Synthesis of Pierce electron guns

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Synthesis of Pierce electron guns

◊ Space-charge-limited and temperature-limited current

◊ Child-Langmuir's law, Langmuir and Blodgett's law

◊ Parallel-flow Pierce gun derived from a flat cathode

◊ Convergent-flow Pierce gun derived from a curved cathode

◊ Conformal transformation mapping of electrode shapes of a Pierce electron gun

Space-charge-limited and temperature-limited conditions of electron emission

Microwave tubes such as the TWT and the klystron operate under the space-charge-limited condition of emission.

Microwave tubes such as the gyrotron and the gyro-TWT operate under the temperature-limited condition of emission.

In a space-charge limited diode, the cathode temperature is raised to a relatively high value and the anode potential, which is positive with respect to the cathode, is raised to such a value that the number of emitted electrons exceeds the number of electrons reaching the anode. Under this condition, the anode current can be increased by increasing the anode potential and cannot be increase by increasing the cathode temperature, until the anode current reaches a saturation value. Beyond this value of the anode saturation current, however, the anode current can be increased by increasing the cathode temperature.

The operating region below this saturation is the space-charge-limited region.
The operating region above this saturation is the temperature-limited region.

The regions below and above the anode saturation current obtained by increasing the anode potential, at a constant cathode temperature, are the space-charge-limited and the temperature-limited regions, respectively.

The regions below and above the anode saturation current obtained by increasing the cathode temperature, at a constant anode potential, are the temperature-limited and the space-charge-limited regions, respectively.

The anode current can be increased by increasing the anode potential in the space-charge-limited region.

The anode current can be increased by increasing the cathode temperature in the temperature-limited region.

Current distribution in a planar diode

$$\left|J\right| = \left(\frac{4}{9}\right)(2|\eta|)^{1/2}\varepsilon_0 \frac{V^{3/2}}{z^2}$$

(*Child-Langmuir's relation* for a planar diode)
(deduced in Appendix)

$$\downarrow \quad \downarrow$$

$$\frac{I_0}{A} = \frac{4}{9}\sqrt{2|\eta|}\varepsilon_0 \frac{V_0^{3/2}}{d^2}$$

J : beam current density at a distance z from the cathode

V : potential at a distance z from the cathode

I_0 : beam current $\quad V_0$: beam voltage

A : cathode cross-sectional area $\quad d$: anode-cathode distance

ε_0 : free-space permittivity $\quad \eta$: charge-to-mass ratio of an electron

Child-Langmuir's relation for a planar diode has been deduced in Appendix.

3/2-power law and beam perveance

$$\frac{I_0}{A} = \frac{4}{9}\sqrt{2|\eta|}\varepsilon_0 \frac{V_0^{3/2}}{d^2}$$ **(following from Child-Langmuir's law)**

$$\downarrow$$

$$\frac{I_0}{V_0^{3/2}} = \frac{4}{9}\sqrt{2|\eta|}\varepsilon_0 \frac{A}{d^2} = \text{beam perveance}$$

$$\downarrow$$

The beam current is proportional to the beam voltage raised to the index of power 3/2**. Hence the name** 3/2**-power law.**

The unit of beam perveance is $A/V^{3/2}$ **or** perv

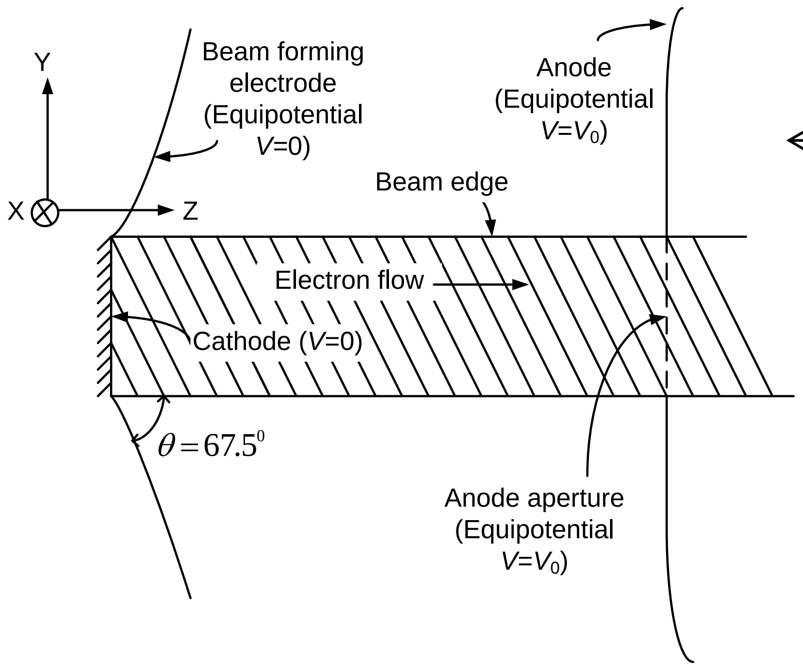
$$\frac{I_0}{A} = \frac{4}{9}\sqrt{2|\eta|}\,\varepsilon_0 \frac{V_0^{3/2}}{d^2}$$ (*Child-Langmuir's relation for a planar diode*)

(deduced in Appendix)

If we increase the distance d between the cathode and the accelerating anode of a diode to provide a hypothetical interaction region and accommodate an interaction structure between them, then the beam current I_0 would reduce to an insignificant value, according to the Child-Langmuir's relation.

With the help of an electron gun, we form an electron beam of the desired beam voltage, current and cross-sectional area with the help of an electron gun and throw it into the interaction region and accommodated an interaction structure between them, then the beam current would reduce to an insignificant value.

Pierce gun derived from a flat cathode giving a strip beam



Beam focusing/forming electrode (BFE) is at the cathode potential ($V = 0$) but thermally insulated from the cathode.

Two conditions at the beam edge:

$$\left. V(z) \right|_{y=0} = \left(\frac{9|J|}{4(2|\eta|)^{1/2} \varepsilon_0} \right)^{2/3} z^{4/3}$$

$y = 0$ **refers to the beam edge**

$$\left. \frac{\partial V}{\partial y} \right|_{y=0} = 0 \quad \leftarrow$$

At the beam edge ($y = 0$) there is no electric field to deflect the beam from the rectilinear flow.

Two-dimensional Laplace's equation is satisfied outside the beam:

$$\frac{\partial^2 V(z,y)}{\partial y^2} + \frac{\partial^2 V(z,y)}{\partial z^2} = 0$$

$$\frac{\partial^2 V(z,y)}{\partial y^2} + \frac{\partial^2 V(z,y)}{\partial z^2} = 0$$ **(two-dimensional Laplace's equation)**

$$\left.\begin{aligned} V(z)\Big|_{y=0} &= \left(\frac{9|J|}{4(2|\eta|)^{1/2}\varepsilon_0}\right)^{2/3} z^{4/3} \\ \left.\frac{\partial V}{\partial y}\right|_{y=0} &= 0 \end{aligned}\right\}$$ **(at the beam edge)**

Solution

(subject to the above conditions)

$$V(z,y) = \mathrm{Re}\ f(z+jy) = \mathrm{Re}\left(\frac{9|J|}{4(2|\eta|)^{1/2}\varepsilon_0}\right)^{2/3} (z+j\,y)^{4/3} \quad (y \geq 0)$$

(potential outside the beam edge)

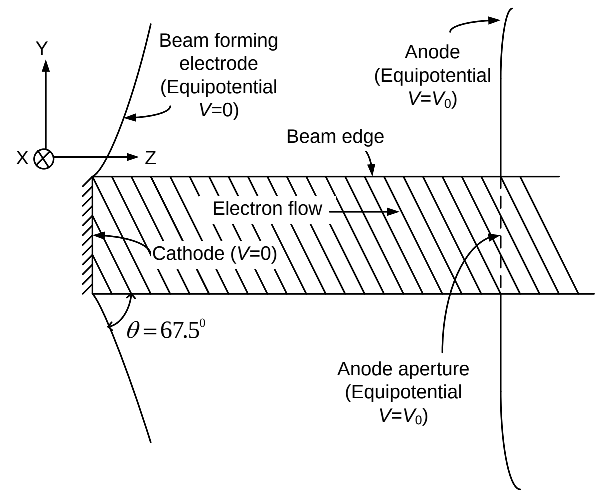
(recalling the relevant mathematical concept that the real part of a complex function $f\,(z+jy)$, known as the analytic function, satisfies the two-dimensional Laplace's equation)

$$V(z,y) = \mathrm{Re}\, f(z+jy) = \mathrm{Re}\left(\frac{9|J|}{4(2|\eta|)^{1/2}\varepsilon_0}\right)^{2/3} (z+j\,y)^{4/3} \quad (y \geq 0)$$

(potential outside the beam edge)

$$z + jy = (z^2 + y^2)^{1/2}(\cos\theta + j\sin\theta)$$

$$\theta = \tan^{-1}\frac{y}{x}$$



$$V(z,y) = \mathrm{Re}\left(\frac{9|J|}{4(2|\eta|)^{1/2}\varepsilon_0}\right)^{2/3} (z^2 + y^2)^{2/3}\left(\cos\frac{4\theta}{3} + j\sin\frac{4\theta}{3}\right) \quad (y \geq 0)$$

(potential outside the beam edge)

Shape of the beam focusing/forming electrode (BFE)

The BFE has to coincide with the equipotential which is put at the cathode potential $V\,(z,y) = 0$**.**

$$V(z,y) = \mathrm{Re}\left(\frac{9|J|}{4(2|\eta|)^{1/2}\varepsilon_0}\right)^{2/3}(z^2+y^2)^{2/3}\left(\cos\frac{4\theta}{3}+j\sin\frac{4\theta}{3}\right)\quad(y\geq 0)$$

(potential outside the beam edge)

$\longleftarrow$ **Shape of the beam focusing electrode (BFE), which is put at the cathode potential:** $V\,(z,y) = 0$

$$\downarrow$$

$$V(z,y) = \mathrm{Re}\left(\frac{9|J|}{4(2|\eta|)^{1/2}\varepsilon_0}\right)^{2/3}(z^2+y^2)^{2/3}\left(\cos\frac{4\theta}{3}+j\sin\frac{4\theta}{3}\right) = 0$$

$$\downarrow$$

$$\cos\frac{4\theta}{3} = 0 \longrightarrow \frac{4\theta}{3} = \frac{\pi}{2} \longrightarrow \theta = \tan^{-1}\frac{y}{x} = \frac{3\pi}{8} = 67.5^0$$ **(BFE angle)**

The BFE will take the 'planar-hat' shape making an angle of 67.5^0 **with the beam edge.**

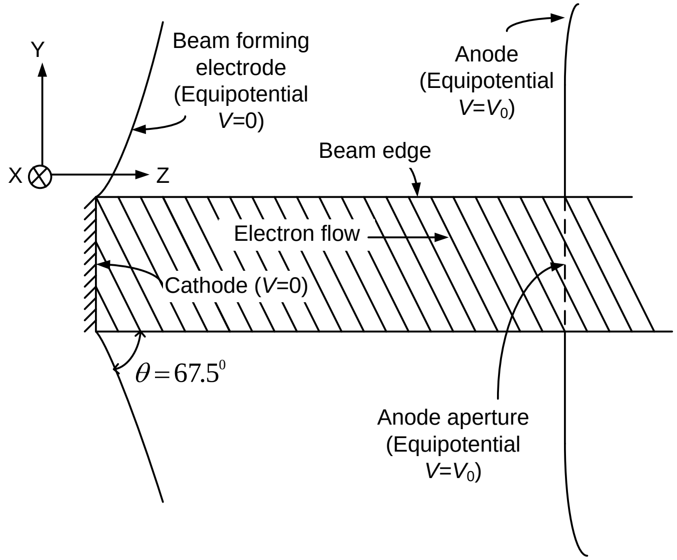
Shape of the anode

The anode voltage $V = V_0$ is applied with respect to the cathode at $V = 0$. The anode has a hole through which the electron beam is allowed to pass through. The anode has to coincide with the equipotential at $V = V_0$ outside the beam starting from the beam edge.

$$V(z,y) = \mathrm{Re}\left(\frac{9|J|}{4(2|\eta|)^{1/2}\varepsilon_0}\right)^{2/3}(z^2+y^2)^{2/3}\left(\cos\frac{4\theta}{3}+j\sin\frac{4\theta}{3}\right)\ (y \geq 0)$$

Potential values in the vicinity of the anode at and outside the beam edge

Generate the locus of points having the same value of potential as that of the anode $V = V_0$, that is, the equipotential corresponding to the anode potential, which becomes the desired shape of the anode.



Pierce gun derived from a curved cathode for a convergent beam

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}=-\frac{\rho}{\varepsilon_0}$$ **(one-dimensional Poisson's equation in spherical-polar coordinates)**

$\downarrow \leftarrow$ **Langmuir-Blogett's solution (deduced in Appendix)**

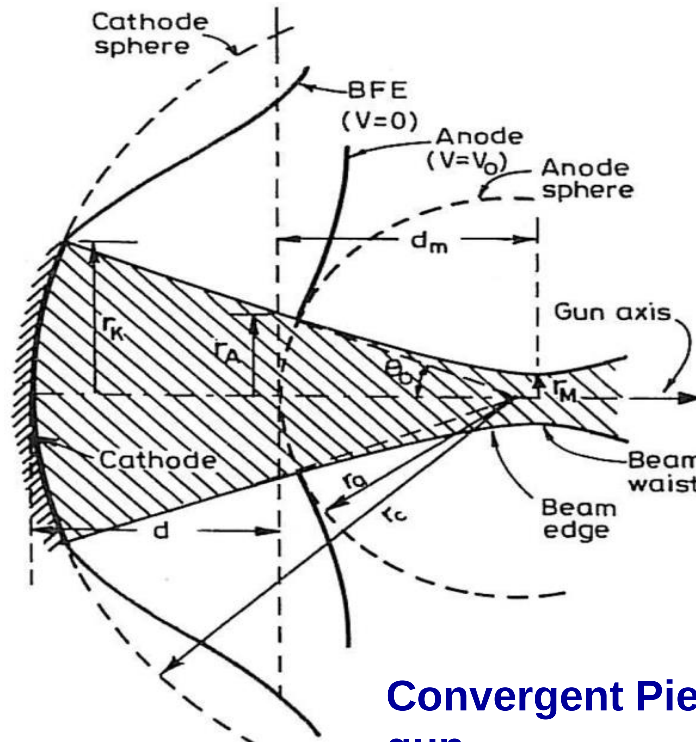
$$V=\left(\frac{9I_0}{16\pi(2|\eta|)^{1/2}\varepsilon_0}\right)^{2/3}G^{4/3}(u)$$

$$u=\ln\frac{r_c}{r}$$

$\downarrow$

$$V=\left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3}r_c^{4/3}G(u)^{4/3}$$

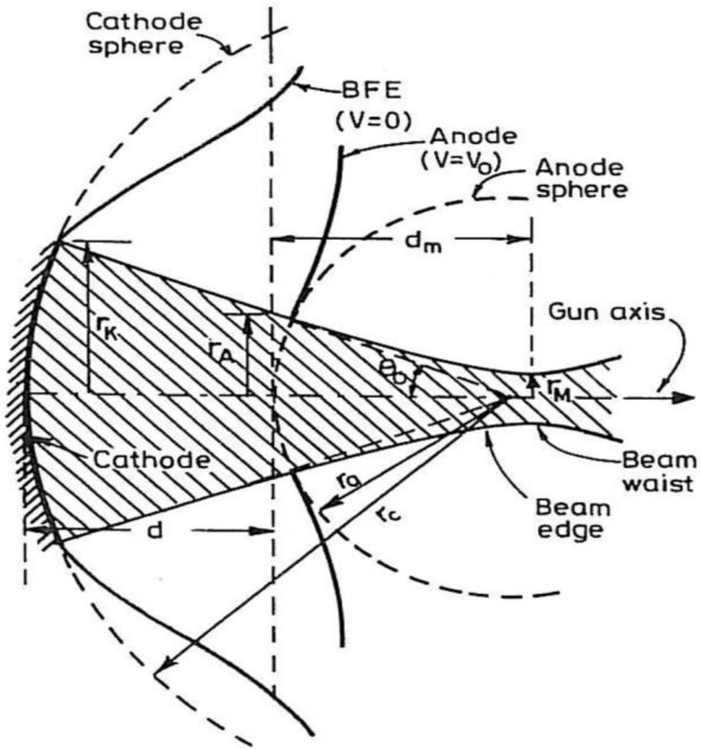
(potential as a function of radial coordinate)



Convergent Pierce gun

$$G(u)=G(\ln\frac{r_c}{r})=\ln\frac{r_c}{r}+\frac{3}{10}\ln^2\frac{r_c}{r}+\frac{3}{40}\ln^3\frac{r_c}{r}+\frac{63}{4400}\ln^4\frac{r_c}{r}-\ldots$$

$A_c=4\pi r_c^2$ **(area of the cathode sphere of radius r_c)**



Convergent Pierce gun

r_c **is the radius of curvature of the cathode.**

r_a **is the radius of curvature of the anode.**

$d = r_c - r_a$ **is the anode-to-cathode distance.**

r_K **is the cathode disc radius.**

r_A **is the anode aperture/hole radius.**

r_M **is the beam waist/throat radius.**

d_m **is the throw of the gun being the distance between the beam waist and the anode.**

θ_0 **is the half cone angle subtended by the spherical cup at the common centre of curvature of the cathode and the anode spheres.**

Relation between the beam voltage V_0 and beam current I_0 for a cathode sphere surrounding a concentric anode sphere

$$V = \left(\frac{9 I_0}{4(2|\eta|)^{1/2} \varepsilon_0 A_c} \right)^{2/3} r_c^{4/3} G(u)^{4/3}$$ **(recalled)**

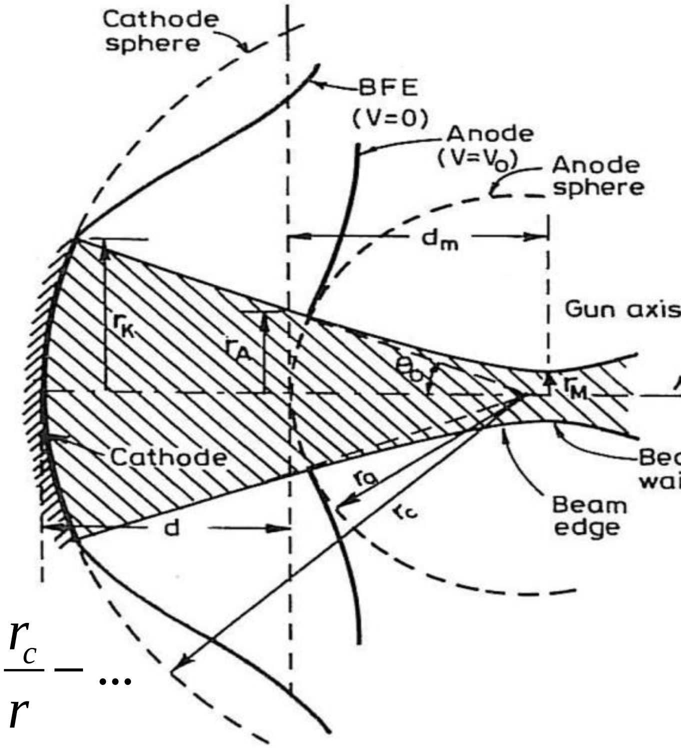
(potential as a function of radial coordinate)

$$G(u) = G(\ln\frac{r_c}{r}) = \ln\frac{r_c}{r} + \frac{3}{10}\ln^2\frac{r_c}{r} + \frac{3}{40}\ln^3\frac{r_c}{r} + \frac{63}{4400}\ln^4\frac{r_c}{r} - \ldots$$

$$V_0 = \left(\frac{9 I_0}{4(2|\eta|)^{1/2} \varepsilon_0 A_c} \right)^{2/3} r_c^{4/3} G_0(u)^{4/3}$$

$$G_0(u) = G_0(\ln\frac{r_c}{r_a}) = \ln\frac{r_c}{r_a} + \frac{3}{10}\ln^2\frac{r_c}{r_a} + \frac{3}{40}\ln^3\frac{r_c}{r_a} + \frac{63}{4400}\ln^4\frac{r_c}{r_a} - \ldots$$

(interpreting V as V_0 and r as r_a)



Convergent Pierce gun

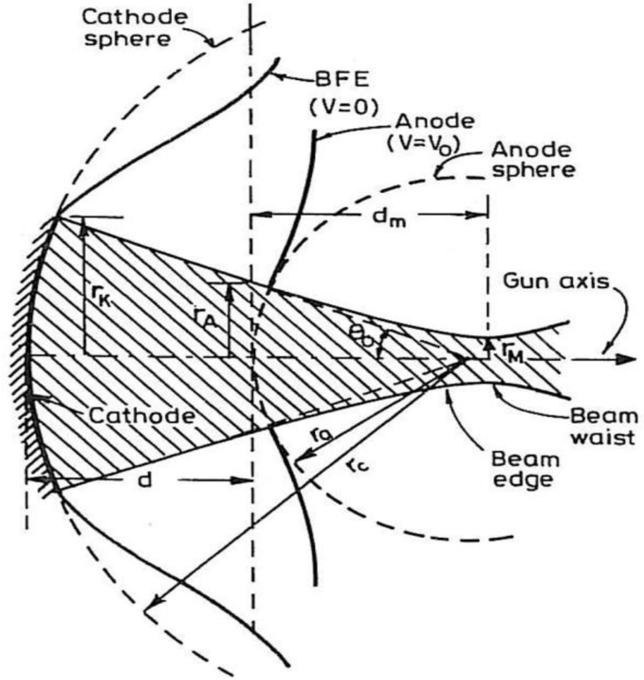
Relation between the beam voltage V_0 and beam current I_0 for a cathode sphere surrounding a concentric anode sphere

$$V_0 = \left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c} \right)^{2/3} r_c^{4/3} G_0(u)^{4/3}$$ **(recalled)**

$$G_0(u) = G_0(\ln\frac{r_c}{r_a}) = \ln\frac{r_c}{r_a} + \frac{3}{10}\ln^2\frac{r_c}{r_a} + \frac{3}{40}\ln^3\frac{r_c}{r_a} + \frac{63}{4400}\ln^4\frac{r_c}{r_a} - \ldots$$

Gives the beam current for the spherical diode consisting of a complete cathode sphere surrounding a complete anode sphere

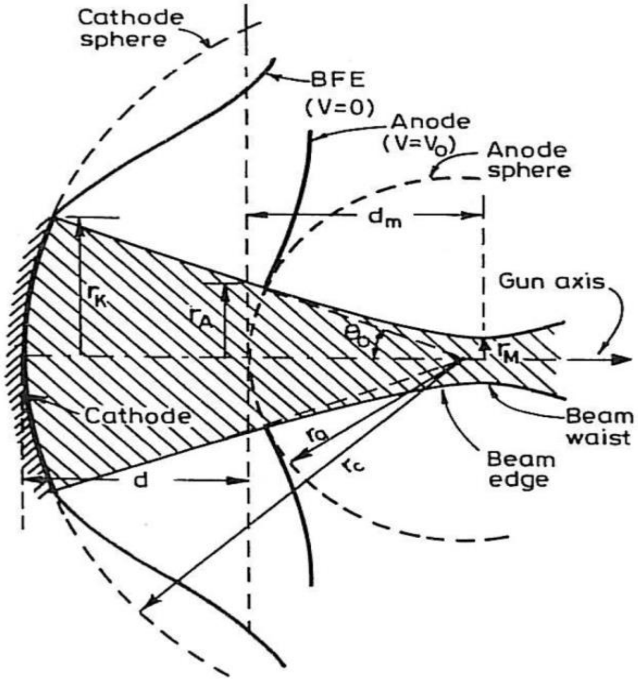
Beam current from spherical-cup cathode in a convergent Pierce gun?



Beam current I_0 for the spherical diode consisting of a complete cathode sphere surrounding a complete concentric anode sphere can be obtained from the relation:

$$V_0 = \left(\frac{9I_0}{4(2|\eta|)^{1/2} \varepsilon_0 A_c} \right)^{2/3} r_c^{4/3} G_0(u)^{4/3} \quad \textbf{(recalled)}$$

$$G_0(u) = G_0(\ln\frac{r_c}{r_a}) = \ln\frac{r_c}{r_a} + \frac{3}{10}\ln^2\frac{r_c}{r_a} + \frac{3}{40}\ln^3\frac{r_c}{r_a} + \frac{63}{4400}\ln^4\frac{r_c}{r_a} - \ldots$$



Beam current from a spherical-cup cathode will reduce from that obtainable from the above relation by a factor of the ratio of the area of the spherical-cup cathode to the area of the complete cathode sphere, that is, by the factor

$$\frac{\text{Spherical-cup cathode area}}{\text{Cathode-sphere area}} = \frac{2\pi r_c^2 (1-\cos\theta_0)}{4\pi r_c^2} = \frac{1-\cos\theta_0}{2}$$

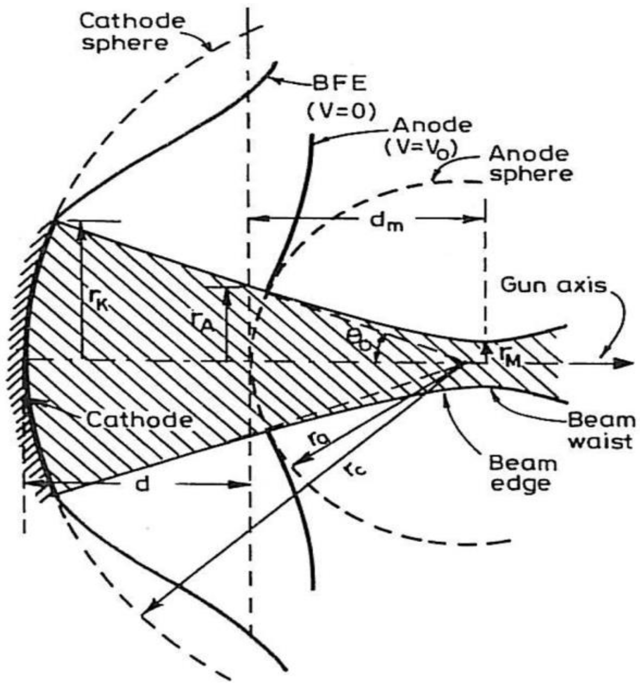
$$V_0 = \left(\frac{9 I_0}{4 (2|\eta|)^{1/2} \varepsilon_0 A_c} \right)^{2/3} r_c^{4/3} G_0(u)^{4/3}$$ **(for a spherical cathode surrounding a concentric spherical anode)**

$$\frac{\text{Spherical-cup cathode area}}{\text{Cathode-sphere area}} = \frac{1 - \cos\theta_0}{2}$$

$$\frac{I_0}{V_0^{3/2}} \frac{1}{2\pi\varepsilon_0 (2|\eta|)^{1/2}} = \left(\frac{4}{9}\right)\left(\frac{1-\cos\theta_0}{G_0^{\,2}(u)}\right)$$ **(for a spherical-cup cathode)**

$$k = \left(\frac{perv}{2\pi\varepsilon_0 (2|\eta|)^{1/2}} \right)^{1/2} \qquad perv = \frac{I_0}{V_0^{3/2}}$$

$$k^2 = \left(\frac{4}{9}\right)\left(\frac{1-\cos\theta_0}{G_0^{\,2}(u)}\right)$$ **(for a spherical-cup cathode)**



Beyond the anode aperture, the electron beam will spread in the field-free region which can be analyzed starting from the following expression for the force on an electron at the edge of the beam supposedly of circular cross section of radius r:

$$m\frac{d^2r}{dt^2} = eE_s$$ **(force equation for beam-edge electron)**

$$E_s = \frac{\eta\rho}{2\varepsilon_0}$$ **(space-charge electric field obtainable using Gauss's law)**

$$\frac{d^2r}{dt^2} = \eta E_s = \frac{\eta\rho}{2\varepsilon_0} r$$

$$dz/dt = v$$

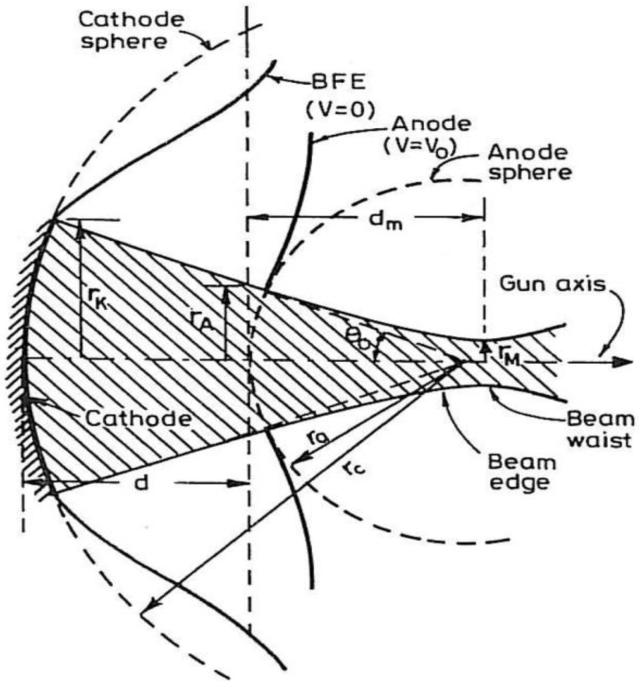
$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left[\left(\frac{dr}{dz}\frac{dz}{dt}\right)\right] = \frac{d}{dt}\left[\left(\frac{dr}{dz}v\right)\right]$$

$$= v\frac{d}{dt}\left(\frac{dr}{dz}\right) = v\frac{d}{dz}\left(\frac{dr}{dz}\right)\frac{dz}{dt} = v^2\frac{d}{dz}\left(\frac{dr}{dz}\right)$$

$$= v^2\frac{d^2r}{dz^2}$$

$$|J| = |\rho|v \longleftarrow J = \rho v$$

$$v^2\frac{d^2r}{dz^2} = \frac{\eta\rho}{2\varepsilon_0} r \longrightarrow \boxed{\frac{d^2r}{dz^2} = \frac{|\eta||J|}{2v^3\varepsilon_0} r}$$



$$\boxed{\frac{d^2 r}{dz^2} = \frac{|\eta||J|}{2v^3 \varepsilon_0} r}$$

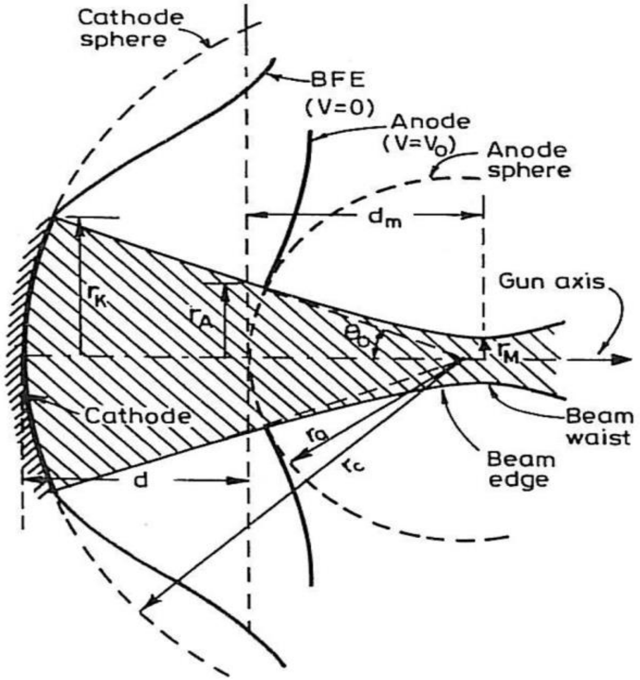
$$|J| = I_0 / (\pi r^2)$$

$$v = (2\eta V_0)^{1/2}$$

r_0 **is a reference value of** r**.**

$$\boxed{\begin{aligned} R &= r / r_0 \\ Z &= k(z / r_0) \end{aligned}}$$

$$k = \left(\frac{perv}{2\pi \varepsilon_0 (2|\eta|)^{1/2}} \right)^{1/2}$$



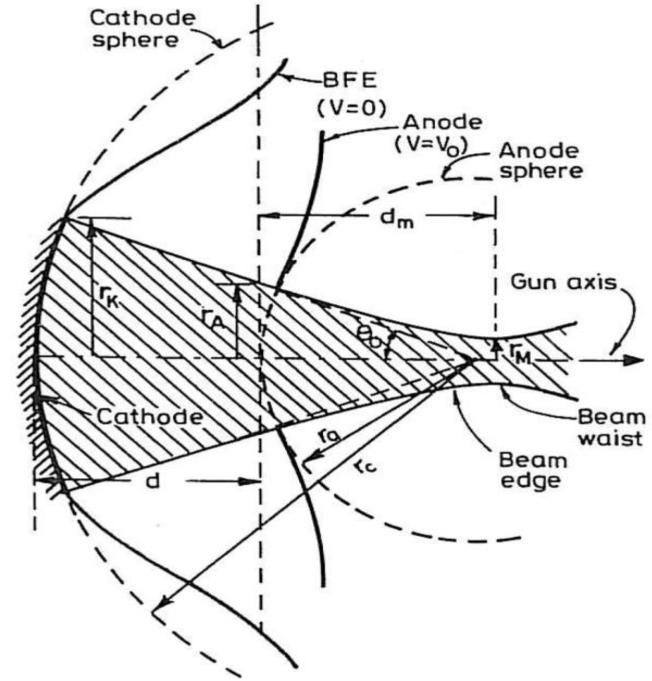
$$\frac{d^2 R}{dZ^2} = \frac{1}{2R}$$

Integration after multiplying by

$$\boxed{\frac{dR}{dZ} = \pm (\ln R)^{1/2}}$$

(setting the integration constant to zero by setting the slope $dr/dz = dR/dZ = 0$ **at the reference taken as** $r = r_0$**, that is,** $R = 1$**)**

Beyond the anode aperture, the electron beam, instead of converging to the common centre of curvature of the cathode and anode spheres, diverge out to take the position of the minimum beam radius $r = r_m$, where the beam waist (or throat) forms. Beyond the beam waist, the beam diverges off from the axis. $dr/dz = dR/dZ$ is negative between the anode aperture and the beam waist. $dr/dz = dR/dZ$ is positive beyond the beam waist.



$$\frac{dR}{dZ} = \pm(\ln R)^{1/2} \longleftarrow \quad R = r/r_0, \quad Z = k(z/r_0)$$

(taking the negative sign)

$$\frac{dR}{dZ} = -(\ln R)^{1/2}$$

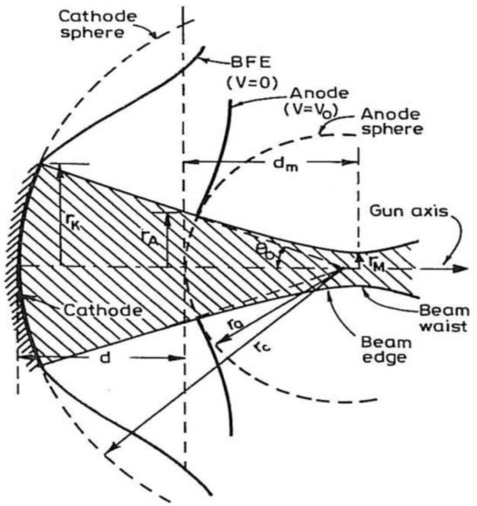
$$\int_0^{-k\frac{d_m}{r_M}} dZ = \int_1^{\frac{r_A}{r_M}} \frac{-dR}{(\ln R)^{1/2}}$$

$$\left.\begin{aligned} z &= -d_m; r = r_A; r_0 = r_M \\ R &= r/r_0 = r_A/r_M \\ Z &= k(z/r_0) = -k(d_m/r_M) \end{aligned}\right\} \text{(anode)}$$

(Upper limit of integration)

$$\left.\begin{aligned} z &= 0; r = r_M; r_0 = r_M \\ R &= r/r_0 = r_M/r_M = 1 \\ Z &= k(z/r_0) = 0 \end{aligned}\right\} \text{(beam waist)}$$

(Lower limit of integration)



$$\int_{0}^{-k\frac{d_m}{r_M}} dZ = \int_{1}^{\frac{r_A}{r_M}} \frac{-dR}{(\ln R)^{1/2}}$$

$$\left.\begin{aligned} z &= -d_m; r = r_A; r_0 = r_M \\ R &= r/r_0 = r_A/r_M \\ Z &= k(z/r_0) = -k(d_m/r_M) \end{aligned}\right\} \text{(anode)}$$

(Upper limit of integration)

$$\left.\begin{aligned} z &= 0; r = r_M; r_0 = r_M \\ R &= r/r_0 = r_M/r_M = 1 \\ Z &= k(z/r_0) = 0 \end{aligned}\right\} \text{(beam waist)}$$

(Lower limit of integration)

$$d_m = \left(\frac{2r_M}{k}\right)\left(\ln^{1/2}\frac{r_A}{r_M} + \frac{\ln^{3/2}\frac{r_A}{r_M}}{3(1!)} + \frac{\ln^{5/2}\frac{r_A}{r_M}}{5(2!)} + \frac{\ln^{7/2}\frac{r_A}{r_M}}{7(3!)} + \right)$$

(throw of the gun)

$$\left(\frac{dr}{dz}\right)_A = -\tan\theta_0$$

$$\left.\begin{aligned} R &= r/r_0 = r_A/r_M \\ Z &= k(z/r_0) = -k(d_m/r_M) \end{aligned}\right\} \text{(anode)}$$

(throw of the gun)

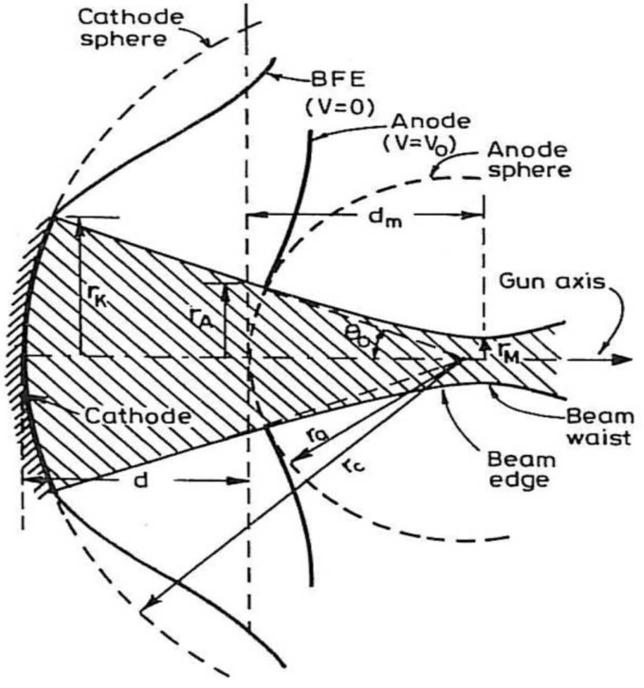
$$\frac{dR}{dZ} = -(\ln R)^{1/2}$$

$$\downarrow$$

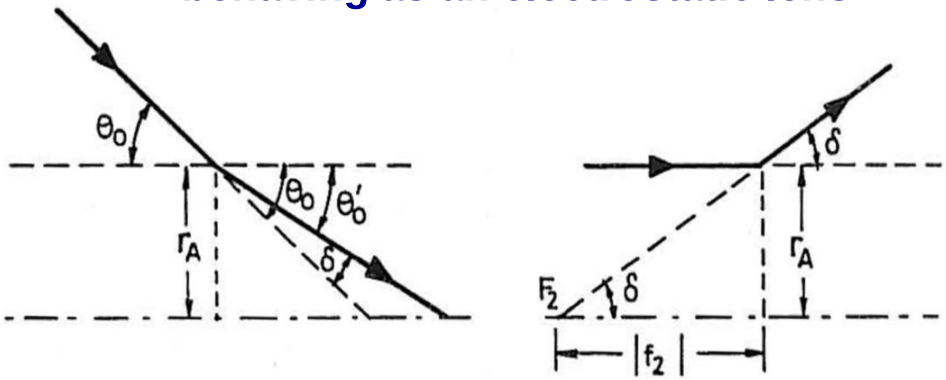
$$k(\log R_A)^{1/2} = k(\log(r_A/r_M)^{1/2} = \tan\theta_0$$

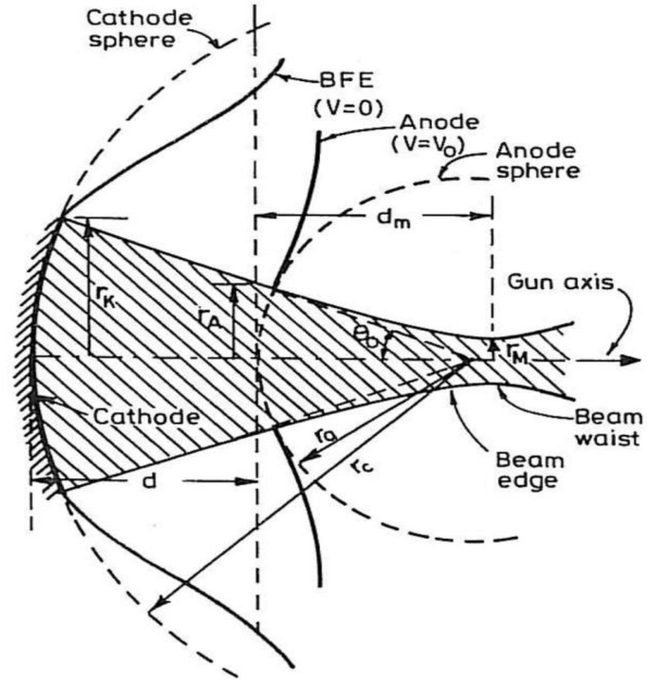
$$\downarrow$$

$$\frac{r_A}{r_M}(= R_A) = \exp\left(\frac{\tan^2\theta_0}{k^2}\right)$$



Modification due to the anode aperture behaving as an electrostatic lens





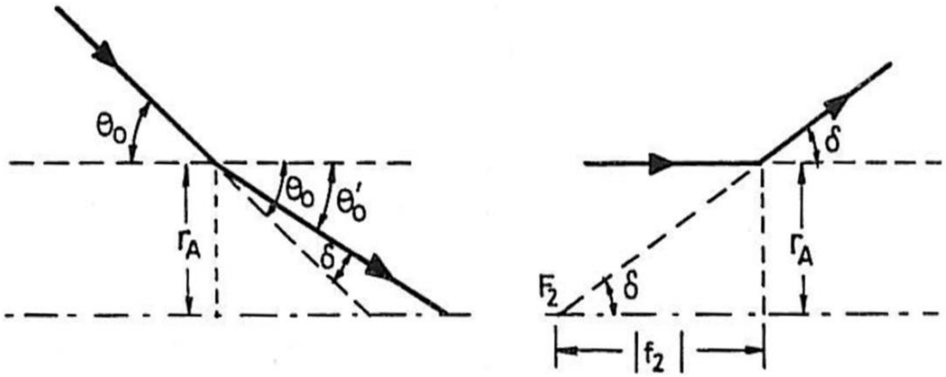
Due to the anode behaving as a lens, the electron path (ray) deviates by an angle δ and θ_0 is modified to θ_0'. F2 is the second focus of the lens. $f_2 = f$ is the second focal length.

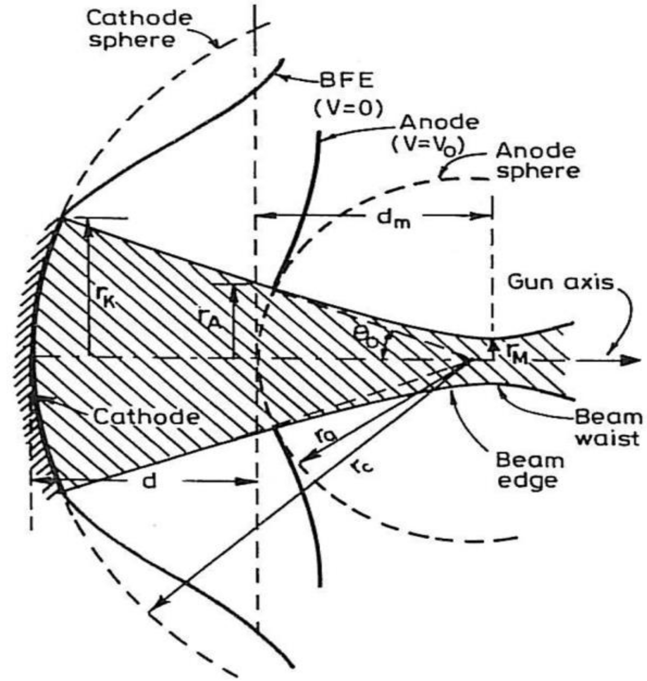
$$\delta = \frac{r_A}{|f_2|} = \frac{r_A}{|f|}$$

$$\frac{r_A}{r_M}(=R_A) = \exp\left(\frac{\tan^2\theta_0}{k^2}\right)$$ **(recalled)**

$$\downarrow \quad \longleftarrow \quad \theta_0' = \theta_0 - \delta = \theta_0 - \frac{r_A}{|f|}$$

$$\frac{r_A}{r_M}(=R_A) = \exp\left(\frac{\tan^2\theta_0'}{k^2}\right)$$





E_1 : **Electric field near the anode in its left-hand side**

E_2 : **Electric field near the anode in its right-hand side**

V_0 : **Anode voltage**

$$f_2 = f = \frac{4V_0}{E_1 - E_2}$$

$$\downarrow \leftarrow E_2 = 0$$

(since the beam is launched in the field-free region beyond the anode aperture)

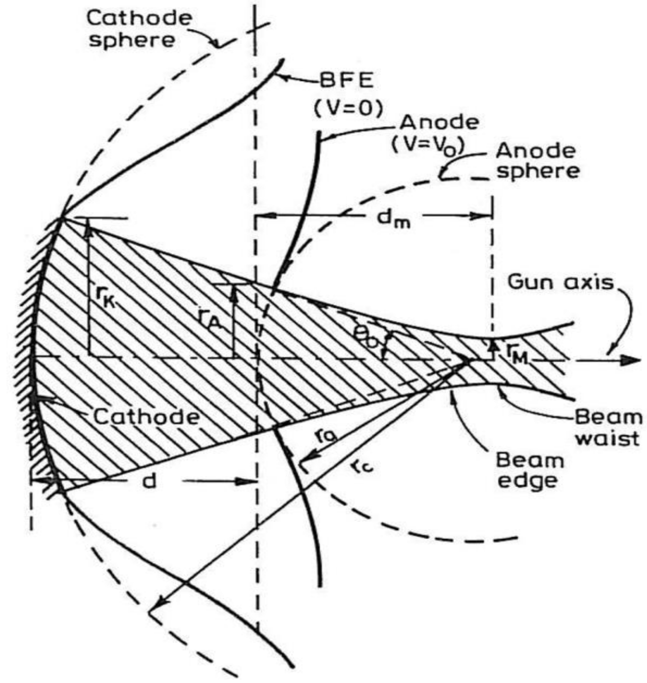
$$f_2 = f = \frac{4V_0}{E_1}$$

$$f_2 = f = \frac{4V_0}{E_1}$$

Along the gun axis, the value of z increases from left to right and that of r from right to left

$$E_1 = -\left(\frac{\partial V}{\partial z}\right)_0 = \left(\frac{\partial V}{\partial r}\right)_0$$

(subscript 0 referring to the quantity at the anode)



$$f_2 = f = \frac{4V_0}{E_1} = \frac{4V_0}{\left|\partial V / \partial r\right|_0}$$

$$V = \left(\frac{9I_0}{16\pi(2|\eta|^{1/2}\varepsilon_0}\right)^{2/3} G^{4/3}(u)$$ **(recalled)**

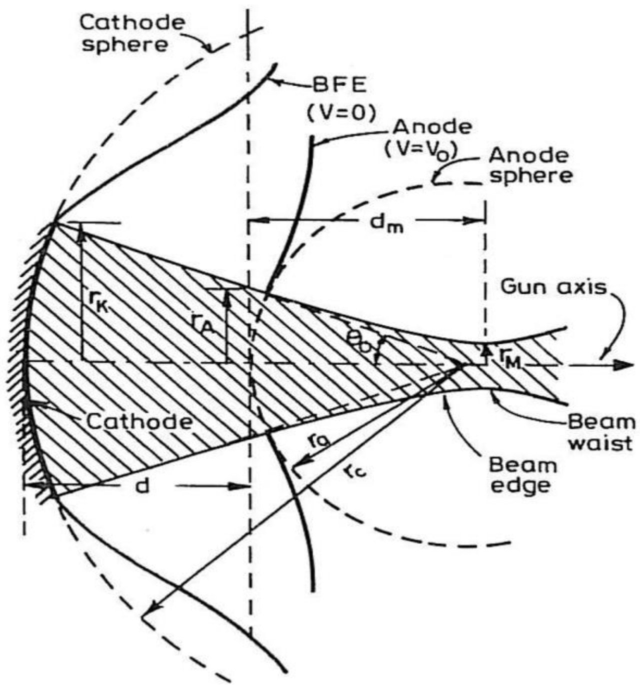
$$f_2 = f = \frac{-6G_0^2(u)}{\left(\frac{r_c}{r_a^2}\right)\left|\frac{dG^2(u)}{d(r_c/r)}\right|_0}$$

$$\theta_0 \cong \frac{r_A}{r_a} = \frac{r_K}{r_c}$$

$$\theta_0' = \theta_0 - \delta = \theta_0 - \frac{r_A}{|f|}$$

$$\frac{r_A}{r_M}(= R_A) = \exp\left(\frac{\tan^2 \theta_0'}{k^2}\right)$$

$$\frac{r_K}{r_M} = \frac{r_K}{r_A} \times \frac{r_A}{r_M} = \frac{r_K}{r_A}\exp\frac{\theta_0'^2}{k^2} = \frac{r_c}{r_a}\exp\frac{\theta_0'^2}{k^2} = \left(\frac{r_c}{r_a}\right)\exp\left[\left(\frac{\theta_0^2}{k^2}\right)\left\{1 - \left(\frac{1}{6}\right)\left(\frac{r_c}{r_a}\right)\left(\frac{1}{G_0^2(u)}\right)\left(\frac{dG^2(u)}{d(r_c/r)}\right)_0\right\}^2\right]$$



$$\frac{r_K}{r_M}=\frac{r_K}{r_A}\times\frac{r_A}{r_M}=\frac{r_K}{r_A}\exp\frac{\theta_0'^2}{k^2}=\frac{r_c}{r_a}\exp\frac{\theta_0'^2}{k^2}=\left(\frac{r_c}{r_a}\right)\exp\left[\left(\frac{\theta_0^2}{k^2}\right)\left\{1-\left(\frac{1}{6}\right)\left(\frac{r_c}{r_a}\right)\left(\frac{1}{G_0^2(u)}\right)\left(\frac{dG^2(u)}{d(r_c/r)}\right)_0\right\}^2\right]$$

$$k^2=\left(\frac{4}{9}\right)\left(\frac{1-\cos\theta_0}{G_0{}^2(u)}\right)$$

$$\frac{\theta_0^2}{k^2}=\left(\frac{9}{2}\right)G_0^2(u)$$

$$\cos\theta_0=1-\theta_0{}^2/2$$

$$\left(\frac{dG^2(u)}{d(r_c/r)}\right)_0=2G_0(u)\left(\frac{dG(u)}{d(r_c/r)}\right)_0$$

$$\frac{r_K}{r_M}=\left(\frac{r_c}{r_a}\right)\exp\frac{9}{2}G_0^2(u)\left[1-\frac{1}{6}\left(\frac{r_c}{r_a}\right)\left(\frac{1}{G_0^2(u)}\right)2G_0(u)\left(\frac{dG(u)}{d(r_c/r)}\right)_0\right]^2$$

$$\frac{r_K}{r_M}=\left(\frac{r_c}{r_a}\right)\exp\frac{9}{2}G_0^2(u)\left[1-\frac{1}{6}\left(\frac{r_c}{r_a}\right)\left(\frac{1}{G_0^2(u)}\right)2G_0(u)\left(\frac{dG(u)}{d(r_c/r)}\right)_0\right]^2$$

$$G(u)=G(\ln\frac{r_c}{r})=\ln\frac{r_c}{r}+\frac{3}{10}\ln^2\frac{r_c}{r}+\frac{3}{40}\ln^3\frac{r_c}{r}+\frac{63}{4400}\ln^4\frac{r_c}{r}-\ldots$$

and its derivative

$$\frac{r_K}{r_M}=\left(\frac{r_c}{r_a}\right)\exp\frac{1}{2}\left[-1+\left(\frac{12}{5}\right)\ln\frac{r_c}{r_a}+\left(\frac{27}{40}\right)\ln^2\frac{r_c}{r_a}+\left(\frac{369}{2200}\right)\ln^3\frac{r_c}{r_a}+\ldots\right]^2$$

(beam convergence)

Inverting the series in the above expression for beam convergence we can find the expression for r_c/r_a **.**

The above expression is useful in the synthesis of the convergent Pierce gun for its output parameters for a given set of input parameters.

Input parameters: **(i) either the operating cathode emission current density or beam convergence r_K/r_m, (ii) the beam voltage V_0, (iii) the beam current I_0, and (iv) the beam-waist radius r_M, which is also the desired beam radius to be retained beyond the beam waist after the beam has been formed by the gun (requiring a focusing structure for beam confinement beyond the beam waist).**

Output parameters: **(i) the cathode radius of curvature r_c, the anode radius of curvature r_a, the inter electrode spacing r_c-r_a, the cathode-disc radius r_K, the anode-aperture radius r_A, and the throw of the gun d_m.**

Relevant design expressions:

$$\frac{r_K}{r_M}=\left(\frac{r_c}{r_a}\right)\exp\frac{1}{2}\left[-1+\left(\frac{12}{5}\right)\ln\frac{r_c}{r_a}+\left(\frac{27}{40}\right)\ln^2\frac{r_c}{r_a}+\left(\frac{369}{2200}\right)\ln^3\frac{r_c}{r_a}+\ldots\right]^2$$

$$k^2=\left(\frac{4}{9}\right)\left(\frac{1-\cos\theta_0}{G_0{}^2(u)}\right) \longleftarrow k=\left(\frac{perv}{2\pi\varepsilon_0(2|\eta|)^{1/2}}\right)^{1/2}$$

$$G_0(u)=G_0(\ln\frac{r_c}{r_a})=\ln\frac{r_c}{r_a}+\frac{3}{10}\ln^2\frac{r_c}{r_a}+\frac{3}{40}\ln^3\frac{r_c}{r_a}+\frac{63}{4400}\ln^4\frac{r_c}{r_a}-\ldots$$

$$d_m=\left(\frac{2r_M}{k}\right)\left(\ln^{1/2}\frac{r_A}{r_M}+\frac{\ln^{3/2}\frac{r_A}{r_M}}{3(1!)}+\frac{\ln^{5/2}\frac{r_A}{r_M}}{5(2!)}+\frac{\ln^{7/2}\frac{r_A}{r_M}}{7(3!)}+\right)$$

$$\frac{r_K}{r_M}=\left(\frac{r_c}{r_a}\right)\exp\frac{1}{2}\left[-1+\left(\frac{12}{5}\right)\ln\frac{r_c}{r_a}+\left(\frac{27}{40}\right)\ln^2\frac{r_c}{r_a}+\left(\frac{369}{2200}\right)\ln^3\frac{r_c}{r_a}+\ldots\right]^2 \quad \text{(1)}$$

$$G_0(u)=G_0(\ln\frac{r_c}{r_a})=\ln\frac{r_c}{r_a}+\frac{3}{10}\ln^2\frac{r_c}{r_a}+\frac{3}{40}\ln^3\frac{r_c}{r_a}+\frac{63}{4400}\ln^4\frac{r_c}{r_a}-\ldots \quad \text{(2)}$$

$$k^2=\left(\frac{4}{9}\right)\left(\frac{1-\cos\theta_0}{G_0{}^2(u)}\right) \quad \text{(3)} \qquad k=\left(\frac{perv}{2\pi\varepsilon_0(2|\eta|)^{1/2}}\right)^{1/2} \quad \text{(4)}$$

$$d_m=\left(\frac{2r_M}{k}\right)\left(\ln^{1/2}\frac{r_A}{r_M}+\frac{\ln^{3/2}\frac{r_A}{r_M}}{3(1!)}+\frac{\ln^{5/2}\frac{r_A}{r_M}}{5(2!)}+\frac{\ln^{7/2}\frac{r_A}{r_M}}{7(3!)}+\right) \quad \text{(5)}$$

Step 1: The cathode disc radius r_K can be found from the operating cathode emission current density: $r_K=(I_0/\pi J_{K,\text{operating}})^{1/2}$

Step 2: Since r_K is known from Step 1, the beam convergence r_K/r_M becomes known with the help of (1), since r_M is one of the given input parameters.

Step 3: Since the beam convergence r_K/r_M is known from Step 2, one may find r_c/r_a by inverting the series in (1).

Step 1: The cathode disc radius can be found from the operating cathode emission current density: $r_K = (I_0 / \pi J_{K,\text{operating}})^{1/2}$.

Step 2: Since r_K is known from Step 1, the beam convergence r_K/r_M becomes known with the help of (1), since r_M is one of the given input parameters.

Step 3: Since the beam convergence r_K/r_M is known from Step 2, we can find r_c/r_a by inverting the series in (1).

Step 4: Since r_c/r_a is known from Step 3, we can find G_0 with the help of (2).

Step 5: Since G_0 is known from Step 4, we can find θ_0 with the help of (3) and (4) noting that $perv$ is known from the given input beam voltage V_0 and beam current I_0.

Step 6: We can take $\theta_0 \approx r_K/r_c$ whence we get $r_c \approx r_K/\theta_0$ and hence we can find r_c knowing θ_0 from Step 5 and r_K from Step 1.

Step 7: That r_c/r_a is known from Step 3 and r_c from Step 6, we can find r_a as $r_a = r_c / (r_c / r_a)$.

Step 8: θ_0 being known from Step 5 and r_a from Step 7, r_A can be found from the relation $\theta_0 \approx r_A/r_a$ as $r_A = \theta_0 r_a$.

Step 9: r_c and r_a being known from Steps 6 and 7, respectively, the interelectrode spacing d can be found as $d = r_c - r_a$.

Step 10: That r_A is known from Step 8 and r_M as well as $perv$, in terms of the beam voltage V_0 and beam current I_0, is known as the given input parameter, the throw of the gun d_m can be found from (5) in conjunction with (4).

Given Input parameters:

(i) **Either operating cathode emission current density or beam convergence** r_K/r_m

(ii) **Beam voltage** V_0

(iii) **Beam current** I_0 **and**

(iv) **Beam-waist radius** r_M**, which is also the desired beam radius to be retained beyond the beam waist after the beam has been formed by the gun (requiring a focusing structure for beam confinement beyond the beam waist)**

Synthesized Output parameters:

1. **Cathode radius of curvature** r_c **(Step 6) (via Step 5 and Step 1)**
2. **Anode radius of curvature** r_a **(Step 7) (via Step 3 and Step 6)**
3. **Inter electrode spacing** r_c-r_a **(Step 9) (via Step 6 and Step 7)**
4. **Cathode-disc radius** r_K **(Step 1)**
5. **Anode-aperture radius** r_A **(Step 8) (via Step 5 and Step 7)**
6. **Throw of the gun** d_m **(Step 10) (via Step 8)**

Conformal mapping of convergent-flow Pierce gun electrode shapes

Conformal transformation

The method of conformal transformation is used to find the solution of a physical problem for a given configuration from the corresponding solution that is known already for a simpler configuration.

In this method, a suitable complex analytic function is used to transform the cross-sectional geometry of the problem from one plane to another. The relative angles between the lines are preserved and the corresponding incremental areas enjoy the similarity in shape (though they differ in scale) by this kind of transformation from one plane to another. Hence the transformation is called *conformal*.

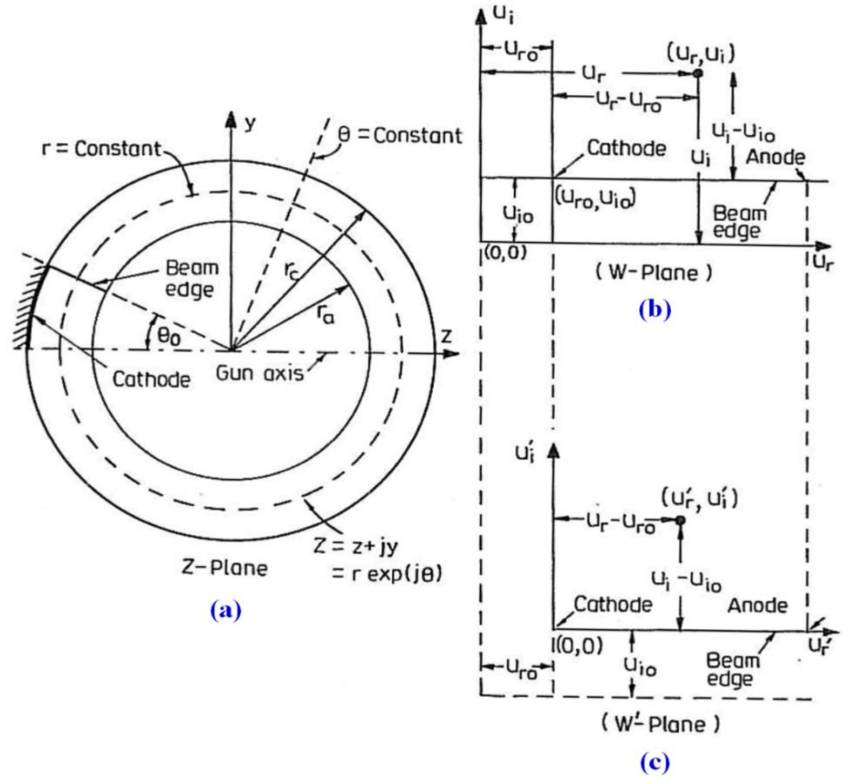
The solution for potential function at a point outside a strip (rectangular) beam was from a flat cathode was already obtained in a simpler rectangular system. Our objective is to exploit the solution for this configuration and use the conformal transformation technique to find the corresponding solution at a point outside a convergent conical beam derived from a spherical-cup cathode for a convergent-flow Pierce gun. We have used typically the logarithmic transformation for this purpose.

Subsequently, the solution for potential function could be interpreted to obtain equipotentials outside the beam at or near its edge and hence the electrode shapes of the convergent-flow Pierce gun.

Conformal mapping of electrode shapes of convergent-flow Pierce gun

Let us take typically <u>logarithmic</u> conformal transformation.

In this approach, the configuration of the convergent conical beam problem is 'conformally' mapped to the configuration of the strip beam of the parallel-flow Pierce gun derived from a flat cathode.



We take a suitable analytic function, here, typically, the logarithmic function:

$$\left.\begin{aligned} W &= f(Z) = \ln Z = u_r + ju_i \\ Z &= z + jy = r\exp(j\theta) \end{aligned}\right\}$$

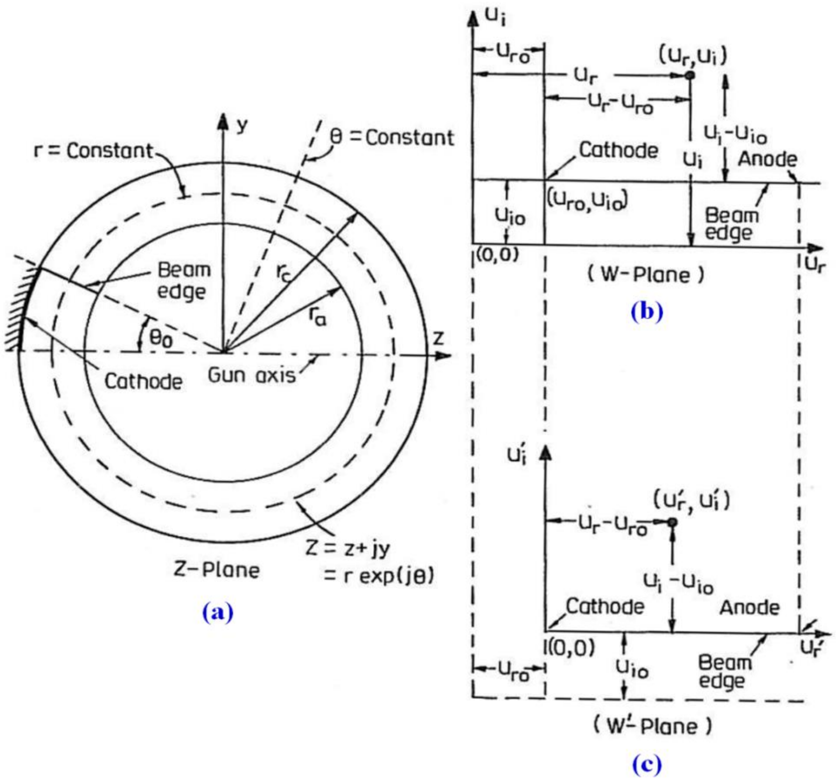
for the transformation of the cross-sectional geometry of the problem from one plane, say, Z-plane (Fig. a) to another, say, W-plane (Fig. b).

Logarithmic function recalled:

$$\left.\begin{aligned} W &= f(Z) = \ln Z = u_r + j u_i \\ Z &= z + jy = r\exp(j\theta) \end{aligned}\right\}$$

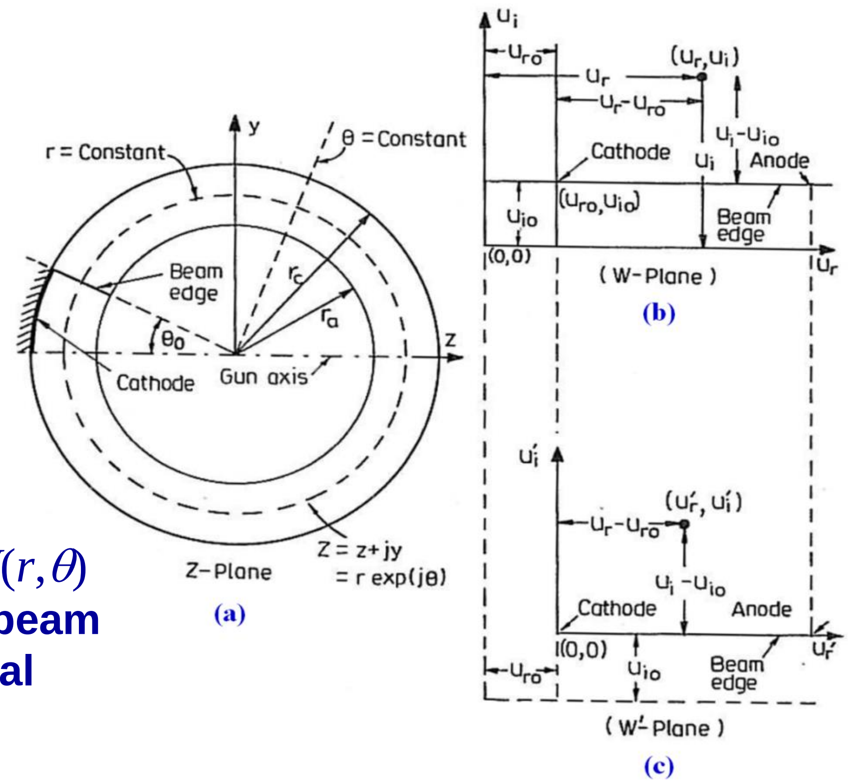
The above function transforms the cross-sectional geometry of the problem from one plane, say, Z-plane (Fig. a) to another, say, W-plane (Fig. b) such that, corresponding to a point (z,y) on the Z-plane with z and y as the real and the imaginary axes, respectively, we have a point (u_r,u_i) on the W-plane with u_r and u_i as the real and the imaginary axes, respectively).

First step: Transformation from Z-plane (Fig. a) to W-plane (Fig. b)



The beam edge of the conical beam after transformation from Z- to W-plane coincides with the real axis of the W-plane (Fig. b), the solution for the potential function above the W-plane (Fig. b) being presumably known.

$$\left.\begin{aligned} W &= f(Z) = \ln Z = u_r + ju_i \\ Z &= z + jy = r\exp(j\theta) \end{aligned}\right\}$$ **(recalled)**



The beam edge of the conical beam after transformation from *Z*- to *W*-plane coincides with the real axis of the *W*-plane (Fig. b), the solution for the potential function above the *W*-plane (Fig. b) being presumably known.

We are looking here for the potential function $V(r,\theta)$ for the conical beam configuration outside the beam $\theta > \theta_0$ which in the limit passes on to the potential function at the beam edge $\theta = \theta_0$:

$$V(r,\theta_0) = \left(\frac{9I_0}{8\pi\varepsilon_0(2|\eta|)^{1/2}(1-\cos\theta_0)}\right)^{2/3} G^{4/3}(u) \longleftarrow V = \left(\frac{9I_0}{16\pi(2|\eta|^{1/2}\varepsilon_0}\right)^{2/3} G^{4/3}(u)$$

(recalled)

$$G(u) = G\left(\ln\frac{r_c}{r}\right) = \ln\frac{r_c}{r} + \frac{3}{10}\ln^2\frac{r_c}{r} + \frac{3}{40}\ln^3\frac{r_c}{r} + \frac{63}{4400}\ln^4\frac{r_c}{r} - \ldots$$

$$\ln Z = \ln[r \exp(j\theta)] = \ln r + \ln(\exp(j\theta)) = \ln r + j\theta$$

$$\left.\begin{aligned} W &= f(Z) = \ln Z = u_r + ju_i \\ Z &= z + jy = r\exp(j\theta) \end{aligned}\right\}$$ **(recalled)**

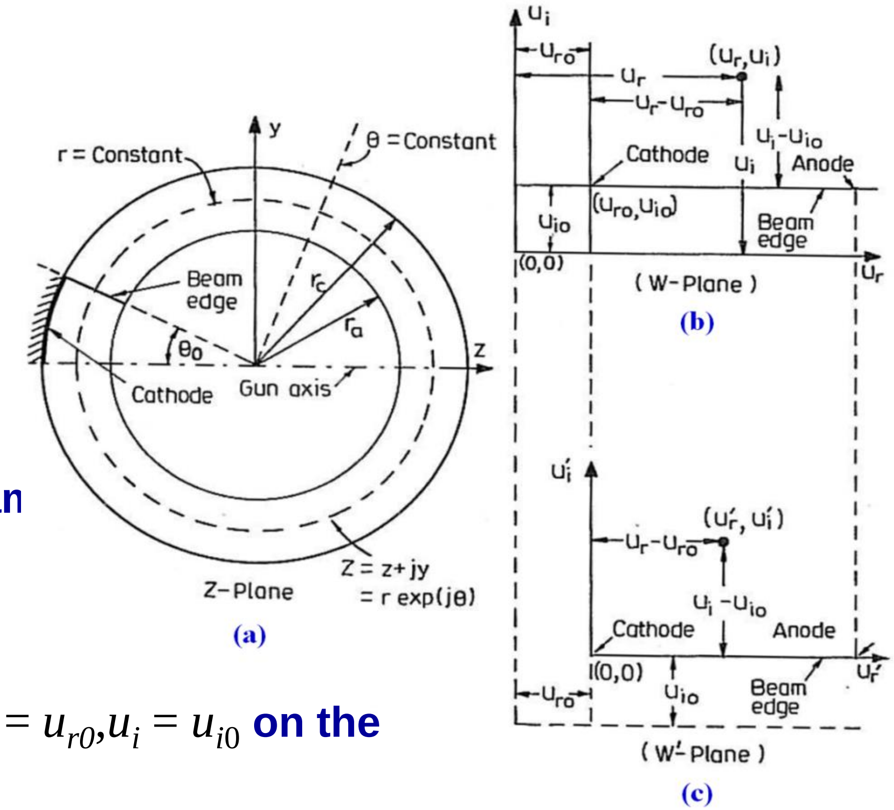
$$\ln Z = u_r + ju_i = \ln r + j\theta$$

$$\left.\begin{aligned} u_r &= \ln r \\ u_i &= \theta \end{aligned}\right\}$$

at the cathode ($r = r_c$) **on the beam edge** ($\theta = \theta_0$) **(Fig. b)**

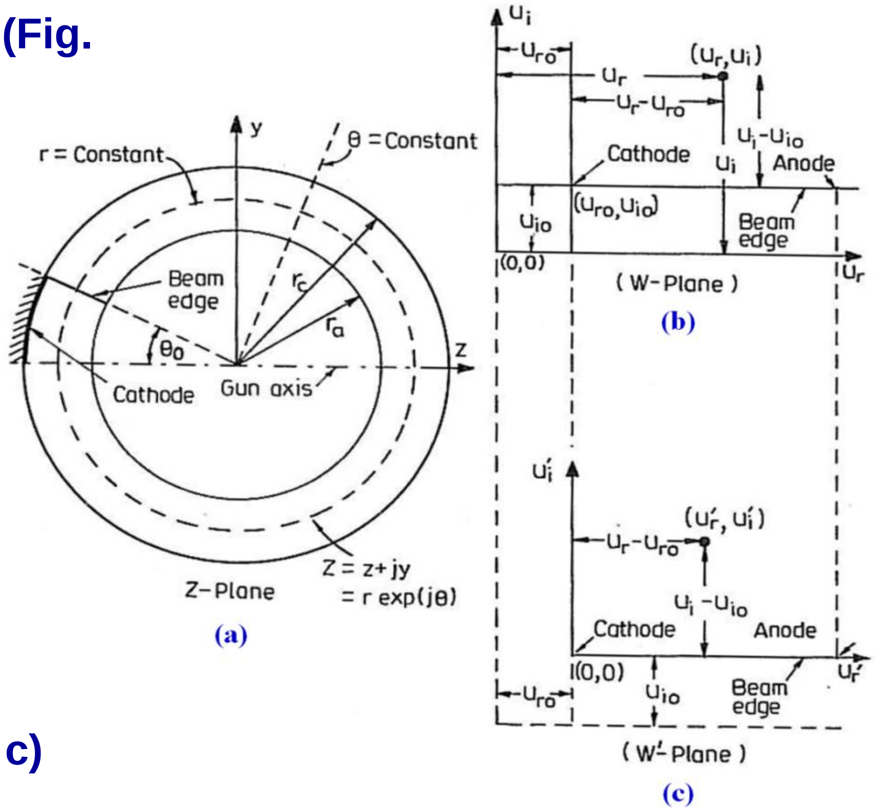
$$\left.\begin{aligned} u_{r0} &= \ln r_c \\ u_{i0} &= \theta_0 \end{aligned}\right\}$$

Reference coordinates $u_r = u_{r0}, u_i = u_{i0}$ **on the** W**-plane (Fig. b)**



Second step: Transformation from W-plane (Fig. b) to W'-plane (Fig. c)

Let us transform the cathode on the beam edge from the plane $W(=u_r + ju_i)$ to the plane $W'(=u'_r + ju'_i)$ such that the origin $(u'_r=0, u'_i=0)$ of the plane W' is located at a point corresponding to the point $(u_r=u_{r0}, u_i=u_{i0})$ of the plane W, and the cathode coincides with the imaginary axis $u'_r=0$ and the beam edge with the real axis $u'_i=0$ of the plane W' (Fig. c).



$$\left.\begin{aligned} u'_r &= u_r - u_0 \\ u'_i &= u_i - u_{i0} \end{aligned}\right\} \textbf{(Fig. c)}$$

$$W' = u'_r + ju'_i = (u_r - u_{r0}) + j(u_i - u_{i0})$$

$$\left.\begin{aligned} u_r &= \ln r \\ u_i &= \theta \end{aligned}\right\} \qquad \left.\begin{aligned} u_{r0} &= \ln r_c \\ u_{i0} &= \theta_0 \end{aligned}\right\}$$

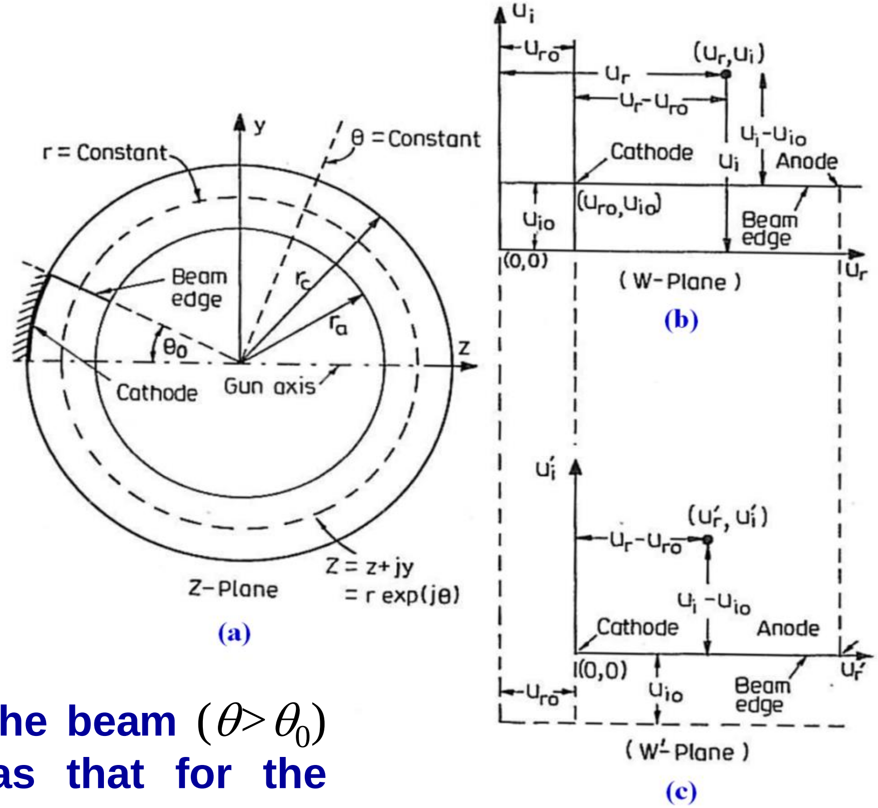
$$W' = (\ln r - \ln r_c) + j(\theta - \theta_0) = \ln(r/r_c) + j(\theta - \theta_0)$$

Second stage of transformation

Beam edge made to coincide with the real axis of the plane W'.

Cathode made to coincide with the imaginary axis of the plane W'.

Configuration of a conical beam changed to that of a rectangular strip beam



Potential function $V(r,\theta)$ at a point outside the beam $(\theta > \theta_0)$ can be written using the same method as that for the formation of rectangular strip beam derived from a flat cathode, as described earlier.

$$V(r,\theta) = \mathrm{Re}\, f(W')$$

$$W' = \ln(r/r_c) + j(\theta - \theta_0) \quad \textbf{(recalled)}$$

$$V(r,\theta) = \mathrm{Re}W' = \mathrm{Re}\, f(\ln(r/r_c) + j(\theta - \theta_0))$$

Potential function $V(r,\theta)$ at a point outside the beam $(\theta > \theta_0)$:

$$V(r,\theta) = \mathrm{Re}\, W' = \mathrm{Re}\, f(\ln(r/r_c) + j(\theta - \theta_0))$$ **(rewritten)**

Potential function to pass on to the potential at the beam edge $(\theta = \theta_0)$:

$$V(r,\theta_0) = \left(\frac{9I_0}{8\pi\varepsilon_0(2|\eta|)^{1/2}(1-\cos\theta_0)}\right)^{2/3} G^{4/3}(u)$$ **(recalled)**

where

$$G(u) = G(\ln\frac{r_c}{r}) = \ln\frac{r_c}{r} + \frac{3}{10}\ln^2\frac{r_c}{r} + \frac{3}{40}\ln^3\frac{r_c}{r} + \frac{63}{4400}\ln^4\frac{r_c}{r} - \dots$$

$$\begin{aligned}
V(r,\theta) &= \mathrm{Re}\, f(\ln r/r_c + j(\theta-\theta_0)) \\
&= \mathrm{Re}\left(\frac{9I_0}{8\pi\varepsilon_0(2|\eta|)^{1/2}(1-\cos\theta_0)}\right)\left[\sum_{n=1,2,3,\dots} a_n(\ln r/r_c + j(\theta-\theta_0))^n\right]^{4/3} \\
&= \mathrm{Re}\left(\frac{9I_0}{8\pi\varepsilon_0(2|\eta|)^{1/2}(1-\cos\theta_0)}\right)\left[\sum_{n=1,2,3,\dots} b_n(\ln r_c/r - j(\theta-\theta_0))^n\right]^{4/3}
\end{aligned}$$

$$V(r,\theta) = \mathrm{Re}\, f(\ln r/r_c + j(\theta-\theta_0))$$

$$= \mathrm{Re}(\frac{9I_0}{8\pi\varepsilon_0(2|\eta|)^{1/2}(1-\cos\theta_0)})[\sum_{n=1,2,3,\dots} a_n(\ln r/r_c + j(\theta-\theta_0))^n]^{4/3}$$

$$= \mathrm{Re}(\frac{9I_0}{8\pi\varepsilon_0(2|\eta|)^{1/2}(1-\cos\theta_0)})[\sum_{n=1,2,3,\dots} b_n(\ln r_c/r - j(\theta-\theta_0))^n]^{4/3}$$

(rewritten)

where

$$a_1 = -1, a_2 = 3/10, a_3 = -3/40, a_4 = 63/4400, \dots\dots$$

$$b_n = (-1)^n a_n \ (n = 1,2,3,4,\dots.)$$

The above potential function satisfies at the beam edge $(\theta = \theta_0)$**:** $(\partial V/\partial\theta)_{\theta=\theta_0} = 0$ **.**

This, in turn, ensures that the azimuthal component of electric field at the beam edge is nil: $-1/r)(\partial V/\partial\theta)_{\theta=\theta_0} = 0$ **. This further ensures the absence of the electrostatic force to cause the deviation of the beam-edge electrons in the azimuthal direction.**

$$V(r,\theta) = \mathrm{Re}\, f(\ln r/r_c + j(\theta-\theta_0))$$

$$= \mathrm{Re}\left(\frac{9I_0}{8\pi\varepsilon_0(2|\eta|)^{1/2}(1-\cos\theta_0)}\right)\left[\sum_{n=1,2,3,\ldots} a_n(\ln r/r_c + j(\theta-\theta_0))^n\right]^{4/3}$$

$$= \mathrm{Re}\left(\frac{9I_0}{8\pi\varepsilon_0(2|\eta|)^{1/2}(1-\cos\theta_0)}\right)\left[\sum_{n=1,2,3,\ldots} b_n(\ln r_c/r - j(\theta-\theta_0))^n\right]^{4/3}$$

(rewritten)

where

$$a_1 = -1, a_2 = 3/10, a_3 = -3/40, a_4 = 63/4400, \ldots..$$

$$b_n = (-1)^n a_n \; (n = 1,2,3,4,\ldots.)$$

The above potential function can be used to find the equipotentials corresponding to the BFE and anode potentials and hence the shapes of the BFE and anode, respectively, for a convergent-flow Pierce gun derived from a curved cathode (as has been done earlier in the case of a parallel-flow Pierce gun derived from a flat cathode).

See Appendix for the deductions of Child-Langmuir's and Langmuir and Blodgett's relations