Some Basic Enabling Concepts in Microwave Tubes

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Some Basic Enabling Concepts

◊ **Electron bunching: relativistic and non-relativistic**

◊ **Conservation of kinetic energy in M-type tubes**

◊ **Induced current on electrodes due to electron beam flow**

◊ **Plasma frequency**

◊ **Space-charge waves**

◊ **Cyclotron waves**

◊ **Devices based on cyclotron resonance maser and Weibel instabilities**

◊ **Space-charge limiting current**

Electron Bunching: Relativistic and Non-Relativistic

In a microwave tube the electrons in a beam of electrons are bunched in an interaction region of the device where

◊ their kinetic energy, for instance, in a TWT, or

◊ their potential energy, for instance, in a magnetron,

is transferred to RF waves in an interaction region of the device.

The bunching mechanism may be

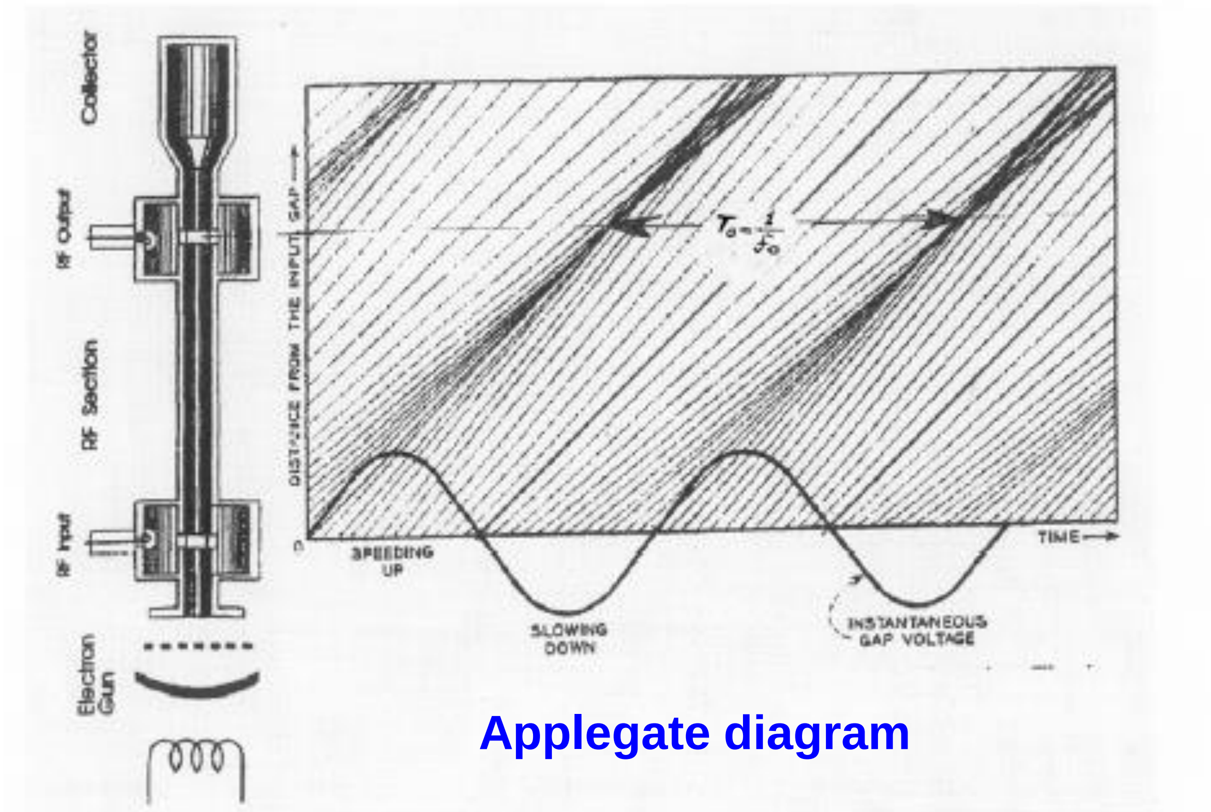
◊ non-relativistic, for instance, in a TWT or

◊ relativistic, for instance, in a gyrotron.

Non-relativistic axial bunching in a klystron

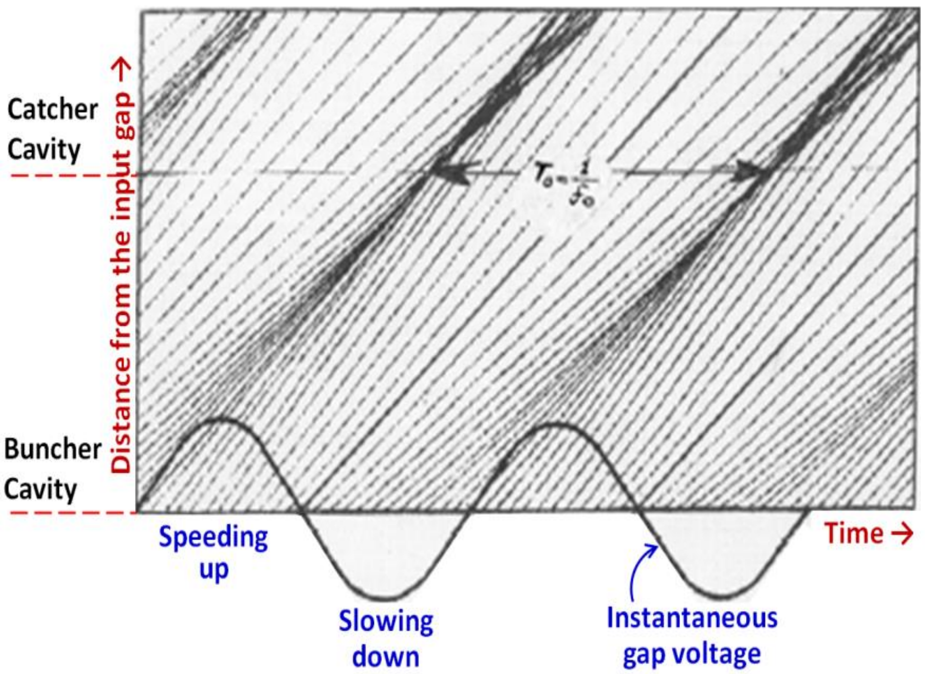
(as explained by Russell Varian and Siguard Varian)

“Just picture a steady stream of cars from San Francisco to Palo Alto; if the cars left San Francisco at equal increments and at the same velocity, then even in Palo Alto they would be evenly spaced and you would call this as direct flow of cars. But suppose somehow the speed of some cars, as they left San Francisco, was increased a bit and others retarded. Then, with time, the fast cars would tend to catch up with the slow ones and they would bunch into groups. Thus, if the velocity of cars was sufficiently different or the time long enough, the steady stream of cars would be broken and, under ideal conditions, would arrive in Palo Alto in clearly defined groups. In the same way an electron tube can be built in which the control of the e-beam is produced by the principle of bunching, rather than the direct control of a grid in a triode…….”



Applegate diagram

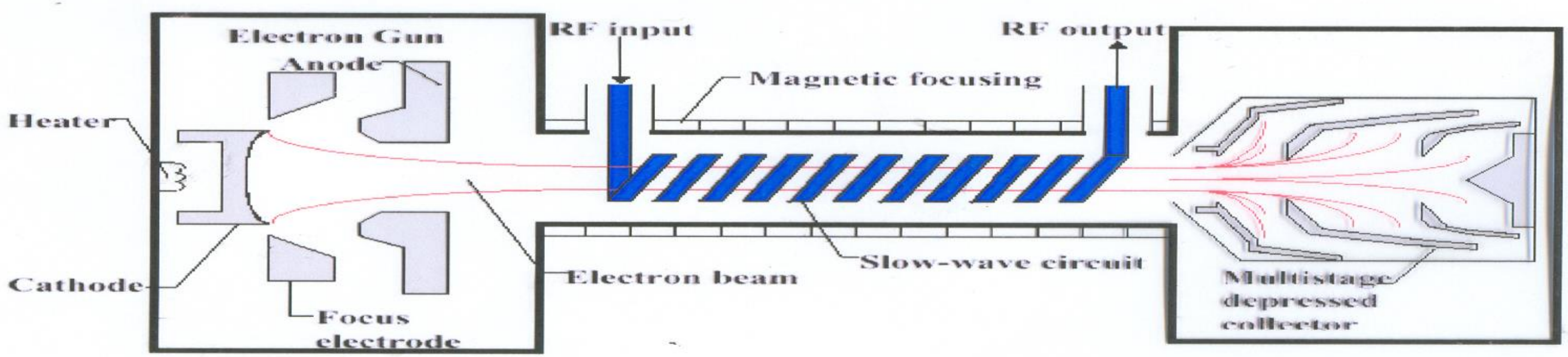
Bunches form at the second/output (catcher) cavity around the electrons that had crossed the first/input (buncher) cavity when the sinusoidal voltage there (input voltage) crossed from negative (decelerating) to positive (accelerating). Bunches arrive at the interval of T_0 (the time period of the input voltage)

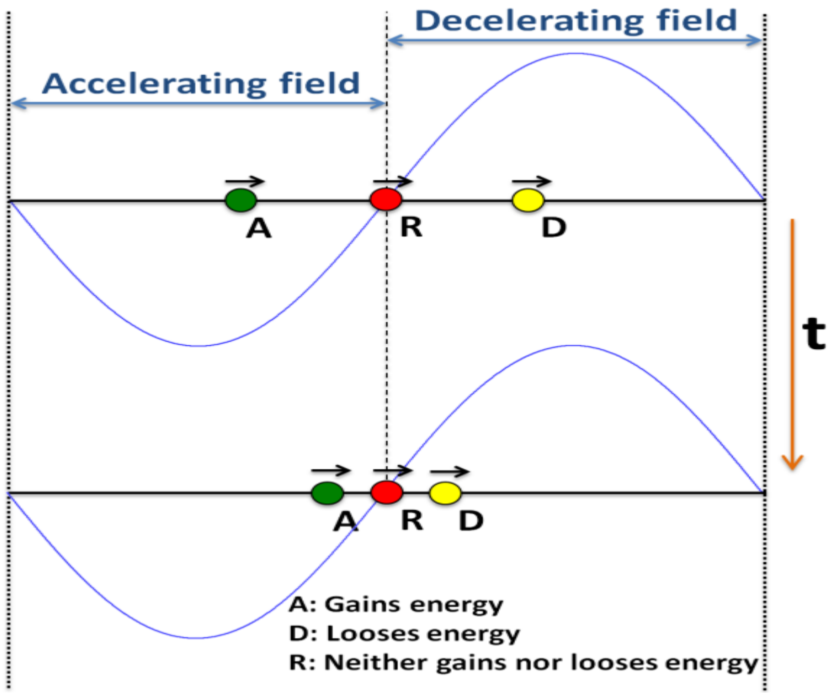


In the Applegate diagram for a two-cavity klystron, a bunch of straight lines of slopes proportional to electron velocities explains the arrival of electron bunches at the location of the crossing of these lines, at the catcher cavity of the klystron that consists of the input buncher and the output catcher cavities in its simplest two-cavity configuration.

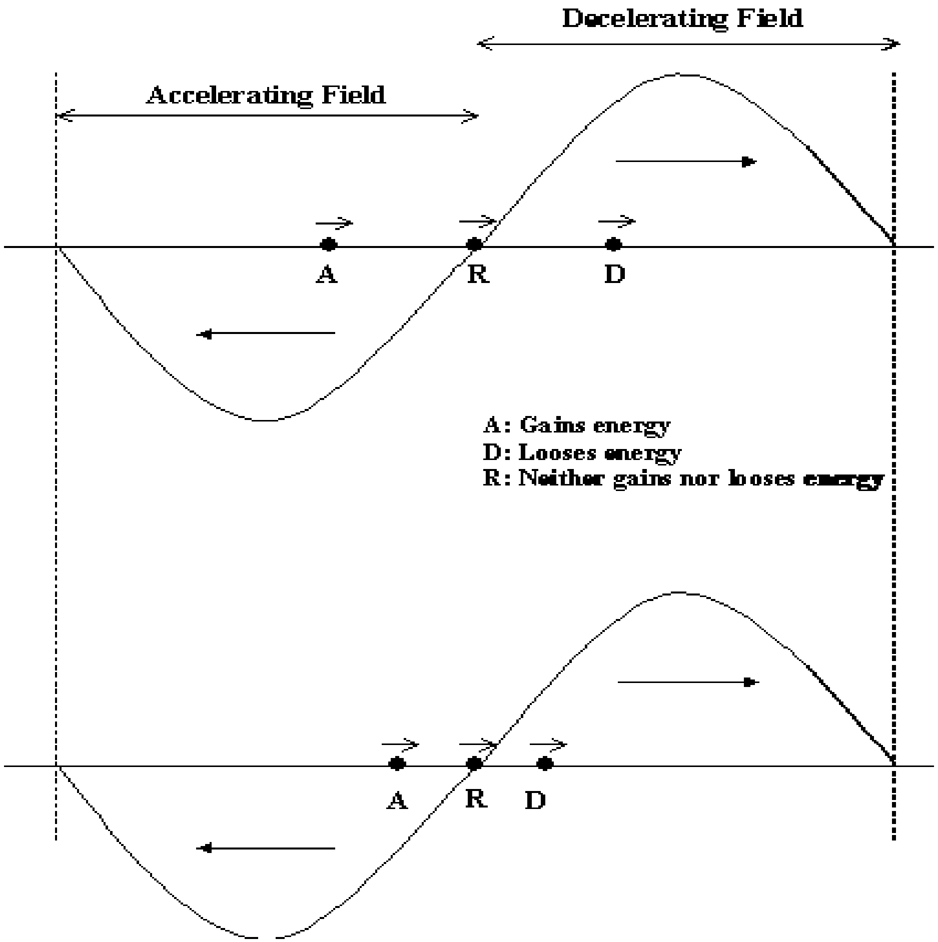
This arrival of electron bunches at the output catcher cavity takes place at the interval of the time period T_0 (=1/f_0), say, of the sinusoidal input voltage of frequency f_0, say, of the buncher cavity around the electrons that had crossed this cavity when the input voltage there crossed from its negative (decelerating) to positive (accelerating) value.

Non-relativistic axial bunching in a travelling-wave tube





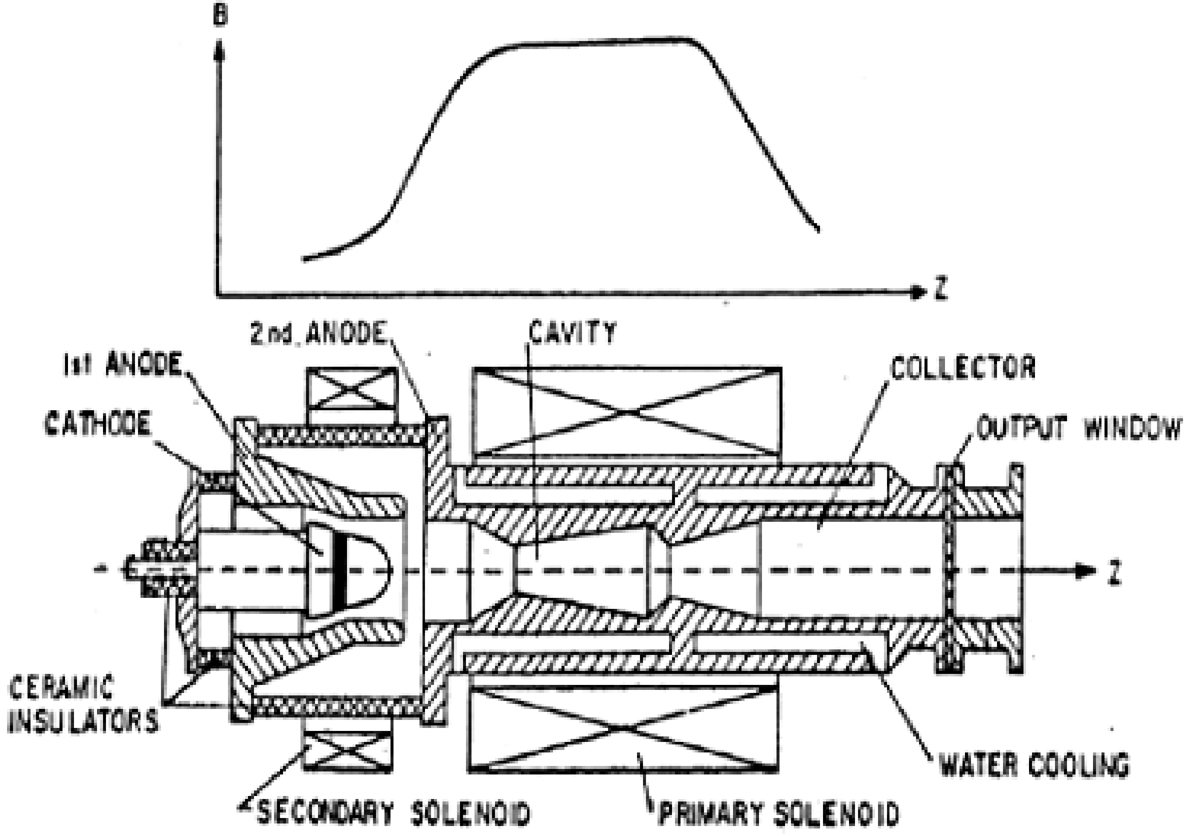
Bunching of typically two electrons 'A' and 'D' subjected to the accelerating and decelerating RF electric fields, respectively, in the interaction region of a TWT around a reference electron 'R' that experiences no such fields.



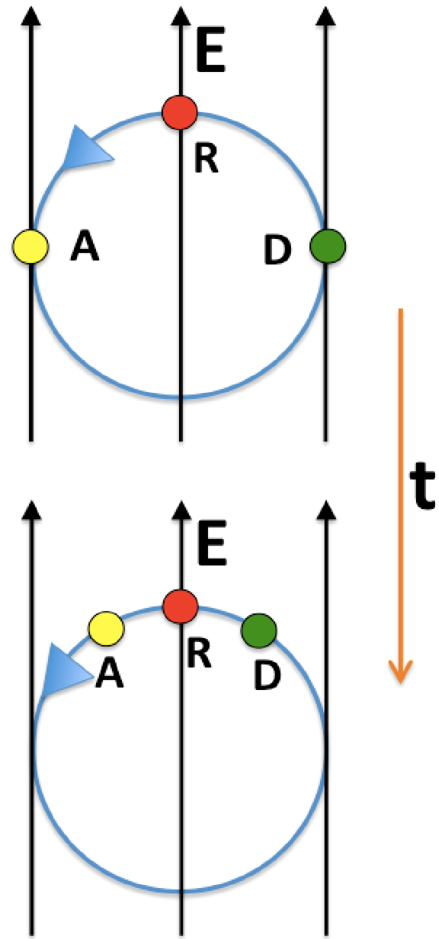
Near-synchronism for net energy transfer from the beam to RF waves:

$$v_0 \underset{\sim}{>} v_{ph}$$

Relativistic azimuthal bunching in a gyrotron



Ref: Kisel' et. al., Radio Eng. Electron. Phys., 19, No.4, 95 (1974)



$$\omega_c = \frac{|e|\,B}{\gamma\, m}$$

Bunching of typically two electrons 'A' and 'D' subjected to the accelerating and the decelerating RF electric fields, respectively, in the interaction region of a gyrotron around a reference electron 'R' that experiences no such fields.

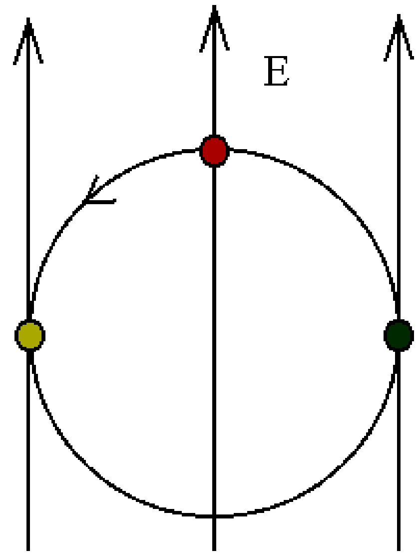
$$\omega_c = \frac{|e|B}{\gamma m}$$

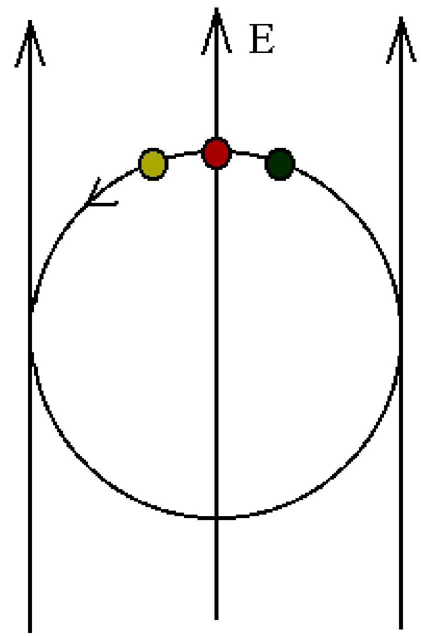
γ **increases**

ω_c **decreases**

T_c **increases**

Electron in accelerating RF electric field moves slower in circular orbit





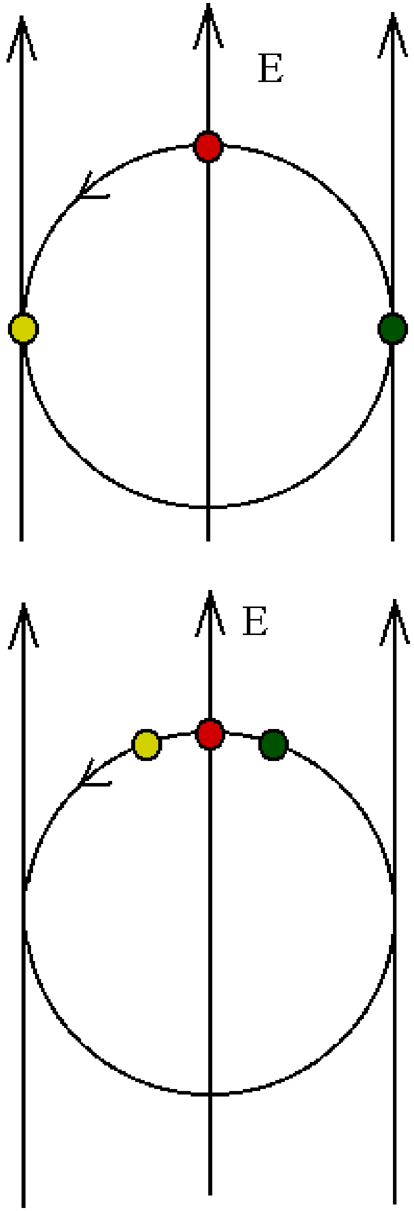
$$\omega_c = \frac{|e|B}{\gamma m}$$

γ **decreases**

ω_c **increases**

T_c **decreases**

Electron in decelerating RF electric field moves faster in circular orbit



Near-cyclotron resonance for net energy transfer from the beam to RF waves:

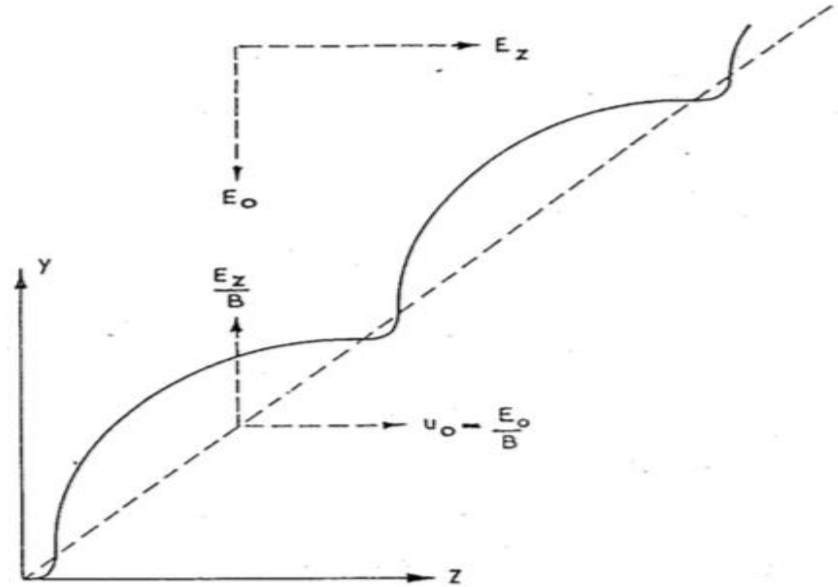
$$T_c \gtrsim T$$

or

$$\omega_c \lesssim \omega$$

Conservation of kinetic energy in an M-type tube

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= 0 \\ m\frac{d^2y}{dt^2} &= e(-E_0 + B_0 v_z) \\ m\frac{d^2z}{dt^2} &= -eB_0 v_y + eE_z \end{aligned}\right\}$$



Initial dc velocity of the electron along z**:** $v_z = u_0$

Crossed DC fields: electric field E_0 **along negative** y **and magnetic field** B_0 **along positive** x

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= 0 \\ m\frac{d^2y}{dt^2} &= e(-E_0 + B_0 v_z) \\ m\frac{d^2z}{dt^2} &= -eB_0 v_y + eE_z \end{aligned}\right\}$$

Initial conditions: $t = 0$

$$\left.\begin{aligned} & x = y = z = 0 \\ & v_x(= dx/dt) = v_y(= dy/dt) = 0 \\ & v_z(= dz/dt) = u_0 \end{aligned}\right\}$$

$$\downarrow$$

$$\left.\begin{aligned} x &= 0 \\ y &= -\frac{1}{\omega_c}\left(u_0 - \frac{E_0}{B_0}\right)(1-\cos\omega_c t) + \frac{E_z}{B_0}\left(t - \frac{\sin\omega_c t}{\omega_c}\right) \\ z &= \frac{E_0}{B_0}t + \left(u_0 - \frac{E_0}{B_0}\right)\frac{\sin\omega_c t}{\omega_c} - \frac{E_z}{\omega_c B_0}(1-\cos\omega_c t) \end{aligned}\right\}$$

$$\omega_c = (-e/m)B_0$$

$$
\left.
\begin{aligned}
& x = 0 \\
& y = -\frac{1}{\omega_c}\left(u_0 - \frac{E_0}{B_0}\right)(1-\cos\omega_c t) + \frac{E_z}{B_0}\left(t - \frac{\sin\omega_c t}{\omega_c}\right) \\
& z = \frac{E_0}{B_0}t + \left(u_0 - \frac{E_0}{B_0}\right)\frac{\sin\omega_c t}{\omega_c} - \frac{E_z}{\omega_c B_0}(1-\cos\omega_c t)
\end{aligned}
\right\} \longleftarrow \quad u_0 = E_0/B_0
$$

$$\downarrow$$

$$z = \frac{E_0}{B_0}t - \frac{E_z}{\omega_c B_0}(1-\cos\omega_c t)$$

$$\downarrow$$

$$\frac{dz}{dt} = \frac{E_0}{B_0} + \frac{E_z \sin\omega_c t}{\omega_c{}^2 B_0}$$

$$\frac{dz}{dt} = \frac{E_0}{B_0} + \frac{E_z \sin \omega_c t}{\omega_c{}^2 B_0} \longrightarrow \left(\frac{dz}{dt}\right)_{\text{time-averaged}} = \frac{E_0}{B_0} \longrightarrow z = \frac{E_0}{B_0} t$$

The electron moving along z **in time** ***t*** **subject to the electric field** E_z **along** z **is decelerated and thus loses its average kinetic energy equal to the amount of work done (force x distance) by the electron (by transferring energy to the field) given by**

$$W\Big|_z = |e| E_z z \quad \textbf{(kinetic energy lost)} \longleftarrow \quad z = \frac{E_0}{B_0} t$$

$$\downarrow$$

$$W\Big|_z = |e| E_z z = \frac{|e| E_z E_0}{B_0} t$$

$$\left.\begin{aligned}
&x = 0 \\
&y = -\frac{1}{\omega_c}\left(u_0 - \frac{E_0}{B_0}\right)(1-\cos\omega_c t) + \frac{E_z}{B_0}\left(t - \frac{\sin\omega_c t}{\omega_c}\right) \\
&z = \frac{E_0}{B_0}t + \left(u_0 - \frac{E_0}{B_0}\right)\frac{\sin\omega_c t}{\omega_c} - \frac{E_z}{\omega_c B_0}(1-\cos\omega_c t)
\end{aligned}\right\} \quad \longleftarrow \quad u_0 = E_0 / B_0$$

$$y = \frac{E_z}{B_0}\left(t - \frac{\sin\omega_c t}{\omega_c}\right)$$

$$\left(\frac{dy}{dt}\right)_{\text{time-averaged}} = \frac{E_z}{B_0}$$

The electron moving along y in time *t* subject to the electric field E_0 directed along negative y direction becomes accelerated and gains an average kinetic energy equal to the amount of work done on it given by

$$W\Big|_y = |e|E_0 y \longleftarrow y = \frac{E_z}{B_0}t \longleftarrow \left(\frac{dy}{dt}\right)_{\text{time-averaged}} = \frac{E_z}{B_0}$$

$$\downarrow$$

$$W\Big|_y = \frac{|e|E_z E_0}{B_0}t$$ **(kinetic energy gained)**

$$W\Big|_z = \frac{|e|E_z E_0}{B_0}t$$ **(kinetic energy lost) (obtained earlier)**

$$W\Big|_y = W\Big|_z$$

The average the kinetic energy gained by an electron in its motion along y is lost by it in its motion along z.

In crossed-field tubes, on the average, the kinetic energy of electrons remains unchanged and their potential energy is converted into RF energy.

Phase bunching in an M-type tube

$$\left(\frac{dz}{dt}\right)_{\text{time-averaged}} = \frac{E_0}{B_0}$$

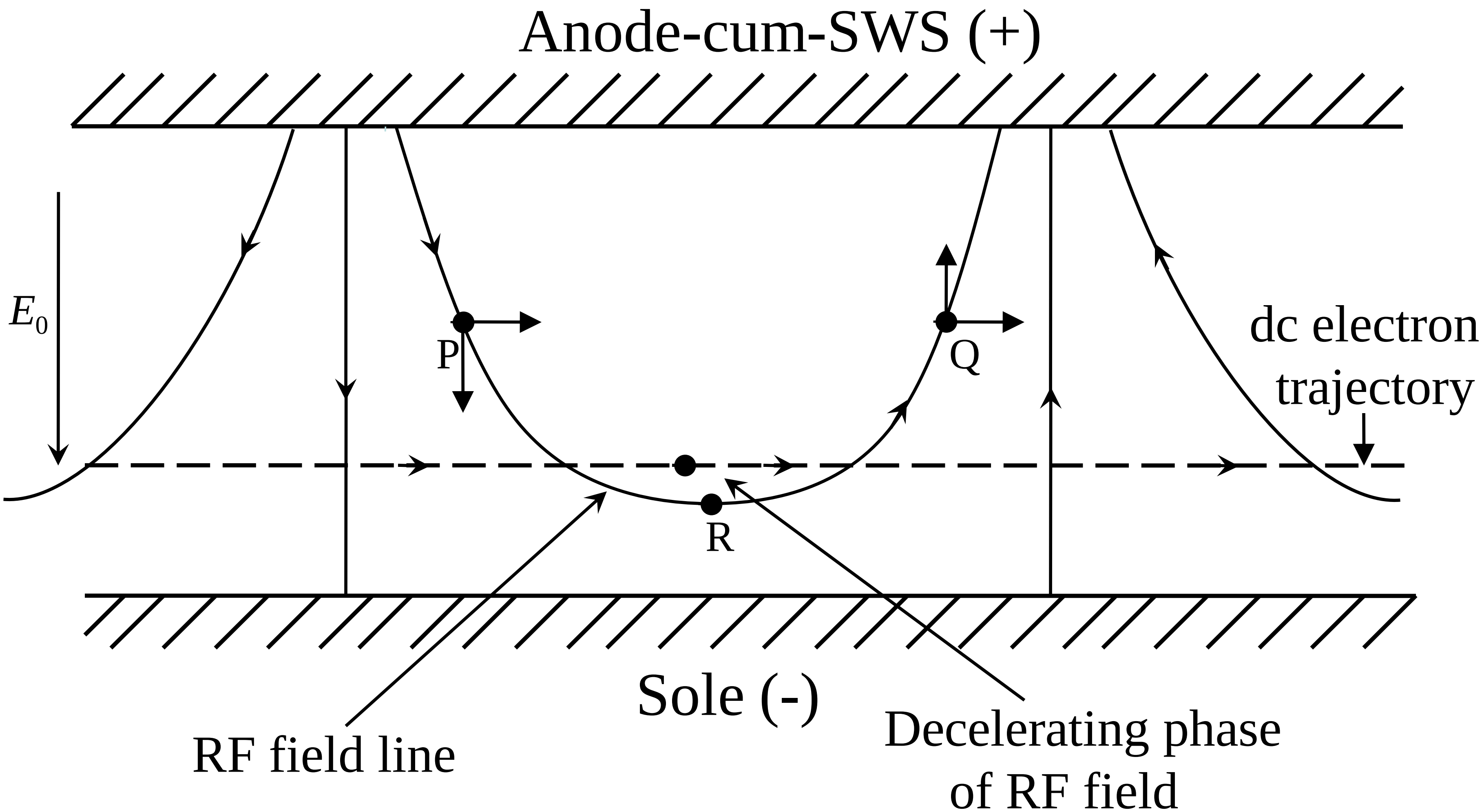
Let us consider three electrons.

The first electron, taken as the reference, enters the system to experience the field at O where the RF electric has no transverse component and has maximum axial decelerating field.

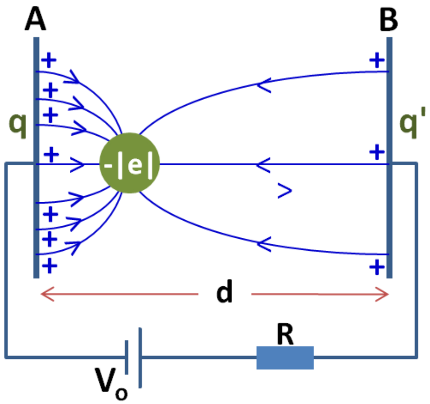
The second electron enters the system to experience the field at P where the transverse component of RF electric adds to the DC electric field in the negative y direction. This makes $(dz/dt)_{\text{time-averaged}}$ more for the second electron at P than for the first electron at O.

The third electron enters the system to experience the field at Q where the transverse component of RF electric in the positive y direction reduces the electric field established by the DC electric field in the negative y direction and thus reduces the field in the negative y direction established by the DC electric field. This makes $(dz/dt)_{\text{time-averaged}}$ less for the third electron at Q than for the first electron at O.

This makes the second electron and the third electron bunch around the first electron.


Anode-cum-SWS (+)
E_0
P
Q
R
dc electron trajectory
Sole (-)
RF field line
Decelerating phase of RF field

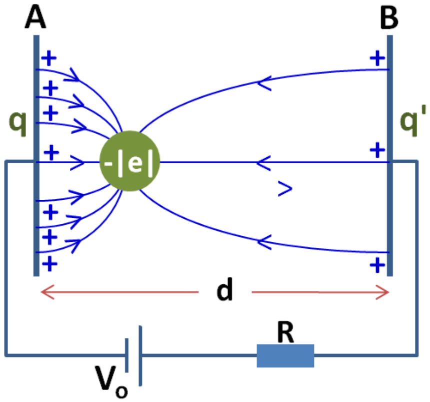
Induced current on electrodes due to electron beam flow



An electron starts from an electrode (A) at a negative potential and is accelerated by another electrode (B) raised to a positive potential (V_0). The electron after the flight between the electrodes A and B finally strikes the electrode B at the positive potential. Consider that there is a finite electron transit time between the electrodes. (The two electrodes (A and B) are connected by a source of potential in the external circuit).

<u>**Question**</u>

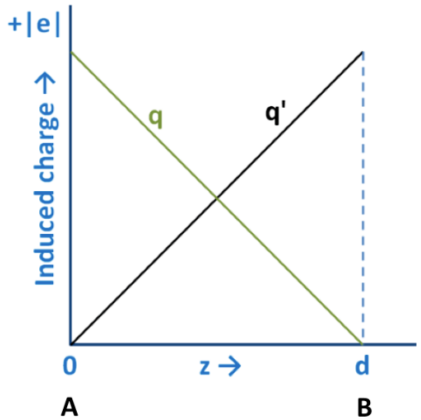
(1) Will there be a current in the external circuit when the electron is in flight between the electrodes? (2) And will there be any current when the electron strikes the electrode?

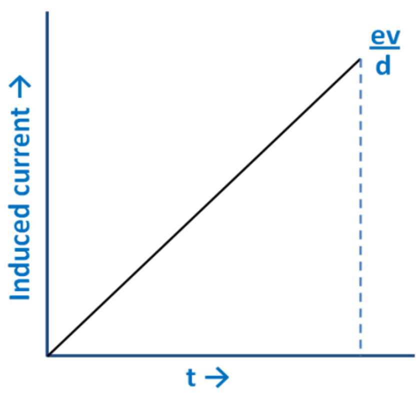


$$q'V_0 = |e|\frac{V_0}{d}z \longrightarrow q' = |e|\frac{z}{d}$$

$$\downarrow$$

$$i' = \frac{dq'}{dt} = |e|\frac{dz}{dt}\frac{1}{d} = \frac{|e|v}{d}$$





Induced current due to the flow of an electron (Triangular current pulse)

For an electron beam, the total induced current is found by adding triangular pulses of current associated with each electron.

Current may even be induced in an electrode to which no electrons flow, if the number or velocity of electrons approaching the electrode is different from the number or velocity of electrons receding from it.

Plasma frequency (Langmuir frequency)

Consider an ensemble of electrons and positive ions maintaining overall charge neutrality

Displacement of electrons from their equilibrium position to a small extent

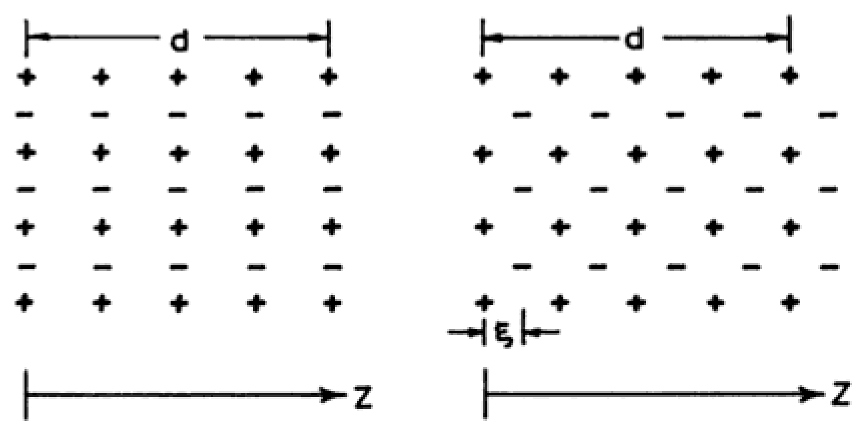
Space-charge electric field in the direction of the displacement of electrons provides a restoring force

Overshoot of electrons

Restoring force again coming into play

Oscillation of electrons about their mean position at the natural angular frequency, called the plasma frequency or Langmuir frequency of electrons

Displacement of electron layers and the resulting space-charge restoring force

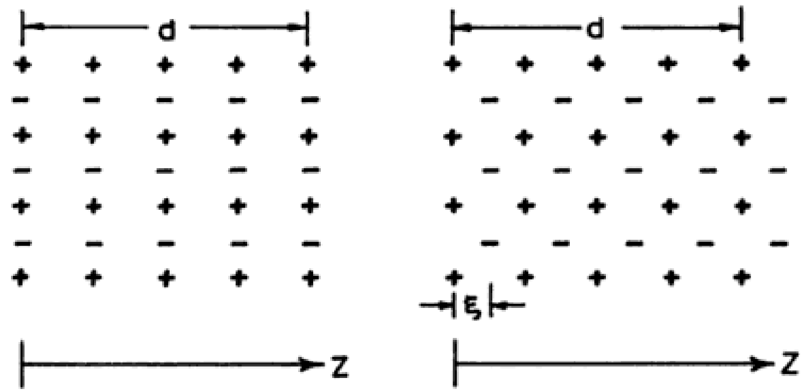


$$\vec{E}=\left(\frac{|\rho_s|}{2\varepsilon_0}\right)(\vec{a}_z)+\left(\frac{-|\rho_s|}{2\varepsilon_0}\right)(-\vec{a}_z)=\frac{|\rho_s|}{\varepsilon_0}\vec{a}_z$$

$$|\rho_s|\alpha=|\rho_0|\xi\alpha$$

$$\vec{E}=\frac{|\rho_s|}{\varepsilon_0}\vec{a}_z \longleftarrow |\rho_s|=|\rho_0|\xi$$

$$E_z=\frac{|\rho_0|\xi}{\varepsilon_0}$$



$$m\frac{d^2\xi}{dt^2}=eE_z \leftarrow \frac{e}{m}=\eta$$

$$\frac{d^2\xi}{dt^2}=\eta E_z=\frac{\eta|\rho_0|}{\varepsilon_0}\xi=-\frac{|\eta||\rho_0|}{\varepsilon_0}\xi$$

$$\omega_p=\sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$

$$\downarrow$$

$$\frac{d^2\xi}{dt^2}=-\omega_p^2\xi$$

$$\frac{d^2\xi}{dt^2} = -\omega_p^2 \xi \qquad\qquad \omega_p = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$

$$\xi = A\exp(j\omega_p t) + B\exp(-j\omega_p t)$$

$$\xi = A(\cos\omega_p t + j\sin\omega_p t) + B(\cos\omega_p t - j\sin\omega_p t)$$

$$= (A+B)\cos\omega_p t + j(A-B)\sin\omega_p t$$

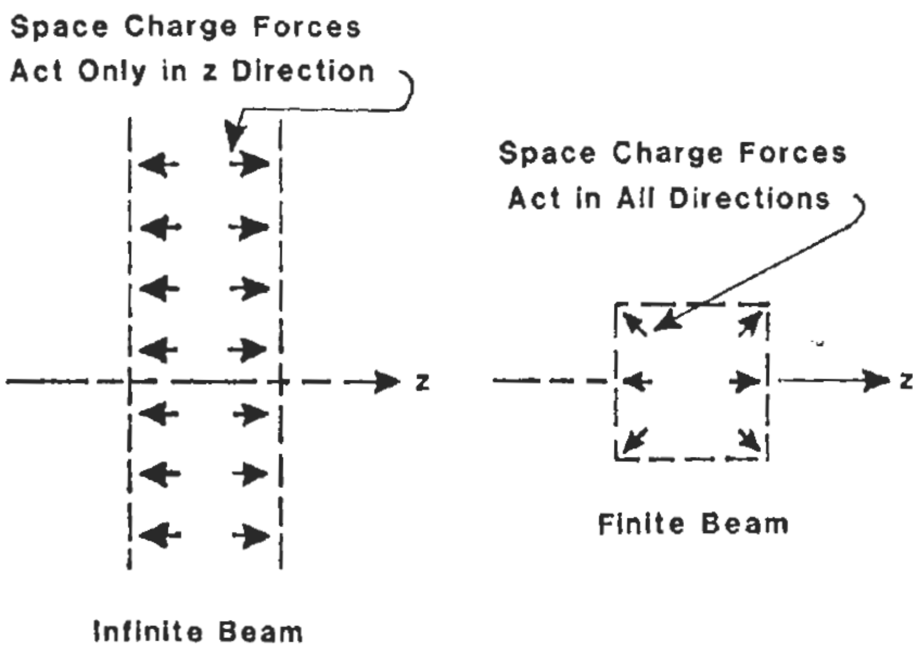
$$\xi = C\cos\omega_p t + D\sin\omega_p t \longleftarrow \xi = 0,\ \text{at } t = 0$$ **(setting initial condition)**

$$C = 0$$

$$\xi = D\sin\omega_p t$$

In an electron beam of infinite cross-sectional area, the electric fields are axial and thus the space-charge fields are constrained in the axial direction.

In a beam of finite cross-sectional area, the electric fields are axial as well as radial, with the result that the axial component is reduced in comparison to the infinite cross-sectional beam



Space-charge waves

Prerequisite

$J = \rho v$ **(Current density equation)**

$$\frac{\partial J_1}{\partial z} + \frac{\partial \rho_1}{\partial t} = 0$$ **(Continuity equation)**

$$\frac{dv_1}{dt} = \frac{e}{m} E_s = \eta E_s$$ **(Force equation)**

$$\frac{\partial E_s}{\partial z} = \frac{\rho_1}{\varepsilon_0}$$ **(Poisson's equation)**

$$J = J_0 + J_1$$

$$\rho = \rho_0 + \rho_1$$

$$v = v_0 + v_1$$

$$J = \rho v$$ **(one-dimensional)** $\longleftarrow$ $$\vec{J} = \rho\vec{v}$$ **(current density equation)**

$\downarrow$ $\longleftarrow$ $$J = J_0 + J_1, \rho = \rho_0 + \rho_1, v = v_0 + v_1$$

$$J_0 + J_1 = (\rho_0 + \rho_1)(v_0 + v_1) = \rho_0 v_0 + \rho_0 v_1 + \rho_1 v_0 + \rho_1 v_1 \quad \longleftarrow \quad J_0 = \rho_0 v_0$$

$$J_1 = \rho_0 v_1 + v_0 \rho_1 + \rho_1 v_1 = \rho_0 v_1 + v_0 \rho_1$$ **(small-signal approximation)**

$$\frac{\partial J_1}{\partial z} = \rho_0 \frac{\partial v_1}{\partial z} + v_0 \frac{\partial \rho_1}{\partial z} \quad \longleftarrow \quad \frac{\partial J_1}{\partial z} + \frac{\partial \rho_1}{\partial t} = 0 \quad \longleftarrow \quad \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

(continuity equation)

$$-\frac{\partial \rho_1}{\partial t} = \rho_0 \frac{\partial v_1}{\partial z} + v_0 \frac{\partial \rho_1}{\partial z}$$

$$\frac{\partial \rho_1}{\partial t} + v_0 \frac{\partial \rho_1}{\partial z} = -\rho_0 \frac{\partial v_1}{\partial z} \quad \longleftarrow \quad D = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}$$

$$D\rho_1 = -\rho_0 \frac{\partial v_1}{\partial z}$$

Force equation

$$m\frac{dv_1}{dt}=eE_s \qquad v=v_0+v_1$$

$$\frac{dv_1}{dt}=\frac{e}{m}E_s=\eta E_s$$

$$\frac{dv_1}{dt}=\frac{\partial v_1}{\partial t}+\frac{dz}{dt}\frac{\partial v_1}{\partial z}=\frac{\partial v_1}{\partial t}+v\frac{\partial v_1}{\partial z}=\frac{\partial v_1}{\partial t}+(v_0+v_1)\frac{\partial v_1}{\partial z}=\frac{\partial v_1}{\partial t}+v_0\frac{\partial v_1}{\partial z}$$

$$\frac{\partial v_1}{\partial t}+v_0\frac{\partial v_1}{\partial z}=\eta E_s \longleftarrow D=\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}$$

$$Dv_1=\eta E_s$$

$$D\rho_1 = -\rho_0 \frac{\partial v_1}{\partial z}$$ **(obtained earlier)** $$[\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z} = D]$$

$$Dv_1 = \eta E_s$$ **(obtained earlier)**

$$D^2\rho_1 = -\rho_0 D \frac{\partial v_1}{\partial z} = -\rho_0 \frac{\partial}{\partial z} Dv_1 = -\rho_0 \frac{\partial}{\partial z}\eta E_s = -\eta\rho_0 \frac{\partial E_s}{\partial z}$$

$$D^2\rho_1 = -\eta\rho_0 \frac{\partial E_s}{\partial z} \longleftarrow \frac{\partial E_s}{\partial z} = \frac{\rho_1}{\varepsilon_0}$$

$$\omega_p = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$

$$D^2\rho_1 = -\eta\rho_0 \frac{\rho_1}{\varepsilon_0} = \frac{-\eta\rho_0}{\varepsilon_0}\rho_1 = \frac{-|\eta||\rho_0|}{\varepsilon_0}\rho_1 = -\omega_p{}^2\rho_1$$

$$D^2 = -\omega_p{}^2$$

$$D = \pm j\omega_p$$

RF quantities vary as $\exp j(\omega t - \beta z)$

$$\frac{\partial}{\partial t} = j\omega, \;\; \frac{\partial}{\partial z} = -j\beta$$

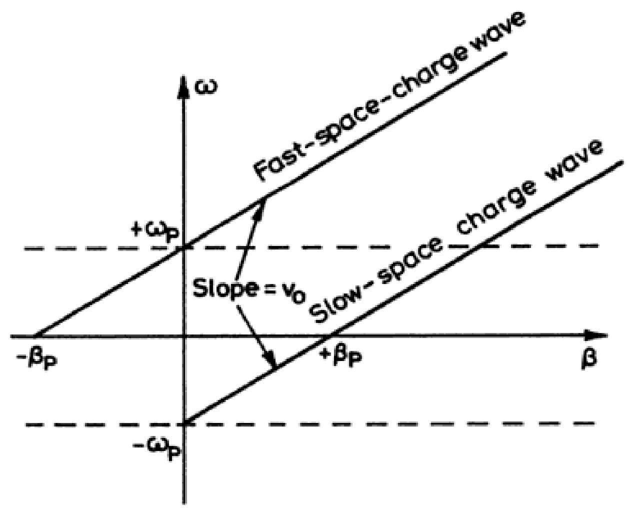
$$D = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z} = j\omega - j\beta v_0 = j(\omega - \beta v_0)$$

$D = \pm j\omega_p$ **(recalled)**

$$\pm j\omega_p = j(\omega - \beta v_0)$$

$$\omega - \beta v_0 = \pm\omega_p$$

Dispersion relation of Hahn and Ramo space-charge waves



$\omega - \beta v_0 = \pm \omega_p$ **(Hahn and Ramo space-charge waves)**

$$\frac{\omega}{v_0} = \beta_e, \; \frac{\omega_p}{v_0} = \beta_p$$

$$\beta = \frac{\omega \mp \omega_p}{v_0} = \beta_e \mp \beta_p$$

$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \mp \omega_p} v_0 = \frac{\omega}{\beta_e \mp \beta_p}$ **(phase velocity of space-charge waves)**

The upper sign $\Rightarrow$ Fast space-charge wave

The lower sign $\Rightarrow$ Slow space-charge wave

$\omega - \beta v_0 = \pm \omega_p$ **(space-charge-wave dispersion relation)**

$v_0 = 0 \Rightarrow \quad \omega = \pm \omega_p$

In a frame of reference which moves with the dc beam velocity v_0, an observer 'sees' a Doppler-shifted frequency ω' given by

$$\omega' = \frac{v_p - v_0}{v_p}\omega = \left(1 - \frac{v_0}{v_p}\right)\omega \quad \longleftarrow \quad v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \mp \omega_p} v_0$$

$$\omega' = \left(1 - \frac{\omega \mp \omega_p}{\omega}\right)\omega = \mp \omega_p$$

Cyclotron waves

Cyclotron waves are supported by an electron beam moving transverse to a DC magnetic field

$$m\frac{dv_{1x}}{dt}=e(\vec{v}_1\times\vec{B})_x$$

$$m\frac{dv_{1y}}{dt}=e(\vec{v}_1\times\vec{B})_y$$

$$\vec{v}_1\times\vec{B}=\begin{vmatrix}\vec{a}_x & \vec{a}_y & \vec{a}_z\\ v_{1x} & v_{1y} & v_{1z}\\ B_x & B_y & B_z\end{vmatrix}$$

$$B_x=0,\; B_y=0,\; B_z=B$$

$$\frac{dv_{1x}}{dt}=\frac{e}{m}(\vec{v}_1\times\vec{B})_x=\eta(\vec{v}_1\times\vec{B})_x \qquad [\frac{d}{dt}=\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial z}=D]$$

$$Dv_{1x}=\eta(\vec{v}_1\times\vec{B})_x=\eta Bv_{1y}=-\omega_c v_{1y}$$

$$[\omega_c=-\eta B]$$

$$Dv_{1y}=\eta(\vec{v}_1\times\vec{B})_y=-\eta Bv_{1x}=\omega_c v_{1x}$$

$$Dv_{1x} = \eta(\vec{v}_1 \times \vec{B})_x = -\omega_c v_{1y} \qquad Dv_{1y} = -\eta B v_{1x} = \omega_c v_{1x}$$

$$D^2 v_{1x} = -\omega_c D v_{1y} = -\omega_c(\omega_c v_{1x}) = -\omega_c{}^2 v_{1x} \qquad [\frac{d}{dt} = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z} = D]$$

For cyclotron waves now we obtain

$$D^2 = -\omega_c{}^2$$

$$\omega - \beta v_0 \mp \omega_c = 0$$

(Dispersion relation for cyclotron waves)

For space-charge waves we obtained

$$D^2 = -\omega_p{}^2$$

$$\omega - \beta v_0 \mp \omega_p = 0$$

(dispersion relation for space-charge waves obtained earlier)

Space-charge and cyclotron waves

Space-charge waves

$$J = \rho v$$ **(Current density equation)**

$$\frac{\partial J_1}{\partial z} + \frac{\partial \rho_1}{\partial t} = 0$$ **(Continuity equation)**

$$\frac{dv_1}{dt} = \frac{e}{m} E_s = \eta E_s$$ **(Force equation)**

$$\frac{\partial E_s}{\partial z} = \frac{\rho_1}{\varepsilon_0}$$ **(Poisson's equation)**

$$\beta = \beta_e \mp \beta_p$$

$$\omega - \beta v_0 \mp \omega_p = 0$$

$$v_p = \frac{\omega}{\omega \mp \omega_p} v_0$$

$$\omega_p = \left(\frac{|\eta||\rho_0|}{\varepsilon_0}\right)^{1/2}$$

$$\beta = \omega / v_p$$

$$\beta_e = \omega / v_0$$

$$\beta_p = \omega_p / v_0$$

Cyclotron waves

Force equations along *x* and *y* for a magnetic field along *z*

$$m\frac{dv_{1x}}{dt} = e(\vec{v} \times \vec{B})_x$$

$$m\frac{dv_{1y}}{dt} = e(\vec{v} \times \vec{B})_y$$

$$\omega - \beta v_0 \mp \omega_c = 0$$

$$v_p = \frac{\omega}{\omega \mp \omega_c} v_0$$

$$\omega_c = -\eta B = |\eta| B$$

Upper sign for the fast wave and lower sign for the slow wave

Space-charge and cyclotron waves

Space-charge waves

$$\omega - \beta v_0 \mp \omega_p = 0$$

$$\beta = \beta_e \mp \beta_p$$

$$\beta = \frac{\omega}{v_p} = \frac{\omega}{v_0} \mp \frac{\omega_p}{v_0} = \frac{\omega \mp \omega_p}{v_0}$$

$$v_p = \frac{\omega}{\omega \mp \omega_p} v_0$$

Cyclotron waves

$$\omega - \beta v_0 \mp \omega_c = 0$$

$$\beta = \beta_e \mp \beta_c$$

$$\beta = \frac{\omega}{v_p} = \frac{\omega}{v_0} \mp \frac{\omega_c}{v_0} = \frac{\omega \mp \omega_c}{v_0}$$

$$v_p = \frac{\omega}{\omega \mp \omega_c} v_0$$

Upper sign for the fast wave and lower sign for the slow wave

$$\beta_e = \frac{\omega}{v_0}; \beta_p = \frac{\omega_p}{v_0}; \omega_p = \left(\frac{|\eta|\rho_0}{\varepsilon_0}\right)^{1/2}$$

$$\beta_e = \frac{\omega}{v_0}; \beta_c = \frac{\omega_c}{v_0}; \omega_c = |\eta| B$$

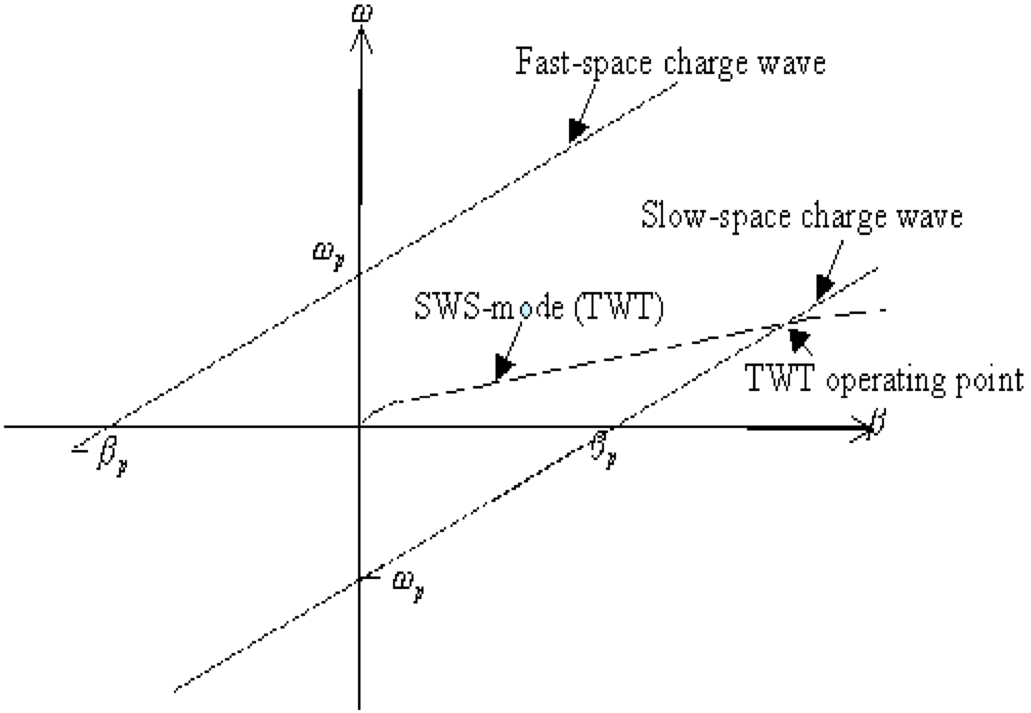
Amplification of space-charge waves

- **An electron beam of uniform diameter in a resistive-wall cylindrical waveguide**
- **An electron beam in a rippled-wall (varying diameter) conducting-wall cylindrical waveguide**
- **An electron beam mixed with another beam of a slightly different DC electron beam velocity (two-stream amplifier)**
- **An electron beam penetrating through a plasma (beam-plasma amplifier)**
- **An electron beam interacting with RF waves supported by a slow-wave structure (TWT)**

Intersection between slow space-charge and circuit waves

$$\omega - \beta v_0 \mp \omega_p = 0$$

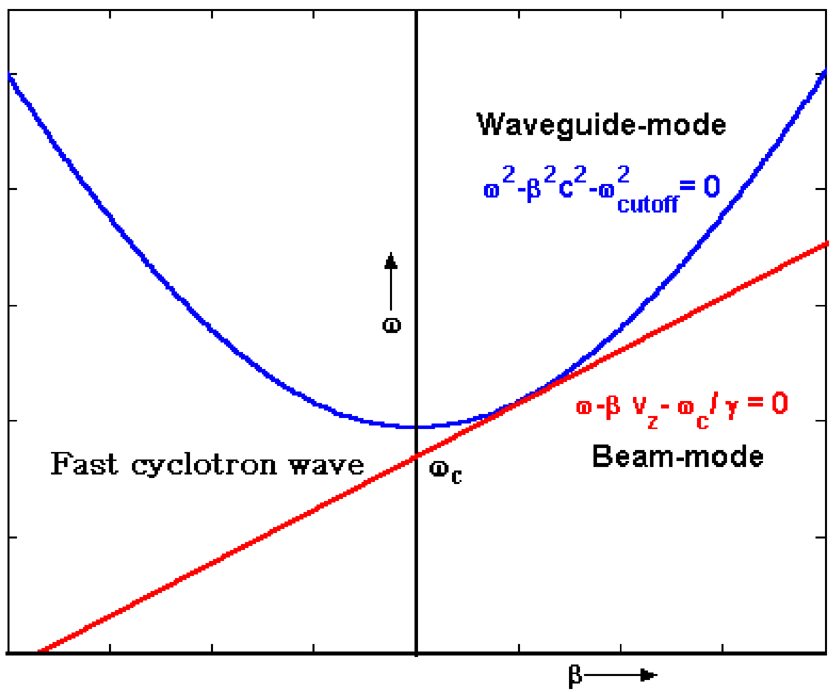
Upper sign for the fast wave and lower sign for the slow wave

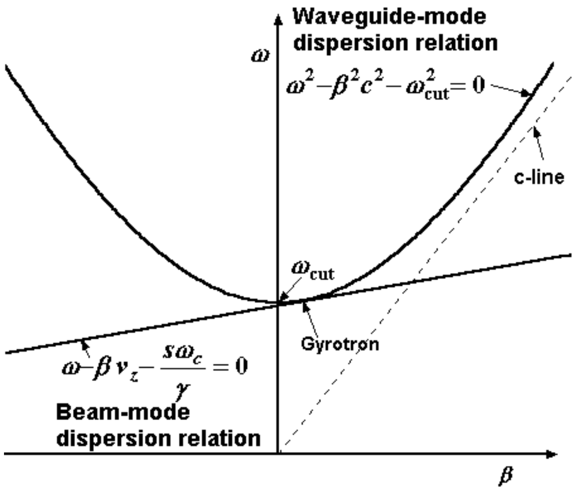


Intersection between fast-cyclotron and circuit waves

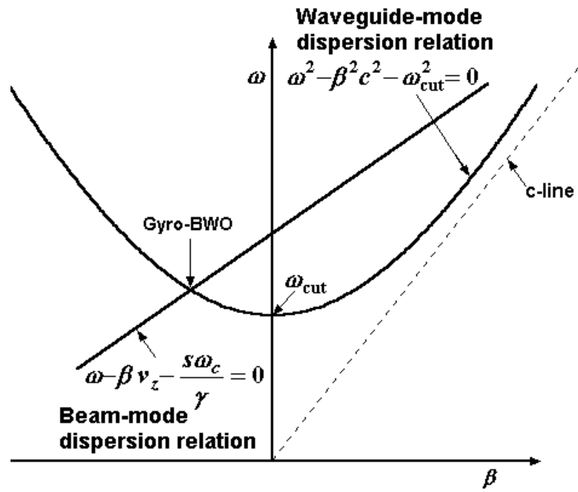
$$\omega - \beta v_0 \mp \omega_c = 0$$

Upper sign for the fast wave and lower sign for the slow wave

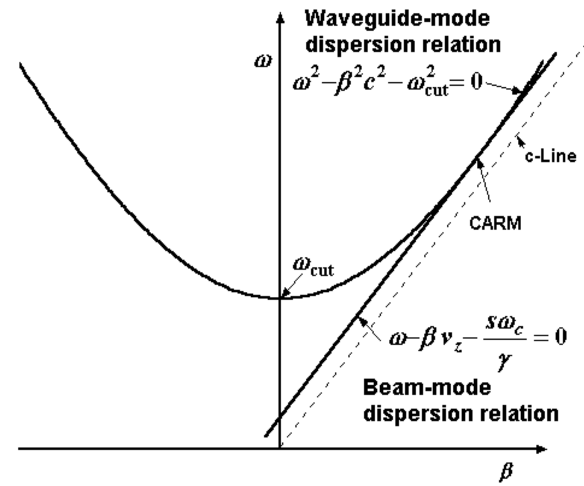




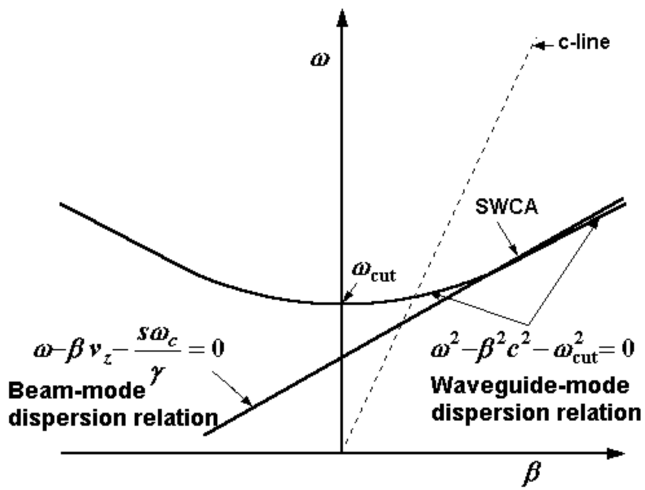
Gyrotron



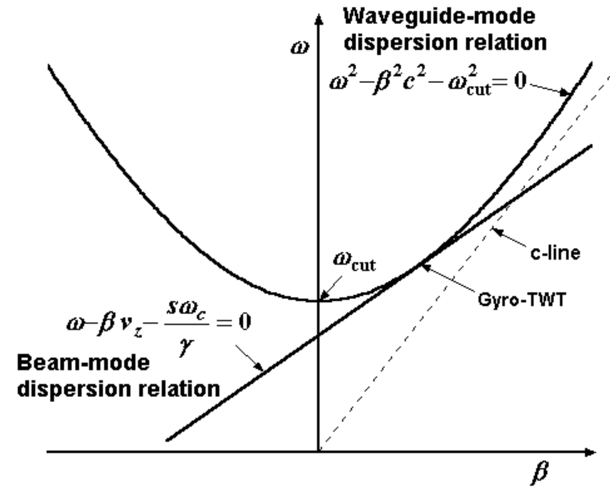
Gyro-BWO



CARM



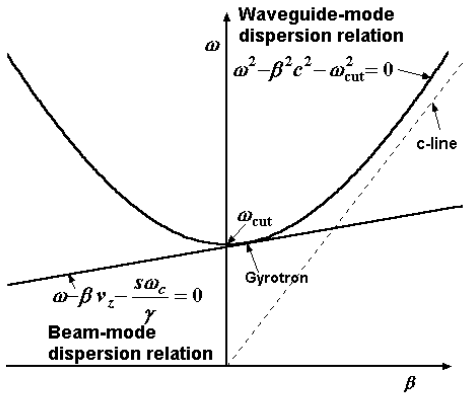
SWCA



Gyro-TWT

Plots of beam-mode and waveguide-mode dispersion characteristics showing the operating point as the intersection between these plots (*Prepared by Vishal Kesari*)

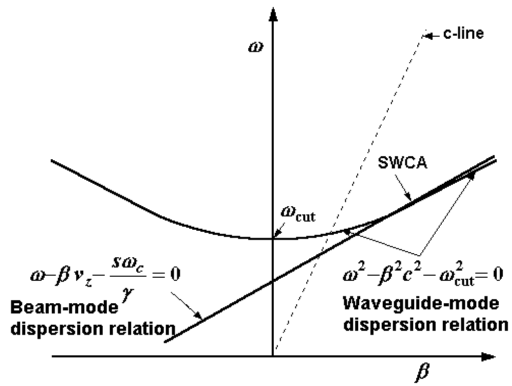
Devices based on cyclotron resonance maser and Weibel instabilities



Gyrotron: cyclotron resonance maser (CRM) instability

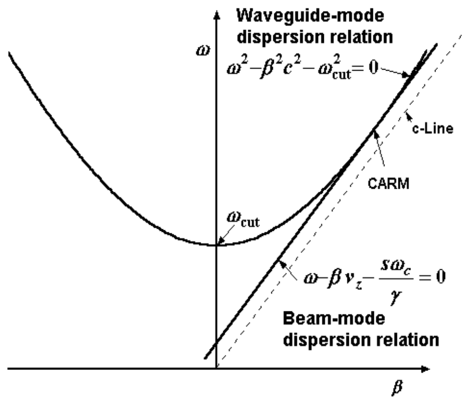
Destabilization of fast wave

$v_p > c$



Slow-wave cyclotron amplifier (SWCA): Weibel instability

$v_p < c$



Cyclotron auto-resonance maser (CARM): CRM and Weibel instabilities in equal proportions

$v_p \gtrsim c$

Beam-mode dispersion relation

$$\omega - \beta v_z - s\omega_c / \gamma = 0$$

$$\omega = \beta v_z + s\omega_c / \gamma = \omega_D$$

$$\Delta\omega_D = \beta \Delta v_z - \frac{s\,\omega_c}{\gamma^2}\Delta\gamma$$

s **is the beam harmonic number**

Doppler-shifted cyclotron angular frequency

Δv_z **is caused by Weibel instability due to Lorentz force on electrons in transverse motion in transverse RF magnetic field**

$\Delta\gamma$ **is caused by energy exchange between electrons with transverse motion in transverse RF electric field**

$$\Delta\omega_D = \beta\,\Delta v_z - \frac{s\,\omega_c}{\gamma^2}\Delta\gamma$$

$$\gamma m \frac{dv_z}{dt}\vec{a}_z = e\vec{v}_\perp \times \vec{B}_\perp$$

$$\gamma m \Delta v_z = e\vec{v}_\perp \times \vec{B}_\perp \Delta t = e\vec{v}_\perp \times (B_r\vec{a}_r + B_\theta\vec{a}_\theta)\Delta t$$

$$\Delta v_z = \frac{e\vec{v}_\perp \times (B_r\vec{a}_r + B_\theta\vec{a}_\theta)}{\gamma m}\Delta t$$

Let us express magnetic field quantities in terms of electric field quantities with the help of Maxwell's equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \rightarrow \quad \frac{1}{r}\begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & \vec{a}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ E_r & rE_\theta & 0 \end{vmatrix} = -\frac{\partial}{\partial t}(B_r\vec{a}_r + B_\theta\vec{a}_\theta + B_z\vec{a}_z)$$

$$\frac{1}{r}\begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & \vec{a}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ E_r & rE_\theta & 0 \end{vmatrix} = -\frac{\partial}{\partial t}(B_r\vec{a}_r + B_\theta\vec{a}_\theta + B_z\vec{a}_z)$$

$$\boxed{\Delta v_z = \frac{e\vec{v}_\perp \times (B_r\vec{a}_r + B_\theta\vec{a}_\theta)}{\gamma m}\Delta t}$$

***r*- component of magnetic field:** $-\frac{\partial E_\theta}{\partial z} = -\frac{\partial B_r}{\partial t} \Rightarrow -j\beta E_\theta = j\omega B_r \Rightarrow B_r = -\frac{\beta E_\theta}{\omega}$

***θ*-component:** $\frac{1}{r}\frac{\partial E_r}{\partial z} = -\frac{\partial B_\theta}{\partial t} \Rightarrow -j\beta E_r = -j\omega B_\theta \Rightarrow B_\theta = \frac{\beta E_r}{\omega}$

$$\Delta v_z = \frac{e\vec{v}_\perp \times (B_r\vec{a}_r + B_\theta\vec{a}_\theta)}{\gamma m}\Delta t = \frac{e\beta\vec{v}_\perp \times (-E_\theta\vec{a}_r + E_r\vec{a}_\theta)}{\gamma m\omega}\Delta t = \frac{e\beta\vec{v}_\perp \times \vec{a}_z \times \vec{E}_\perp}{\gamma m\omega}\Delta t$$

$$\vec{a}_z \times \vec{E}_\perp = \vec{a}_z \times (E_r\vec{a}_r + E_\theta\vec{a}_\theta) = E_r\vec{a}_\theta - E_\theta\vec{a}_r = -E_\theta\vec{a}_r + E_r\vec{a}_\theta$$

$$\Delta v_z = \frac{e\beta \vec{v}_\perp \times \vec{a}_z \times \vec{E}_\perp}{\gamma m \omega} \Delta t$$

$$\Delta v_z = \frac{e\beta \vec{v}_\perp \times \vec{a}_z \times \vec{E}_\perp}{\gamma m \omega} \Delta t = \frac{e\beta [(\vec{v}_\perp \cdot \vec{E}_\perp)\vec{a}_z - (\vec{v}_\perp \cdot \vec{a}_z)\vec{E}_\perp]}{\gamma m \omega} \Delta t = \frac{e\beta [(\vec{v}_\perp \cdot \vec{E}_\perp)\vec{a}_z]}{\gamma m \omega} \Delta t$$

$$\vec{v}_\perp \cdot \vec{a}_z = 0$$

$$\Delta v_z = \frac{e\beta [(\vec{v}_\perp \cdot \vec{E}_\perp)\vec{a}_z]}{\gamma m \omega} \Delta t$$

$$\Delta \omega_D = \beta \, \Delta v_z - \frac{s\,\omega_c}{\gamma^2} \Delta \gamma$$

$$\Delta \gamma = ?$$

$$\Delta\gamma = ?$$

Rate of change of kinetic energy with time

$$\frac{d}{dt}(\gamma mc^2) = e(\vec{v}_\perp \cdot \vec{E}_\perp)$$

$$\Delta v_z = \frac{e\beta[(\vec{v}_\perp \cdot \vec{E}_\perp)\vec{a}_z]}{\gamma m\omega}\Delta t$$

(Weibel instability)

$$mc^2(\Delta\gamma) = e(\vec{v}_\perp \cdot \vec{E}_\perp)\Delta t$$

$$\boxed{\Delta\gamma = \frac{e}{mc^2}(\vec{v}_\perp \cdot \vec{E}_\perp)\Delta t}$$

(CRM instability)

$$\Delta\omega_D = \beta\,\Delta v_z - \frac{s\,\omega_c}{\gamma^2}\Delta\gamma$$

$$\Delta\omega_D = \beta\frac{e\beta}{\gamma m\omega}(\vec{v}_\perp \cdot \vec{E}_\perp)\Delta t - \frac{s\omega_c}{\gamma^2}\frac{e}{mc^2}(\vec{v}_\perp \cdot \vec{E}_\perp)\Delta t = [\frac{e\beta^2}{\gamma m\omega} - \frac{s\omega_c}{\gamma^2}\frac{e}{mc^2}]\Delta t(\vec{v}_\perp \cdot \vec{E}_\perp)$$

(Weibel instability) **(CRM instability)**

$$\Delta\omega_D = \beta\frac{e\beta}{\gamma m\omega}(\vec{v}_\perp\cdot\vec{E}_\perp)\Delta t - \frac{s\omega_c}{\gamma^2}\frac{e}{mc^2}(\vec{v}_\perp\cdot\vec{E}_\perp)\Delta t = [\frac{e\beta^2}{\gamma m\omega} - \frac{s\omega_c}{\gamma^2}\frac{e}{mc^2}]\Delta t(\vec{v}_\perp\cdot\vec{E}_\perp)$$

(Weibel instability) **(CRM instability)**

$$\Delta\omega_D = [\frac{e\beta^2}{\gamma m\omega} - \frac{s\omega_c}{\gamma^2}\frac{e}{mc^2}]\Delta t(\vec{v}_\perp\cdot\vec{E}_\perp)$$

$$\frac{s\omega_c}{\gamma} \lesssim \omega \qquad \frac{s\omega_c}{\gamma} = F\omega \qquad F \lesssim 1$$

$$\Delta\omega_D = [\frac{e\beta^2}{\gamma m\omega} - F\omega\frac{e}{\gamma mc^2}]\Delta t(\vec{v}_\perp\cdot\vec{E}_\perp)$$

$$\Delta\omega_D = [\frac{e\beta^2}{\gamma m\omega} - F\omega\frac{e}{\gamma mc^2}]\Delta t(\vec{v}_\perp \cdot \vec{E}_\perp)$$

$$\frac{s\omega_c}{\gamma} = F\omega$$

(Weibel instability)

(CRM instability)

If Weibel instability dominates over CRM instability

$$\frac{e\beta^2}{\gamma m\omega} > F\omega\frac{e}{\gamma mc^2} \qquad (F \approx 1)$$

$$\frac{\beta^2}{\omega^2} > \frac{1}{c^2} \longrightarrow \frac{\omega^2}{\beta^2} < c^2$$

$$\frac{\omega}{\beta} < c$$

$$v_p < c$$

Destabilization of slow waves: slow-wave cyclotron amplifier (SWCA)

If CRM instability dominates over Weibel instability

$$F\omega\frac{e}{\gamma mc^2} > \frac{e\beta^2}{\gamma m\omega} \qquad (F \approx 1)$$

$$\frac{\omega^2}{c^2} > \beta^2 \longrightarrow \frac{\omega^2}{\beta^2} > c^2$$

$$\frac{\omega}{\beta} > c$$

$$v_p > c$$

Destabilization of fast waves: Gyro-TWT

If both Weibel instability and CRM instability are present in equal proportions (auto-resonance)

$$\frac{e\beta^2}{\gamma m\omega} = F\omega \frac{e}{\gamma mc^2} \qquad (\Delta\omega_D = 0) \qquad \frac{s\omega_c}{\gamma} \lesssim \omega \qquad \frac{s\omega_c}{\gamma} = F\omega \qquad F \lesssim 1$$

$$\frac{\beta^2}{\omega^2} = F\frac{1}{c^2} \qquad F \lesssim 1$$

$$\frac{\omega}{\beta} = c\frac{1}{\sqrt{F}}$$

$$v_p = c\frac{1}{\sqrt{F}} \qquad F \lesssim 1$$

$$v_p \gtrsim c$$

Destabilization of slightly fast waves: Cyclotron auto-resonance maser

Space-charge limiting current

The motion of the electrons in a vacuum tube is possible due to the presence of the positive ions that are always practically present in the tube and that neutralize the negative charges of the electrons.

If the neutralizing positive ions were absent, the negative space-charge of the electrons would have caused a depression in the potential in the region of the traversal of electrons thereby causing the electrons to slow down, stop and even return.

At sufficiently high electron beam currents, a virtual cathode forms due to the potential depression and the electrons in excess of the maximum number are reflected back to the electron source. The presence of the negative space-charge of the electrons in an electron tube sets the upper limit of the beam current in the tube called the space-charge-limiting (SCL) current. This limit is imposed by the potential depression caused by the negative space-charge in the tube that retards the flow of electrons or reflects the flow back to form what is known as the virtual cathode.

The plasma is effective in compensating for these space-charge effects in high-current beams allowing higher beam current operation of plasma-filled tubes than vacuum tubes.

Space-charge limiting current of an electron beam in a metal surrounding

$$I_{SC\lim} = \frac{I_A}{2\ln\frac{r_0}{r_b}}(\gamma_c^{2/3} - 1)^{3/2}\frac{1}{1-\frac{n_i}{n_e}}$$

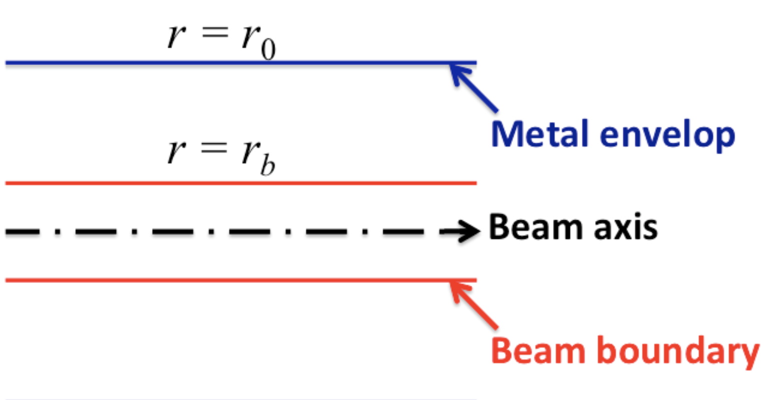
(infinitesimally thin hollow electron beam)

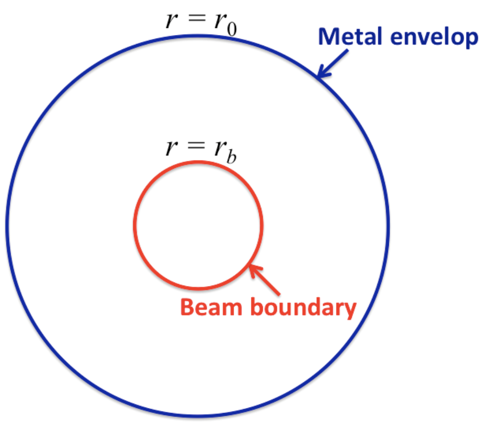
$$I_{SC\lim} = \frac{I_A}{1+2\ln\frac{r_0}{r_b}}(\gamma_c^{2/3} - 1)^{3/2}\frac{1}{1-\frac{n_i}{n_e}}$$

(thick solid electron beam)

$$I_A = 4\pi\varepsilon_0 mc^3/|e| = 17.1\ \mathrm{kA};\ \gamma_c = 1 + |e|V_c/mc^2$$

(relativistic mass factor of an electron corresponding to an accelerating potential V_c)





Space-charge limiting current of an infinitesimally thin 'hollow' electron beam in a drift tube (metal envelope)

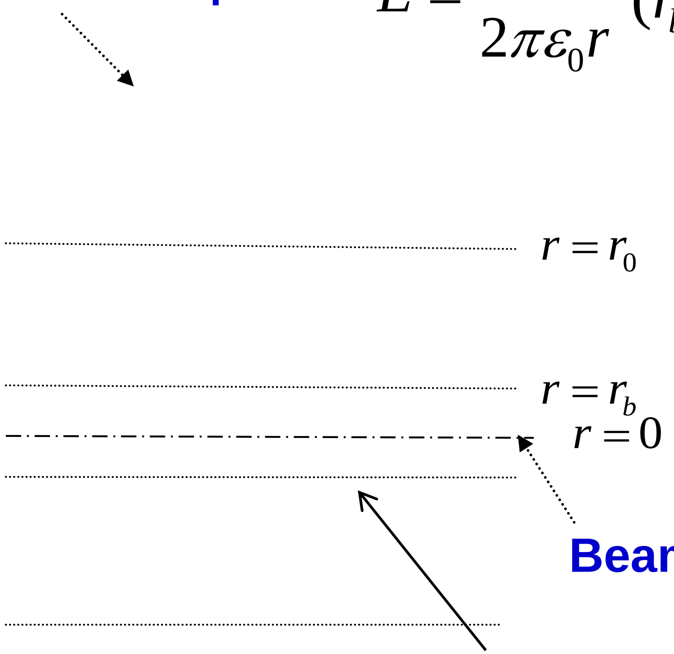
Infinitesimally thin hollow electron beam $(r_b \le r \le r_0)$

$$\varepsilon_0 E 2\pi r l = \rho_l l \quad (r_b \le r \le r_0) \quad \longleftarrow$$ **Gauss's law** ρ_l **= charge per unit length**

Metal envelope

$$E = \frac{\rho_l}{2\pi\varepsilon_0 r} \; (r_b \le r \le r_0)$$

PE = potential energy of the electron of a *thin beam* being the work done in moving the electron along the electric field line from the metal envelope to the position of the thin beam



Beam axis

hollow-beam boundary

$$\mathrm{PE} = W_{thin\,beam} = \int_{r_b}^{r_0} eE\,dr$$

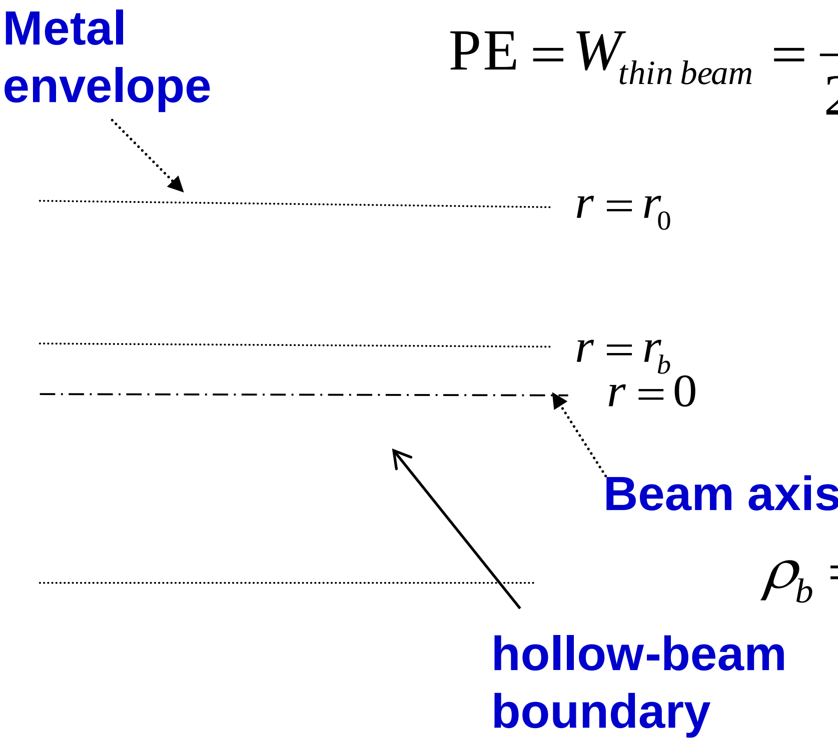
$$\mathrm{PE} = W_{thin\ beam} = \int_{r_b}^{r_0} eE\,dr \longleftarrow \quad E = \frac{\rho_l}{2\pi\varepsilon_0 r}$$

$\rho_l =$ charge per unit length

$$\mathrm{PE} = W_{thin\ beam} = \int_{r_b}^{r_0} e\frac{\rho_l}{2\pi\varepsilon_0 r}\,dr = \frac{e\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$

$$\rho = \rho_b - \rho_i$$

$$\mathrm{PE} = W_{thin\ beam} = \frac{e\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e(\rho_b - \rho_i)\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$



$\rho =$ charge per unit volume

$=$ volume charge density

$\alpha =$ beam cross-sectional area

$\rho_b =$ volume charge density of beam electrons

$\rho_i =$ volume charge density of beam ions

The subscripts *b* and *i* refer to the beam and the ion of the plasma, respectively.

$$\mathrm{PE} = W_{thin\ beam} = \frac{e\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e(\rho_b - \rho_i)\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$

$$\mathrm{PE} = \frac{e(\rho_b - \rho_i)\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho_b(1 - \frac{\rho_i}{\rho_b})\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$

$$\mathrm{PE} = \frac{e\rho_b(1 - \frac{\rho_i}{\rho_b})\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho_b(1 - \frac{n_i e}{n_b e})\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$

$$= \frac{e\rho_b(1 - \frac{n_i}{n_b})\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho_b\alpha(1 - f_n)}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$

$$f_n = \frac{\rho_i}{\rho_b} = \frac{n_i}{n_b}$$

(charge neutralizing factor)

n_i **= number density of plasma ions**

n_b **= number density of beam electrons**

$$\mathrm{PE} = \frac{e\rho_b(1-\frac{n_i}{n_b})\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho_b\alpha(1-f_n)}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$

beam current interpreted as positive

$$I_b = -J\alpha$$

$$J = \rho_b v_b$$

$$-\frac{I_b}{\alpha} = \rho_b v_b$$

$$\mathrm{PE} = -e\frac{I_b}{2\pi\varepsilon_0 v_b}(\ln\frac{r_0}{r_b})(1-f_n)$$

$$\mathrm{PE} = |e|\frac{I_b}{2\pi\varepsilon_0 v_b}(\ln\frac{r_0}{r_b})(1-f_n)$$

$$\mathrm{PE} = W_{thin\,beam} = |e|\frac{I_b}{2\pi\varepsilon_0 v_b}(\ln\frac{r_0}{r_b})(1-f_n)$$

Relativistic kinetic energy of the beam

Relativistic kinetic energy of the beam $= \gamma_c mc^2 - mc^2 = mc^2(\gamma_c - 1) = |e|V_c$

$$\downarrow$$

$$\gamma_c = 1 + |e|V_c / mc^2$$

V_c = Cathode potential
γ_c = Relativistic mass factor corresponding to the potential V_c.

It is assumed that the drift space is grounded and that the electron beam is launched from a cathode at a negative voltage, V_c <0 **with respect to the wall of the drift tube such that the potential within the drift tube will be the potential of the wall.**

The transport of an electron beam of high current is possible in a drift tube when the relativistic kinetic energy of the beam in the limit exceeds its potential energy PE. The expression for the kinetic energy needs to be found giving due consideration to its reduction caused by the space-charge depression in the beam resulting from the depression of the effective accelerating potential in the drift tube. As the beam current increases, the electrons are increasingly slowed down due to the space-charge depression until, at some point, the electron velocity is retarded from v_b **to** nil (=0) **and the beam current reached its limiting value—the so-called space-charge limiting current** $I_{SC\mathrm{lim}}$.

(As the beam current increases, the electrons are increasingly slowed down due to the space-charge depression until, at some point, the electron velocity is retarded from v_b to nil and the beam current reached its limiting value—the so-called space-charge limiting current I_{SClim}).

This in turn corresponds to the reduction of the relativistic kinetic energy of the electron by an amount $mc^2(\gamma_b\text{-}1)$ from its value $mc^2(\gamma_c\text{-}1)$, the latter corresponding to the potential V_c of the drift tube relative to the cathode.

It is assumed that the drift space is grounded and that the electron beam is launched from a cathode at a negative voltage, $V_c<0$, with respect to the wall such that the potential within the drift tube will be the potential of the wall.

(γ_c is the relativistic mass factor corresponding to the potential V_c).

$$\mathrm{KE} = (\gamma_c mc^2 - mc^2) - (\gamma_b mc^2 - mc^2) = \gamma_c mc^2 - \gamma_b mc^2 = mc^2(\gamma_c - \gamma_b)$$

$$\gamma_b = (1 - {v_b}^2 / c^2)^{-1/2}$$

$$\mathrm{PE} = W_{thin\,beam} = |e|\frac{I_b}{2\pi\varepsilon_0 v_b}(\ln\frac{r_0}{r_b})(1-f_n)$$ **(recalled)**

$$\mathrm{KE} = mc^2(\gamma_c - \gamma_b)$$ **(recalled)**

$$\gamma_b = (1 - {v_b}^2/c^2)^{-1/2} \longrightarrow v_b = [({\gamma_b}^2 - 1)^{1/2}/\gamma_b]c$$

$$\gamma_c mc^2 - mc^2 = |e|V_c \longrightarrow \gamma_c = 1 + |e|V_c/mc^2$$

For beam transport, KE>PE. The space-charge limiting current corresponds to

$$\mathrm{KE} = \mathrm{PE}$$

$$\downarrow$$

$$|e|\frac{I_b}{2\pi\varepsilon_0 v_b}(\ln\frac{r_0}{r_b})(1-f_n) = mc^2(\gamma_c - \gamma_b)$$

$$|e|\frac{I_b}{2\pi\varepsilon_0 v_b}(\ln\frac{r_0}{r_b})(1-f_n)=mc^2(\gamma_c-\gamma_b)$$

$$I_b=\frac{2\pi\varepsilon_0 v_b mc^2(\gamma_c-\gamma_b)}{|e|\ln\frac{r_0}{r_b}}\frac{1}{1-f_n} \longleftarrow \quad v_b=[(\gamma_b{}^2-1)^{1/2}/\gamma_b]c$$

$$I_b=\frac{2\pi\varepsilon_0 mc^3(\gamma_b{}^2-1)^{1/2}(\gamma_c-\gamma_b)}{\gamma_b|e|\ln\frac{r_0}{r_b}}\frac{1}{1-f_n} \longleftarrow \quad I_A=\frac{4\pi\varepsilon_0 mc^3}{|e|}=17.1\text{kA}$$

$$\downarrow$$

$$I_b=\frac{I_A}{2\ln\frac{r_0}{r_b}}(\gamma_b{}^2-1)^{1/2}(\frac{\gamma_c-\gamma_b}{\gamma_b})\frac{1}{1-f_n}$$

$$I_b = \frac{I_A}{2\ln\frac{r_0}{r_b}}(\gamma_b{}^2 - 1)^{1/2}\left(\frac{\gamma_c - \gamma_b}{\gamma_b}\right)\frac{1}{1 - f_n}$$

$$\downarrow \quad \longleftarrow \quad \gamma_b = \gamma_c{}^{1/3}$$

I_b **can be maximized**

$$I_{SC\lim} = I_{b\max} = I_b\Big|_{\gamma_b = \gamma_c{}^{1/3}} = \frac{I_A}{2\ln\frac{r_0}{r_b}}(\gamma_c{}^{2/3} - 1)^{3/2}\frac{1}{1 - f_n}$$

$$f_n = \frac{n_i}{n_b}$$

$$\downarrow$$

$$I_{SC\lim} = \frac{I_A}{2\ln\frac{r_0}{r_b}}(\gamma_c{}^{2/3} - 1)^{3/2}\frac{1}{1 - \frac{n_i}{n_e}}$$

$$\downarrow \quad \longleftarrow \quad I_A = \frac{4\pi\varepsilon_0 mc^3}{|e|} = 17.1\mathrm{kA}$$

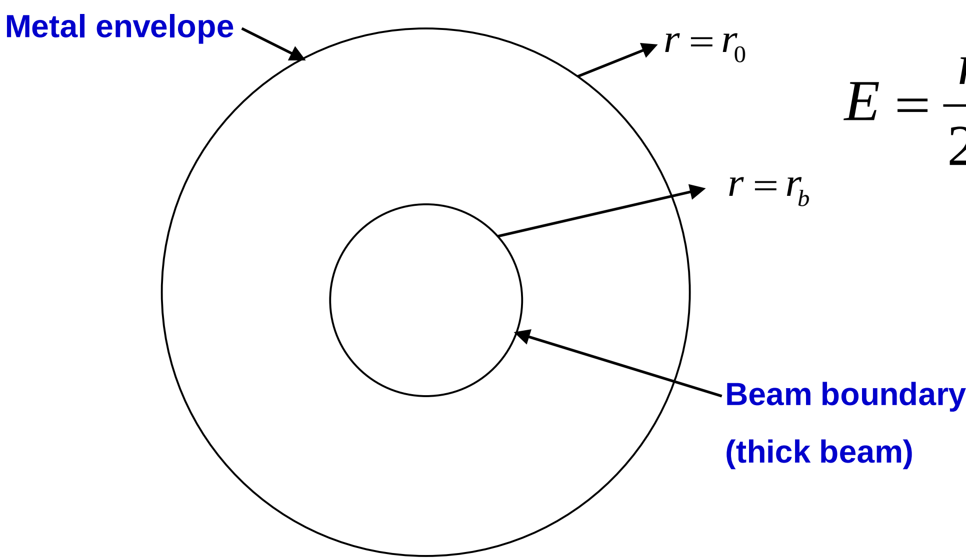
$$I_{SC\lim} = \frac{8.5}{\ln\frac{r_0}{r_b}}(\gamma_c{}^{2/3} - 1)^{3/2}\frac{1}{1 - f_n}\ \mathrm{kA}$$

(Infinitesimally thin hollow electron beam)

Space-charge limiting current of a thick solid electron beam in a metal envelope

Thick solid electron beam in a metal envelope

Electric field inside the beam: $(0 \le r \le r_b)$ $\quad \varepsilon_0 E 2\pi r l = \rho \pi r^2 l$ **(Gauss's law)**



$$E = \frac{r\rho}{2\varepsilon_0} \quad (0 \le r \le r_b)$$

Electric field outside the beam and within the metal envelope: $(r_b \le r \le r_0)$

$$\varepsilon_0 E 2\pi r l = \rho_l l \longrightarrow E = \frac{\rho_l}{2\pi\varepsilon_0 r} = \frac{\rho\alpha}{2\pi\varepsilon_0 r} \quad (r_b \le r \le r_0)$$

Solid thick beam

$$\rho = \rho_b - \rho_i$$

$$E = \frac{r\rho}{2\varepsilon_0} \; (0 \le r \le r_b) \qquad\qquad E = \frac{\rho\alpha}{2\pi\varepsilon_0 r} \; (r_b \le r \le r_0)$$

Potential energy of the electron of a *thick beam* is the work done in moving the electron along the electric field line from the metal envelope to the position of the beam at its axis = PE

$$\mathrm{PE} = W_{thick\ beam} = \int_0^{r_W} eE dr = \int_0^{r_b} \frac{er\rho}{2\varepsilon_0} dr + \int_{r_b}^{r_0} \frac{e\rho\alpha}{2\pi\varepsilon_0 r} dr$$

,

$$= \frac{e\rho}{2\varepsilon_0}\frac{r_b^{\,2}}{2} + \frac{e\rho\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$

$$\mathrm{PE} = \frac{e\rho}{2\varepsilon_0}\frac{r_b^{\,2}}{2} + \frac{e\rho\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho}{2\pi\varepsilon_0}\frac{\pi r_b^{\,2}}{2} + \frac{e\rho\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho}{2\pi\varepsilon_0}\frac{\alpha}{2} + \frac{e\rho\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b}$$

$$\mathrm{PE} = \frac{e\rho}{2\pi\varepsilon_0}\frac{\alpha}{2} + \frac{e\rho\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho\alpha}{4\pi\varepsilon_0} + \frac{e\rho\alpha}{2\pi\varepsilon_0}\ln\frac{r_0}{r_b} = \frac{e\rho\alpha}{4\pi\varepsilon_0}\left(1 + 2\ln\frac{r_0}{r_b}\right)$$

$$\mathrm{PE} = \frac{e\rho\alpha}{4\pi\varepsilon_0}(1 + 2\ln\frac{r_0}{r_b})$$

$$\rho = \rho_b - \rho_i$$

$$f_n = \frac{\rho_i}{\rho_b}$$

$$I_b = -J\alpha$$

$$J = \rho_b v_b$$

$$-\frac{I_b}{\alpha} = \rho_b v_b$$

,

$$\mathrm{PE} = \frac{-eI_b}{4\pi\varepsilon_0 v_b}(1 + 2\ln\frac{r_0}{r_b})(1 - f_n)]$$

$$e = -|e|$$

$$\mathrm{PE} = \frac{|e|I_b}{4\pi\varepsilon_0 v_b}(1 + 2\ln\frac{r_0}{r_b})(1 - f_n)]$$

(solid thick beam)

Solid thick beam | **Infinitesimally thin hollow beam**

Solid thick beam

$$\mathrm{KE} = mc^2(\gamma_c - \gamma_b)$$

$$\mathrm{PE} = \frac{|e| I_b}{4\pi\varepsilon_0 v_b}(1 + 2\ln\frac{r_0}{r_b})(1 - f_n)]$$

Infinitesimally thin hollow beam

$$\mathrm{KE} = mc^2(\gamma_c - \gamma_b)$$

$$\mathrm{PE} = |e| \frac{I_b}{2\pi\varepsilon_0 v_b}(\ln\frac{r_0}{r_b})(1 - f_n)$$

The expression for KE of a solid thick beam and that of an infinitesimally thin beam are the same. However, the corresponding expressions for PE differ only in one factor. The concerned factor for a solid thick beam and that of an infinitesimally thin hollow beam are:

$1 + 2\ln\frac{r_0}{r_b}$ **are** $2\ln\frac{r_0}{r_b}$ **, respectively.**

Therefore, the space-charge limiting current derived from the expression for the potential energy of the solid beam will differ from the corresponding expression for the infinitesimally thin hollow beam only in respect of this factor, the expressions for the kinetic energy remaining unchanged.

Infinitesimally thin hollow beam

$$\mathrm{KE} = mc^2(\gamma_c - \gamma_b)$$

$$\mathrm{PE} = |e|\frac{I_b}{2\pi\varepsilon_0 v_b}(\ln\frac{r_0}{r_b})(1 - f_n)$$

$$I_{SC\lim} = \frac{I_A}{2\ln\frac{r_0}{r_b}}(\gamma_c^{\ 2/3} - 1)^{3/2}\frac{1}{1 - f_n}$$

Solid thick beam

$$\mathrm{KE} = mc^2(\gamma_c - \gamma_b)$$

$$\mathrm{PE} = \frac{|e|I_b}{4\pi\varepsilon_0 v_b}(1 + 2\ln\frac{r_0}{r_b})(1 - f_n)]$$

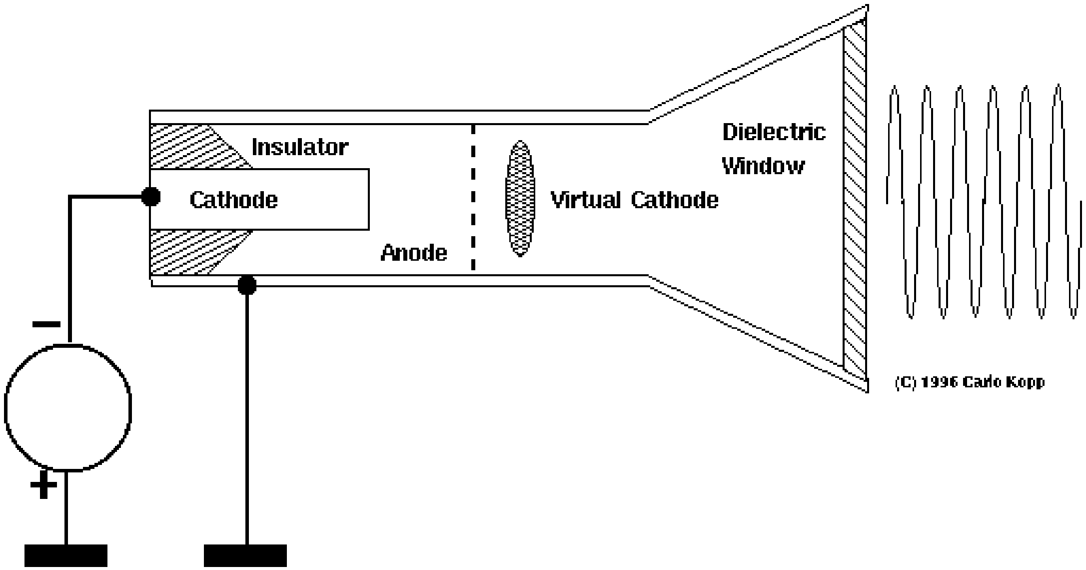
$$I_{SC\lim} = \frac{I_A}{1 + 2\ln\frac{r_0}{r_b}}(\gamma_c^{\ 2/3} - 1)^{3/2}\frac{1}{1 - f_n}$$

$$I_A = \frac{4\pi\varepsilon_0 mc^3}{|e|} = 17.1\ \mathrm{kA}$$

$$f_n = \frac{\rho_i}{\rho_b} = \frac{n_i}{n_b}$$ **(charge neutralizing factor)**

The space-charge limiting current can be increased by placing the metal envelope closer to the beam and/or by increasing the charge neutralizing factor.

In a virtual cathode oscillator, the metal envelope is located far off from the electron beam to increase the ratio r_0/r_b and hence to reduce the value of the space-charge limiting current with a view to forming a virtual cathode.



Virtual cathode oscillator (VIRCATOR)