Appendix to

Lecture on

Synthesis of Pierce Electron Gun

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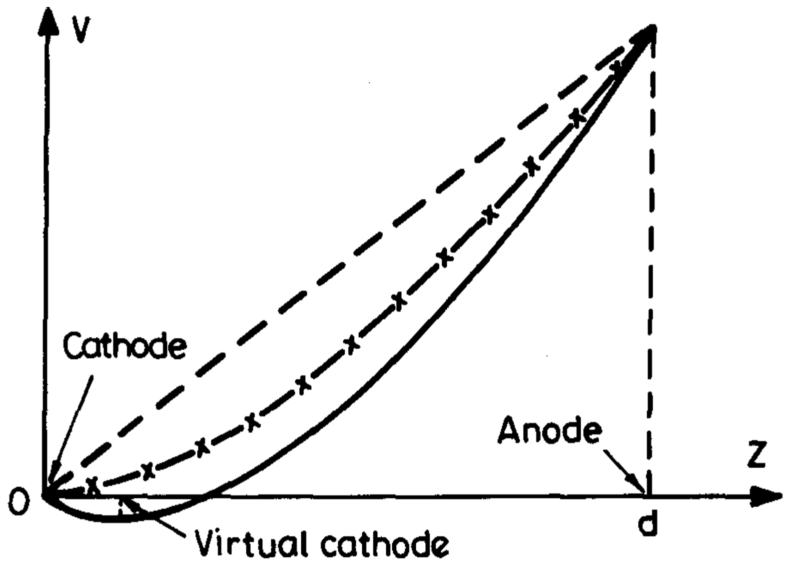
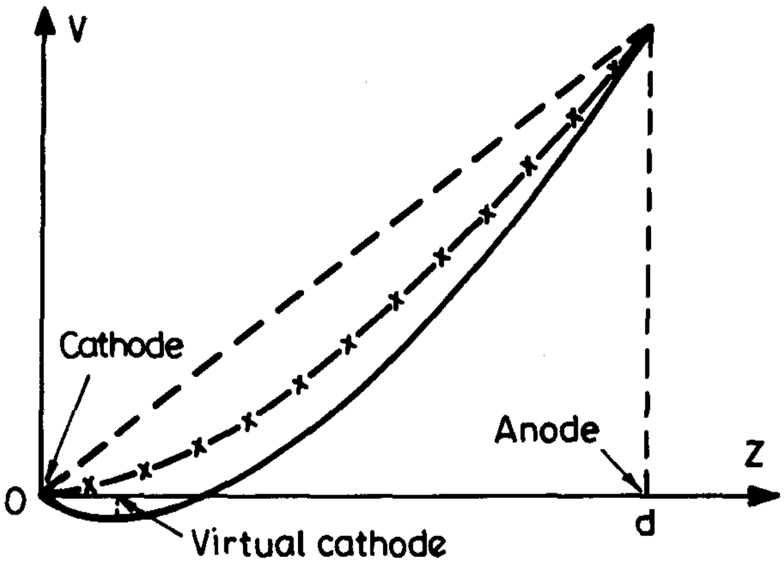
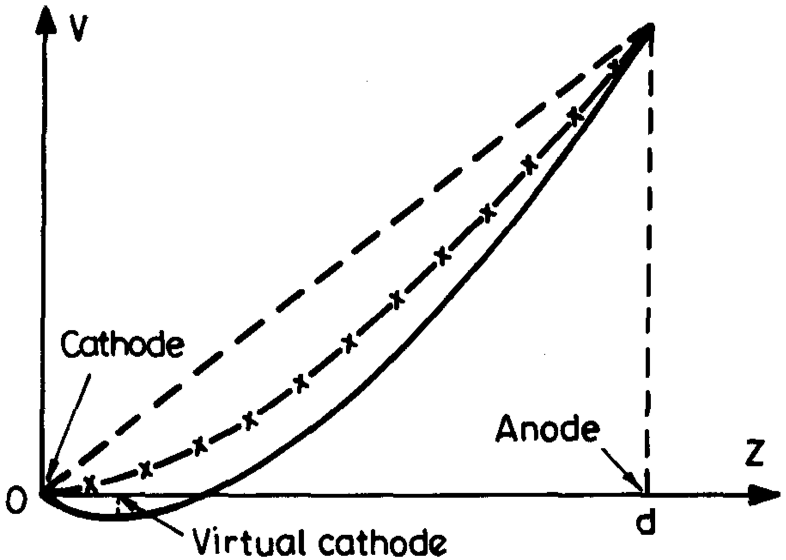
Child-Langmuir's Relation

Potential distribution in a planar diode in the region between two large conducting plates separated by a small distance d and kept at difference of potential V_0. The plate at higher potential $V=V_0$ is the anode. The plate at lower potential $V=0$ is the cathode.

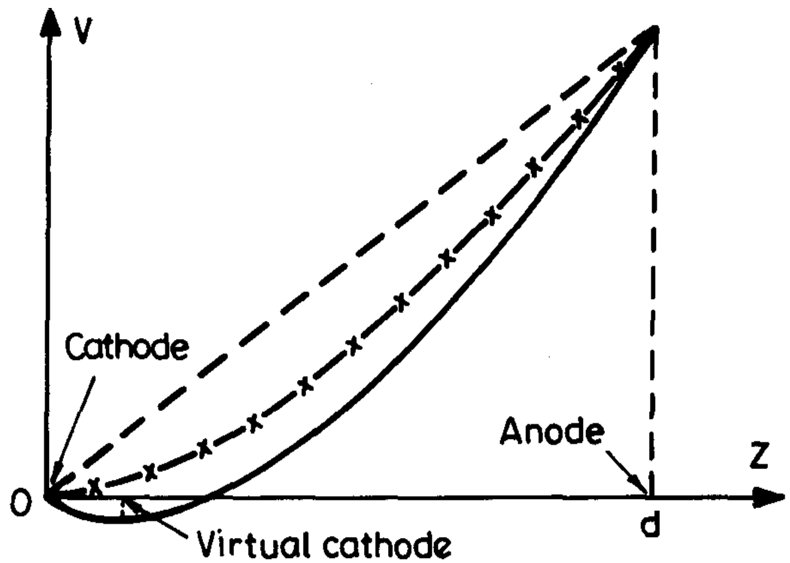




The broken line shows the linear variation of potential in the absence of space charge. The line with crosses gives the variation of potential in the presence of space charge considering a zero velocity of emission at the cathode showing a zero slope at the cathode. The solid line gives the potential variation in the presence of space charge considering a finite velocity of emission at the cathode showing a voltage minimum and zero slope at the *virtual cathode.*



Under the space-charge limited condition, the number of electrons in flight between the cathode and the anode is such that the effect of the *negative* space-charge field due to them at the cathode is neutralized by that of the electrostatic field due to the *positive* potential applied on the anode. The electrons in excess over this number are repelled back into the cathode. This would correspond to a zero slope in the potential variation, and hence to a zero electric field intensity, at the cathode.



The electric flux lines would terminate on the electrons rather than thread into the cathode. The distribution of potential in this case is shown as the line with crosses in the accompanying figure. If the slope of potential variation is positive, more electrons would leave the vicinity of cathode which would increase the negative space charge in the region and hence depress the potential distribution curve towards the zero slope at the cathode. On the other hand, if the slope overshoots to a negative value, the emitted electrons would be forced back to the cathode which would reduce the negative space charge in the region and consequently lift the potential distribution curve to have a zero slope at the cathode.

Potential distribution in the planar diode in the absence of space-charge

$$\frac{\partial^2 V}{\partial^2 z} = 0 \quad \leftarrow \quad \nabla^2 V = \frac{\partial^2 V}{\partial^2 x} + \frac{\partial^2 V}{\partial^2 y} + \frac{\partial^2 V}{\partial^2 z} = 0$$

One-dimensional Laplace's equation

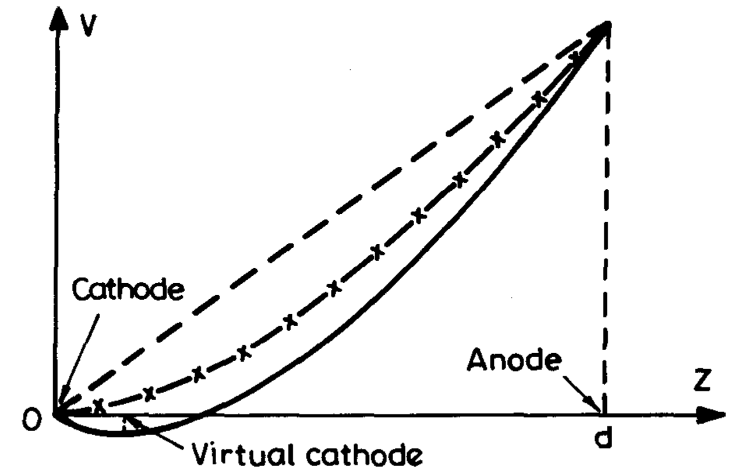
$$\frac{d^2 V}{d^2 z} = 0 \rightarrow \frac{dV}{dz} = A$$

$$V = Az + B \; (A, B : \text{constants})$$

$$\left. \begin{aligned} V &= 0 \text{ at } z = 0 \, (\text{cathode}) \\ V &= V_0 \text{ at } z = d \, (\text{anode}) \end{aligned} \right\}$$

$$\left. \begin{aligned} B &= 0 \\ A &= \frac{V_0}{d} \end{aligned} \right\}$$

$$V = \frac{V_0}{d} z$$



Potential distribution in the cathode-anode region. The broken line shows the linear variation of potential in the absence of space charge.

⇨ **The corresponding electric field is the potential gradient:** V_0 / d

Potential distribution in the planar diode in the presence of space-charge

Solution of Poisson's equation under the potential distribution at equilibrium corresponding to a zero slope at the cathode

$$\rho = \frac{J}{v} = \frac{J}{\left(2\frac{|e|}{m}V\right)^{1/2}} = \frac{J}{(2|\eta|)^{1/2}} V^{-1/2} = \frac{-|J|}{(2|\eta|)^{1/2}} V^{-1/2}$$

$$\left.\begin{aligned} J &= \rho v \\ \frac{1}{2} m v^2 &= |e| V \end{aligned}\right\}$$

$$\frac{\partial^2 V}{\partial^2 z} = \frac{d^2 V}{d^2 z} = -\frac{\rho}{\varepsilon_0} \quad \longleftarrow \quad \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

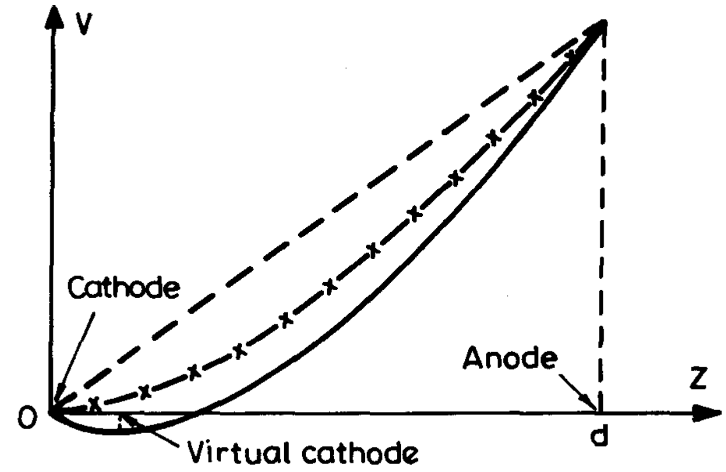
(one-dimensional Poisson's equation)

$$\frac{d^2 V}{d^2 z} = \frac{|J|}{(2|\eta|)^{1/2} \varepsilon_0} V^{-1/2}$$

Solution is sought subject to boundary conditions at the cathode

$$\left.\begin{aligned} \left.\frac{dV}{dz}\right|_{z=0} &= 0 \\ \left.V\right|_{z=0} &= 0 \end{aligned}\right\}$$

(at the cathode boundary)



Potential distribution in the cathode-anode region. **The line with crosses gives the variation of potential in the presence of space charge considering a zero velocity of emission at the cathode and showing a zero slope at the cathode (z=0) held at zero reference potential (V=0).**

$$\frac{d^2V}{d^2z} = \frac{|J|}{(2|\eta|)^{1/2}\varepsilon_0} V^{-1/2}$$

Solution is sought subject to boundary conditions at the cathode

$$\left.\frac{dV}{dz}\right|_{z=0} = 0, \qquad V\big|_{z=0} = 0$$

(at the cathode boundary)

Multiplying both sides by $2dV/dz$

$$2\frac{dV}{dz}\frac{d^2V}{d^2z} = 2\frac{dV}{dz}\frac{|J|}{(2|\eta|)^{1/2}\varepsilon_0} V^{-1/2}$$

$$\frac{d}{dz}\left(\frac{dV}{dz}\right)^2 = 2\frac{dV}{dz}\frac{|J|}{(2|\eta|)^{1/2}\varepsilon_0} V^{-1/2}$$

$$\frac{d}{dz}\left(\frac{dV}{dz}\right)^2 = 2\frac{dV}{dz}\frac{|J|}{(2|\eta|)^{1/2}\varepsilon_0}V^{-1/2}$$

$\downarrow$ $\leftarrow$ **Integrating and putting the integration constant equal to 0 subject to the boundary conditions:** $dV/dz\big|_{z=0} = 0;\ V\big|_{z=0} = 0$

$$\left(\frac{dV}{dz}\right)^2 = 2\frac{|J|}{(2|\eta|)^{1/2}\varepsilon_0}\frac{V^{1/2}}{1/2} = \frac{4|J|}{(2|\eta|)^{1/2}\varepsilon_0}V^{1/2}$$

$\downarrow$

$$\frac{dV}{dz} = \left(\frac{4|J|}{(2|\eta|)^{1/2}\varepsilon_0}\right)^{1/2} V^{1/4}$$

$\downarrow$

$$V^{-1/4}dV = \left(\frac{4|J|}{(2|\eta|)^{1/2}\varepsilon_0}\right)^{1/2} dz$$

$$V^{-1/4}dV = \left(\frac{4|J|}{(2|\eta|)^{1/2}\varepsilon_0}\right)^{1/2} dz$$ **(rewritten)**

↓ ← **Integrating and putting the integration constant equal to** 0 **subject to the boundary conditions:** $V\big|_{z=0} = 0$

$$\frac{V^{3/4}}{\frac{3}{4}} = \left(\frac{4|J|}{(2|\eta|)^{1/2}\varepsilon_0}\right)^{1/2} z$$

↓

$$V^{3/4} = \frac{3}{4}\left(\frac{4|J|}{(2|\eta|)^{1/2}\varepsilon_0}\right)^{1/2} z$$

↓

$$|J| = \left(\frac{4}{9}\right)(2|\eta|)^{1/2}\varepsilon_0 \frac{V^{3/2}}{z^2}$$ **(Child-Langmuir's law)**

$$|J| = \left(\frac{4}{9}\right)(2|\eta|)^{1/2}\varepsilon_0 \frac{V^{3/2}}{z^2}$$ **(rewritten)**

Current distribution in a planar diode

$$|J| = \left(\frac{4}{9}\right)(2|\eta|)^{1/2}\varepsilon_0 \frac{V^{3/2}}{z^2}$$ **(Child-Langmuir's law) (rewritten)**

$\downarrow$

$$\frac{I_0}{A} = \frac{4}{9}\sqrt{2|\eta|}\varepsilon_0 \frac{V_0^{3/2}}{d^2}$$

(Child-Langmuir's relation for a planar diode)

J : **beam current density at a distance z from the cathode**

V : **potential at a distance z from the cathode**

I_0 : **beam current** V_0 : **beam voltage**

A: **cathode cross-sectional area** d : **anode-cathode distance**

ε_0 : **free-space permittivity** η : **charge-to-mass ratio of an electron**

3/2-power law and beam perveance

$$\frac{I_0}{A} = \frac{4}{9}\sqrt{2|\eta|}\varepsilon_0 \frac{V_0^{3/2}}{d^2}$$ **(following from Child-Langmuir's law)**

↓

$$\frac{I_0}{V_0^{3/2}} = \frac{4}{9}\sqrt{2|\eta|}\varepsilon_0 \frac{A}{d^2} = \text{beam perveance}$$

↓

The beam current is proportional to the beam voltage raised to the index of power 3/2. Hence the name 3/2-power law.

The unit of beam perveance is $A/V^{3/2}$ **or** perv

$$\frac{I_0}{A} = \frac{4}{9}\sqrt{2|\eta|}\varepsilon_0 \frac{V_0^{3/2}}{d^2}$$ **(Child-Langmuir's relation for a planar diode)**

⇓

If we increase the distance d between the cathode and the accelerating anode of a diode to provide a hypothetical interaction region and accommodate an interaction structure between them, then the beam current I_0 would reduce to an insignificant value, according to the Child-Langmuir's relation.

With the help of an electron gun, we form an electron beam of the desired beam voltage, current and cross-sectional area with the help of an electron gun and throw it into the interaction region and accommodated an interaction structure between them, then the beam current would reduce to an insignificant value.

Langmuir-Blogett's Relation

Deduction of Langmuir-Blogett's relation

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}=-\frac{\rho}{\varepsilon_0}$$ **(one-dimensional Poisson's equation in spherical-polar coordinates)**

↓ ← **Langmuir-Blogett's solution to obtain**

$$V=\left(\frac{9I_0}{16\pi(2|\eta|^{1/2}\varepsilon_0}\right)^{2/3}G^{4/3}(u)$$

$$u=\ln\frac{r}{r_c}$$

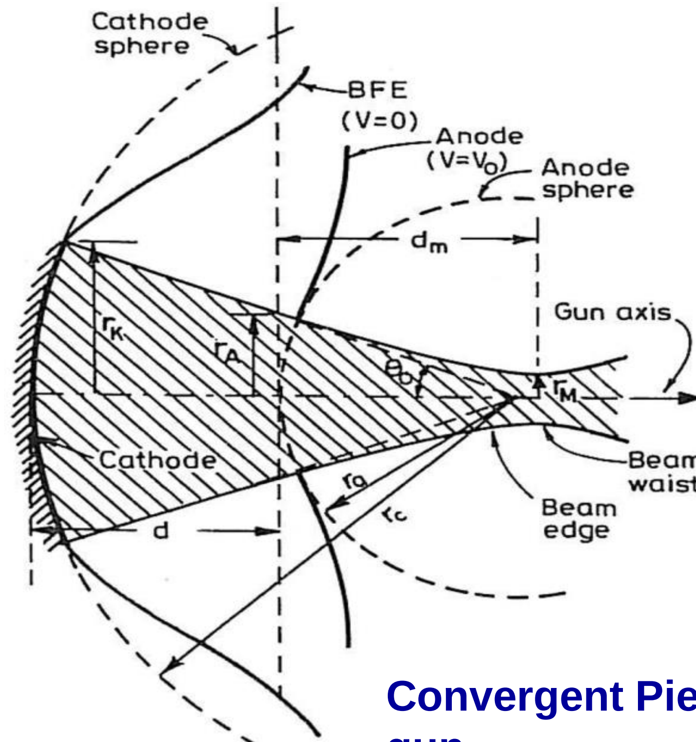
↓

$$V=\left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} r_c^{4/3}G(u)^{4/3}$$

(potential as a function of radial coordinate)

$$G(u)=\ln\frac{r_c}{r}+\frac{3}{10}\ln^2\frac{r_c}{r}+\frac{3}{40}\ln^3\frac{r_c}{r}+\frac{63}{4400}\ln^4\frac{r_c}{r}\ \ldots$$

$A_c=4\pi r_c^2$ **(area of the cathode sphere of radius r_c)**



Convergent Pierce gun

We are going to find this series for a convergent beam

$$V^{3/4} = \frac{3}{4}\left(\frac{4|J|}{(2|\eta|)^{1/2}\varepsilon_0}\right)^{1/2} z$$ (recalled for a planar diode with large cathode and anode)

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = -\frac{\rho}{\varepsilon_0}$$ (Poisson's equation)

$$|J| = \frac{I_0}{A_c}$$ A_c : Cathode area

$$V = \left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} z^{4/3}$$ ← The solution should pass on to

Hence let us choose the following Tailor's series function for the potential:

$$V = \left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} r_c^{4/3} G(u)^{4/3}$$

$$G(u) = G(0) + uG'(0) + \frac{u^2}{2!}G''(0) + \frac{u^3}{3!}G'''(0) + \frac{u^4}{4!}G''''(0) + ...$$

$$u = \ln(r/r_c)$$

$$A_c = 4\pi r_c^2$$

We are going to justify the choice of this potential function.

$$V=\left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} r_c^{4/3} G(u)^{4/3}$$

Function chosen → **Dimensionally same as**

$$V = \left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} z^{4/3}$$

(planar diode Child-Langmuir's relation)

$$K' = \frac{I_0}{4\pi\varepsilon_0(2|\eta|\varepsilon_0)^{1/2}}$$

z **is the distance of a point in the cathode-anode region measured from, and perpendicular to, the cathode:** $z = r_c - r$ **.**

$$V=\left(\frac{4}{9}K'\right)^{2/3} G^{4/3}(u)$$

$$u = \ln(r/r_c) = \ln(r_c - z)/r_c)$$

$$z/r_c \ll 1$$

$$u = \ln\left(1-\frac{z}{r_c}\right) \approx \frac{-z}{r_c}$$

At the cathode: z=0, V=0**,** u=0

$$G(u) \approx G(0) + uG'(0)$$

$$G(0)=0$$

$$G(u) = G(0) + uG'(0) + \frac{u^2}{2!}G''(0) + \frac{u^3}{3!}G'''(0) + \frac{u^4}{4!}G''''(0) + \ldots\ldots$$

$$G(u) = uG'(0)$$

$$u = -z/r_c$$

$$G(u) = -(z/r_c)G'(0)$$

$$V=\left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} r_c^{4/3}G(u)^{4/3}$$

(rewritten)

$\longleftarrow$

$$G(u)=-(z/r_c)G'(0)$$

(rewritten)

$\downarrow$

$$V=\left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} G'(0)^{4/3}z^{4/3}$$

$\downarrow$ $\longleftarrow$ **Provided we put** $\longleftarrow$ $\boxed{G'(0)=\pm 1}$ $\longleftarrow$ $G'(0)^{4/3}=1$

$$V=\left(\frac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} z^{4/3}$$

$\Rightarrow$ **Identical with the planar diode expression**

$\downarrow$

$$G(u)$$

$\downarrow$

Series solution we are looking for $\longrightarrow$

$$\boxed{V=\left(\frac{4}{9}K'\right)^{2/3} G^{4/3}(u)}$$

$$V = \left(\frac{4}{9}K'\right)^{2/3} G^{4/3}(u) \longleftarrow G'(0) = \pm 1$$

We are looking for the series: $G(u)$

Let us start with Poisson's equation:

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = -\frac{\rho}{\varepsilon_0} \longleftarrow \rho = \frac{J}{2|\eta|^{1/2}} V^{-1/2}$$

$$\frac{1}{2}mv^2 = |e|V$$

$$\left.\begin{aligned} \rho &= -|\rho| \\ J &= -|J| \end{aligned}\right\}$$

$$v = (2|\eta|V)^{1/2}$$

$$J = \rho v$$

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = \frac{|J|V^{-1/2}}{(2|\eta|)^{1/2}\varepsilon_0}$$

$$\left.\begin{aligned} |J| &= \frac{I_0}{4\pi r^2} \\ K' &= \frac{I_0}{4\pi\varepsilon_0(2|\eta|\varepsilon_0)^{1/2}} \end{aligned}\right\}$$

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = \frac{K'}{r^2}V^{-1/2}$$

$$V=\left(\frac{4}{9}K'\right)^{2/3}G^{4/3}(u) \quad \longleftarrow \quad G'(0)=\pm 1$$

We are looking for the series: $G(u)$

$$\downarrow$$

$$\left.\begin{aligned}\frac{dV}{dr}&=\left(\frac{4}{3}\right)\left(\frac{9}{4}K'\right)^{\frac{2}{3}}G(u)^{\frac{1}{3}}\frac{dG(u)}{dr}\\ \frac{d^2V}{dr^2}&=\left(\frac{4}{3}\right)\left(\frac{9}{4}K'\right)^{2/3}\left[\frac{1}{3}\right]G(u)^{-2/3}\left(\frac{dG(u)}{dr}\right)^2+G(u)^{1/3}\frac{d^2G(u)}{dr^2}\end{aligned}\right\}$$

$$\downarrow$$

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}=\frac{K'}{r^2}V^{-1/2}$$ **(recalled)**

$$\downarrow$$

$$r^2\left(\frac{dG(u)}{dr}\right)^2+3r^2G(u)\frac{d^2G(u)}{dr^2}+6rG(u)\frac{dG(u)}{dr}-1=0$$

$$\frac{dG}{dr} = \left(\frac{dG(u)}{du}\right)\left(\frac{du}{dr}\right) = \frac{G'(u)}{r}$$

$$\frac{d^2G(u)}{dr^2} = \frac{1}{r}G''(u)\frac{du}{dr} - \frac{1}{r^2}G'(u) = G''(u) - \frac{G'(u)}{r^2}$$

$\longleftarrow$

$$u = \ln\frac{r}{r_c}$$

$$\frac{du}{dr} = \left(\frac{1}{r/r_c}\right)\left(\frac{1}{r_c}\right) = \frac{1}{r}$$

$\downarrow$

$$r^2\left(\frac{dG(u)}{dr}\right)^2 + 3r^2G(u)\frac{d^2G(u)}{dr^2} + 6rG(u)\frac{dG(u)}{dr} - 1 = 0$$ **(rewritten)**

$\downarrow$

$$G'(u)^2 + 3G(u)G''(u) + 3G(u)G'(u) - 1 = 0$$

$$G'(u)^2 + 3G(u)G''(u) + 3G(u)G'(u) - 1 = 0$$

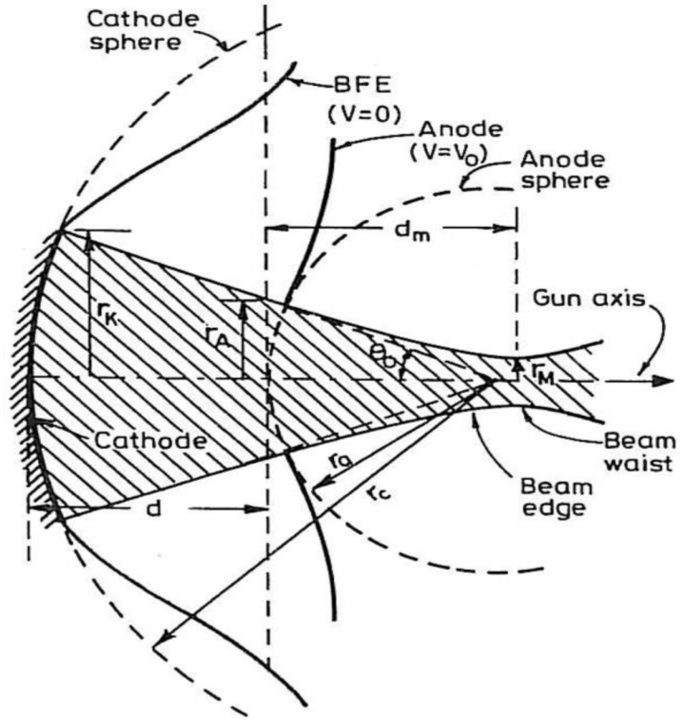
(rewritten)

$$\left.\begin{array}{l}\ln(r/r_c) = \ln 1 = 0 \\ G(0) = 0\end{array}\right\} \text{(cathode)}$$

(recalled)

$$G'(0)^2 = 1 \longrightarrow \boxed{G'(0) = \pm 1}$$

Minus sign corresponds to the convergent beam in which the cathode is outside the anode ($r_c > r_a$).



For a convergent beam, with the decrease of r, that is, with the decrease of u, the potential increases from the cathode towards the anode. That makes at the cathode: $\boxed{G'(0) = -1}$.

$$\left.\begin{array}{l} V = \left(\dfrac{9I_0}{4(2|\eta|)^{1/2}\varepsilon_0 A_c}\right)^{2/3} r_c^{4/3} G(u)^{4/3} \\ u = \ln(r/r_c) \end{array}\right\}$$

Similarly, for a divergent beam: $\boxed{G'(0) = 1}$.

Thus, we can obtain the first two terms of the series:

$$G(u) = G(0) + uG'(0) + \frac{u^2}{2!}G''(0) + \frac{u^3}{3!}G'''(0) + \frac{u^4}{4!}G''''(0) + ...$$

$$G(0) = 0$$

$G'(0) = -1$ **(convergent beam)**

$G'(0) = 1$ **(divergent beam)**

$G'(u)^2 + 3G(u)G''(u) + 3G(u)G'(u) - 1 = 0$ **(recalled)**

Upon differentiation

$$2G'(u)G''(u) + 3G(u)G'''(u) + 3G''(u)G'(u) + 3G(u)G''(u) + 3G'(u)^2 = 0$$

Thus, we can obtain the third term of the series:

$$2G'(u)G''(u)+3G(u)G'''(u)+3G''(u)G'(u)+3G(u)G''(u)+3G'(u)^2=0$$

$G(0)=0$

$G'(0)=-1$ **(convergent beam)**

$G'(0)=1$ **(divergent beam)**

$G''(0)=\dfrac{3}{5}$ **(convergent beam)**

$G''(0)=-\dfrac{3}{5}$ **(divergent beam)**

Thus, similarly we can obtain the fourth term of the series:

$$2G'(u)G''(u)+3G(u)G'''(u)+3G''(u)G'(u)+3G(u)G''(u)+3G'(u)^2=0$$

$$G(0)=0$$

$$G'(0)=-1$$ **(convergent beam)**

$$G'(0)=1$$ **(divergent beam)**

$$G''(0)=\frac{3}{5}$$ **(convergent beam)**

$$G''(0)=-\frac{3}{5}$$ **(divergent beam)**

$$G'''(0)=-\frac{18}{40}$$ **(convergent beam)**

$$G'''(0)=\frac{18}{40}$$ **(divergent beam)**

Following the same procedure we can then find

$$G''''(0)=\frac{189}{550}$$ **(convergent beam)**

$$G''''(0)=-\frac{189}{550}$$ **(divergent beam)**

$$G(u) = G(0) + uG'(0) + \frac{u^2}{2!}G''(0) + \frac{u^3}{3!}G'''(0) + \frac{u^4}{4!}G''''(0) + \ldots\ldots$$

$$u = \ln(r/r_c) \qquad G(0) = 0$$

$$\left.\begin{aligned} G'(0) &= -1 \\ G''(0) &= \frac{3}{5} \\ G'''(0) &= -\frac{18}{40} \\ G''''(0) &= \frac{189}{550} \end{aligned}\right\}$$

(convergent beam)

$$\left.\begin{aligned} G'(0) &= 1 \\ G''(0) &= -\frac{3}{5} \\ G'''(0) &= \frac{18}{40} \\ G''''(0) &= -\frac{189}{550} \end{aligned}\right\}$$

(divergent beam)

$$G(u) = G(\ln\frac{r}{r_c}) = -\ln\frac{r}{r_c} + \frac{3}{10}\ln^2\frac{r}{r_c} - \frac{3}{40}\ln^3\frac{r}{r_c} + \frac{63}{4400}\ln^4\frac{r}{r_c}\ldots$$

(convergent beam)

$$G(u) = G(\ln\frac{r}{r_c}) = \ln\frac{r}{r_c} - \frac{3}{10}\ln^2\frac{r}{r_c} + \frac{3}{40}\ln^3\frac{r}{r_c} - \frac{63}{4400}\ln^4\frac{r}{r_c}\ldots$$

(divergent beam)

$$G(u) = -\ln\frac{r}{r_c} + \frac{3}{10}\ln^2\frac{r}{r_c} - \frac{3}{40}\ln^3\frac{r}{r_c} + \frac{63}{4400}\ln^4\frac{r}{r_c}\ldots$$

(convergent beam)

$$G(u) = \ln\frac{r_c}{r} + \frac{3}{10}\ln^2\frac{r_c}{r} + \frac{3}{40}\ln^3\frac{r_c}{r} + \frac{63}{4400}\ln^4\frac{r_c}{r}\ldots$$

(convergent beam)

$$\left.\begin{aligned} V &= \left(\frac{4}{9}K'\right)^{2/3} G^{4/3}(u) \\ K' &= \frac{I_0}{4\pi\varepsilon_0(2|\eta|\varepsilon_0)^{1/2}} \end{aligned}\right\}$$

Langmuir and Blodgett's relation that can be used in the synthesis of convergent Pierce gun