\gamma^2}{\beta}V$$

$$I_0(\gamma a)K_1(\gamma a) + K_0(\gamma a)I_1(\gamma a) = \frac{1}{\gamma a}$$

$$-j\beta I_{za} + j\omega CV = 0$$

$$C = \frac{2\pi\varepsilon_0}{I_0(\gamma a)K_0(\gamma a)}$$

(Capacitance per unit length of a transmission line equivalent of a helix)

Inductance per unit length *L*

$$-\frac{j\omega\mu_0}{\gamma}C_1 I_1(\gamma a)=\frac{j\omega\mu_0}{\gamma}D_2 K_1(\gamma a)$$

$$C_1 I_0(\gamma a)-D_2 K_0(\gamma a)=\frac{I_{\theta a}}{2\pi a}$$

$$E_{\theta 1}=E_{\theta 2}$$

$$H_{z1}-H_{z2}=\frac{I_\theta}{2\pi a}$$

$$\boxed{C_1=\gamma a K_1(\gamma a)\frac{I_{\theta a}}{2\pi a}}$$

$$\boxed{E_{\theta a}\cos\psi+E_{za}\sin\psi=0} \longleftarrow E_{\theta 1}\cos\psi+E_{z1}\sin\psi=0$$

$$E_{\theta 1}=-\frac{j\omega\mu_0}{\gamma}C_1 I_1(\gamma r)$$

$$\downarrow$$

$$\boxed{E_{\theta a}=-\frac{j\omega\mu_0}{\gamma}C_1 I_1(\gamma a)}$$

$$J_{s\perp}=\vec{J}_s\cdot\vec{a}_\perp=(J_{s\theta}\vec{a}_\theta+J_{sz}\vec{a}_z)\cdot\vec{a}_\perp$$

$$=J_{s\theta}\,\vec{a}_\theta\cdot\vec{a}_\perp+J_{sz}\vec{a}_z\cdot\vec{a}_\perp=-J_{s\theta}\sin\psi+J_{sz}\cos\psi=0$$

(no sheath-helix current perpendicular to the winding direction)

$$\downarrow$$

$$\boxed{-I_{\theta a}\sin\psi+I_{za}\cos\psi=0}$$

$$\frac{\partial V}{\partial z} + L\frac{\partial I_{za}}{\partial t} = 0$$

(Telegrapher's equation)

$$\frac{\partial}{\partial t} = j\omega, \quad \frac{\partial}{\partial z} = -j\beta$$

RF quantities vary as $\exp j(\omega t - \beta)z$

$$-j\beta V + j\omega L I_{za} = 0$$

$$L = \frac{\mu_0}{2\pi}\left(\frac{\beta}{\gamma}\right)^2 \cot^2\psi \, I_1(\gamma a) K_1(\gamma a)$$

$$E_{\theta a} = -\frac{j\omega\mu_0}{\gamma} C_1 I_1(\gamma a)$$

$$E_{za} = \frac{j\gamma^2}{\beta} V$$

$$C_1 = \gamma a K_1(\gamma a) \frac{I_{\theta a}}{2\pi a}$$

$$E_{\theta a}\cos\psi + E_{za}\sin\psi = 0$$

$$-I_{\theta a}\sin\psi + I_{za}\cos\psi = 0$$

$$I_0(\gamma a)K_1(\gamma a) + K_0(\gamma a)I_1(\gamma a) = \frac{1}{\gamma a}$$

$$C = \frac{2\pi\varepsilon_0}{I_0(\gamma a)K_0(\gamma a)}$$

$$L = \frac{\mu_0}{2\pi}\left(\frac{\beta}{\gamma}\right)^2 \cot^2\psi\, I_1(\gamma a)K_1(\gamma a)$$

$$\beta^2 = \omega^2 LC$$

$$\frac{k\cot\psi}{\gamma} = \left(\frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\right)^{1/2}$$

(Dispersion relation of a helix in free-space obtained by the equivalent circuit analysis)

$$\gamma^2 = \beta^2 - \omega^2\mu_0\varepsilon_0 = \beta^2 - \frac{\omega^2}{c^2} = \beta^2 - k^2$$

$$\gamma = \sqrt{\beta^2 - k^2}$$

$$k = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c}$$

$$\beta = \frac{\omega}{v_p}$$

The electromagnetic field analysis and the equivalent circuit analysis yield one and the same dispersion relation.

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Thank you!