Engineering Electromagnetics Essentials

Chapter 1

Introduction

and

Chapter 2

Vector calculus expressions for gradient, divergence, and curl

Electromagnetics

The subject is **based on** the fundamental principles of electricity and magnetism

Finds applications in electrical engineering, electronics and communication engineering, and related disciplines.

Objective of the Book

- To make the subject easy to understand

- To give the students the overall essence of the subject

- To uncover in a reasonable period of time the elementary concepts of the subject as is required to appreciate the engineering application of the subject

- To make studying the subject enjoyable and entertaining.

Prerequisites

Electromagnetics is simple because it is mathematical, and it is mathematics that makes the subject simple.

However, very little mathematics is required to develop the understanding of the subject

Required Background

- ❑High-school-level algebra, trigonometry, and calculus
- ❑Definition of vector and scalar quantities
- ❑Meaning of dot and cross products of two vector quantities

Vector Calculus Expressions

It is worth developing the vector calculus expressions at the outset:

- ❑ **Gradient of a scalar quantity**
- ❑ **Divergence of a vector quantity**
- ❑ **Curl of a vector quantity**

Let us develop these expressions in generalized curvilinear system of coordinates.

But why to choose generalized curvilinear system of coordinates?

Why to choose curvilinear system of coordinates for vector calculus expressions?

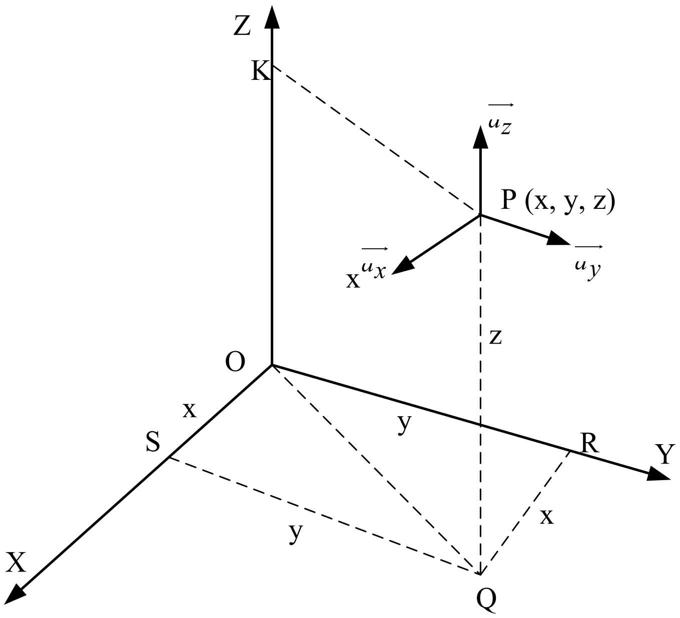
✓Very easy to deduce vector calculus expressions and remember them

✓Much easier to deuce than in cylindrical or spherical system of coordinates

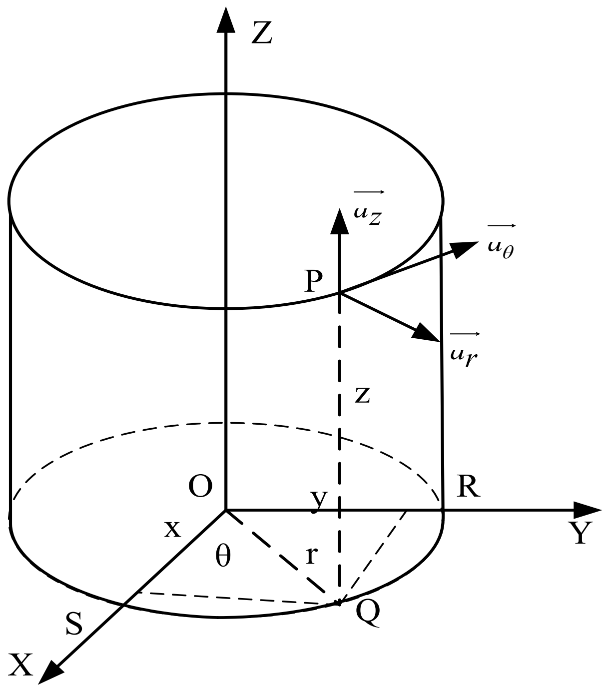
✓Can be written by permuting the first term of a vector calculus expression

✓Can be easily translated in rectangular, cylindrical or spherical system of coordinates

Point P (x,y,z) represented in rectangular system of coordinates

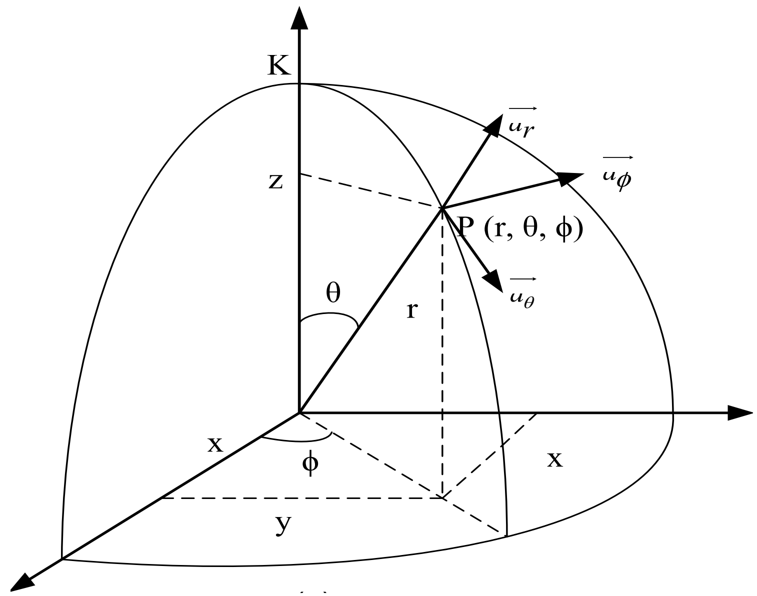


Point P (r,θ,z) represented in cylindrical system of coordinates



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$$

Point P (r,θ,ϕ) represented in spherical system of coordinates



$$
\left.\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}\right\}
$$

Cylindrical vis-à-vis rectangular coordinate

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$$

Spherical vis-à-vis rectangular coordinate

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ \mathrm{z} &= \mathrm{r}\cos\theta \end{aligned}\right\}$$

UNIT VECTORS

Rectangular system

$\vec{a}_x$ $\vec{a}_y$ $\vec{a}_z$ in the directions of increasing *x*, *y*, and *z* respectively

Cylindrical system

$\vec{a}_r$, $\vec{a}_\theta$, $\vec{a}_z$ in the directions of increasing *r*, θ, and *z* respectively

Spherical system

$\vec{a}_r$, $\vec{a}_\theta$, $\vec{a}_\phi$ in the directions of increasing *r*, θ, and ϕ respectively

CROSS PRODUCTS OF UNIT VECTORS

$\vec{a}_x \times \vec{a}_y = \vec{a}_z$ (Rectangular system)

$\vec{a}_r \times \vec{a}_\theta = \vec{a}_z$ (Cylindrical system)

$\vec{a}_r \times \vec{a}_\theta = \vec{a}_\phi$ (Spherical system)

Element of volume in different systems of coordinates

Rectangular system of coordinates

A (x, y, z)

B $(x, y+dy, z)$

C $(x, y, z+dz)$

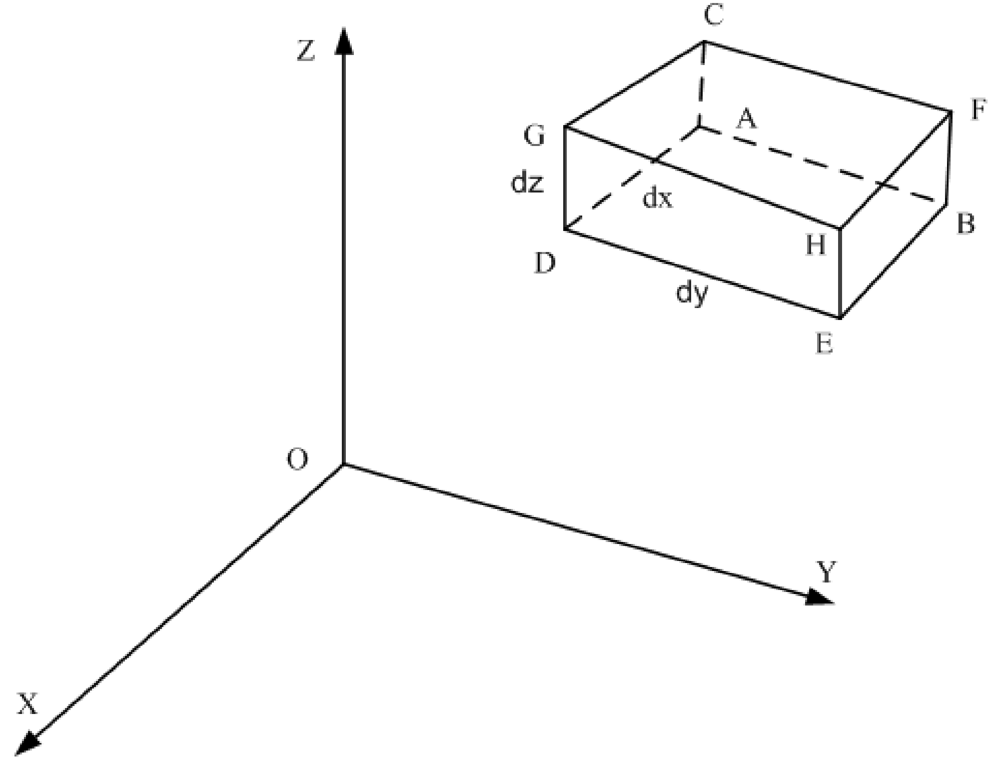
D $(x+dx, y, z)$

E $(x+dx, y+dy, z)$

F $(x, y+dy, z+dz)$

G $(x+dx, y, z+dz)$

H $(x+dx, y+dy, z+dz)$



Element of volume $d\tau = dxdydz$

Cylindrical system of coordinates

A (r,θ,z)

B $(r,\theta+d\theta,z)$

C $(r,\theta,z+dz)$

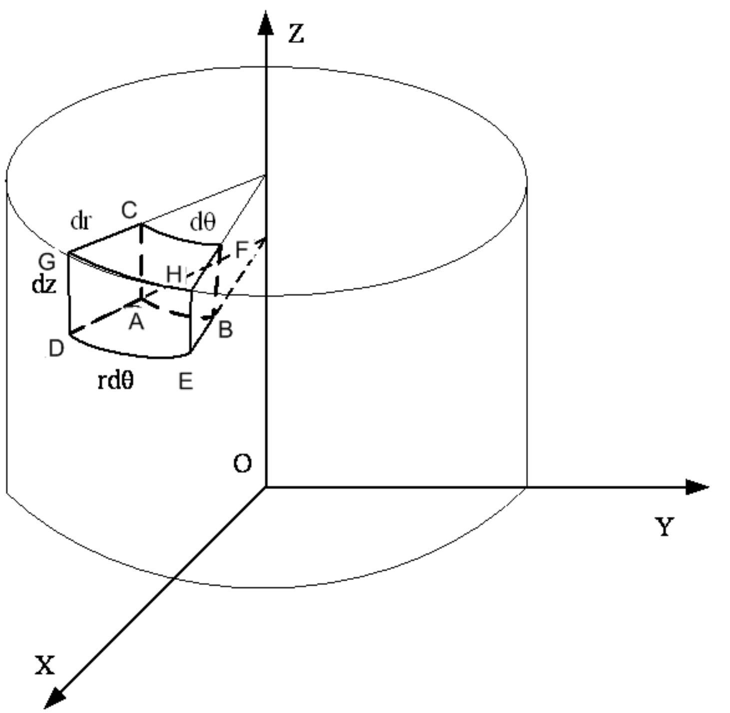
D $(r+dr,\theta,z)$

E $(r+dr,\theta+d\theta,z)$

F $(r,\theta+d\theta,z+dz)$

G $(r+dr,\theta,z+dz)$

H $(r+dr,\theta+d\theta,z+dz)$



C

Element of volume $d\tau = dr\, r d\theta\, dz$

Spherical system of coordinates

A (r, θ, ϕ)

B $(r, \theta + d\theta, \phi)$

C $(r, \theta, \phi + d\phi)$

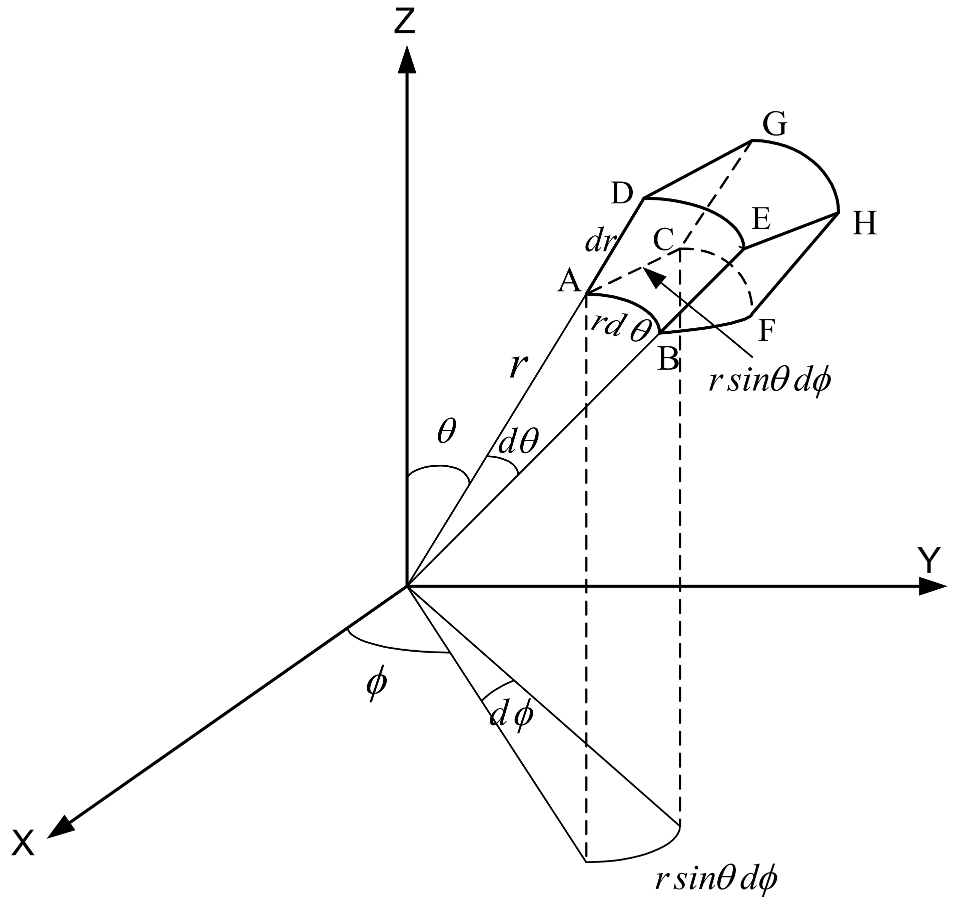
D $(r + dr, \theta, \phi)$

E $(r + dr, \theta + d\theta, \phi)$

F $(r, \theta + d\theta, \phi + d\phi)$

G $(r + dr, \theta, \phi + d\phi)$

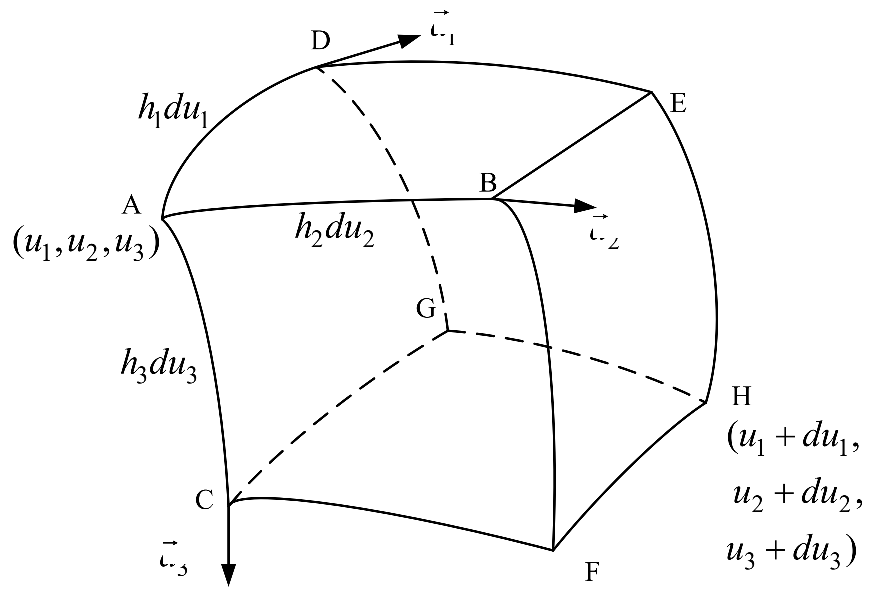
H $(r + dr, \theta + d\theta, \phi + d\phi)$



(c)

Element of volume $d\tau = dr\, rd\theta\; r\sin\theta\, d\phi$

Generalized curvilinear system of coordinates



Element of volume $d\tau = h_1 du_1 h_2 du_2 h_3 du_3$

Generalized curvilinear system Element of volume $d\tau = h_1 du_1 h_2 du_2 h_3 du_3$

Rectangular system Element of volume $d\tau = dxdydz$

$$h_1 = 1, h_2 = 1, h_3 = 1;\ u_1 = x, u_2 = y, u_3 = z$$

Cylindrical system Element of volume $d\tau = dr\, rd\theta\, dz$

$$h_1 = 1, h_2 = r, h_3 = 1;\ u_1 = r, u_2 = \theta, u_3 = z$$

Spherical system Element of volume $d\tau = dr\, rd\theta\ r\sin\theta\, d\phi$

$$h_1 = 1, h_2 = r, h_3 = r\sin\theta;\ u_1 = r, u_2 = \theta, u_3 = \phi$$

Element of distance vector $d\vec{R}$ directed from one point (A) to another (H)

Rectangular system of coordinates

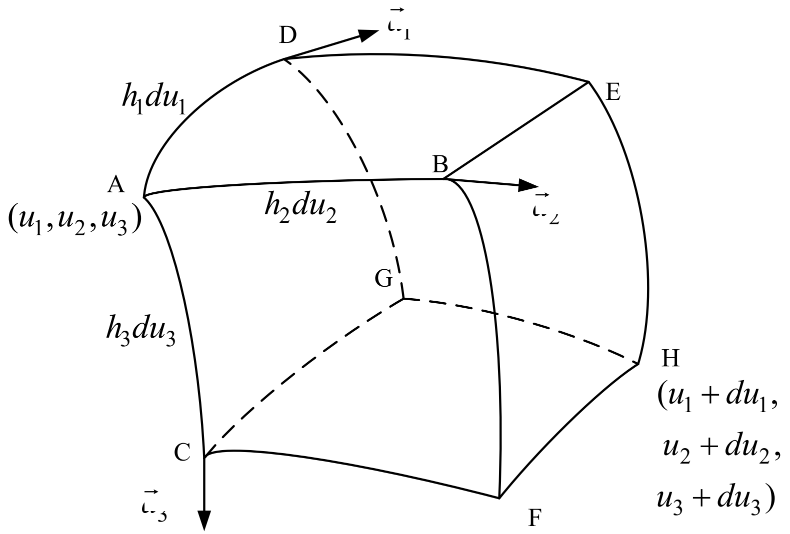
$$d\vec{R} = dx\vec{a}_x + dy\vec{a}_y + dz\vec{a}_z$$

Cylindrical system of coordinates

$$d\vec{R} = dr\vec{a}_r + rd\theta\,\vec{a}_\theta + dz\vec{a}_z$$

Spherical system of coordinates

$$d\vec{R} = dr\vec{a}_r + rd\theta\,\vec{a}_\theta + r\sin\theta\,\vec{a}_\phi$$



Generalized curvilinear system of coordinates

$$d\vec{R} = h_1du_1\vec{a}_1 + h_2du_2\vec{a}_2 + h_3du_3\vec{a}_3$$

Gradient of a scalar quantity

The gradient of a scalar quantity such as electric potential V at a point is a vector quantity.

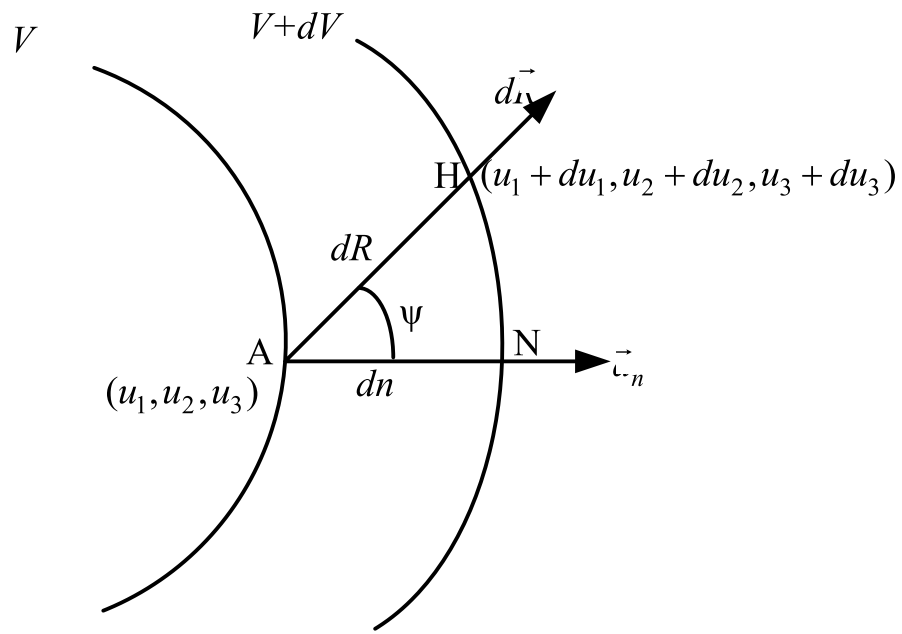
In general, the scalar quantity, here V, varies in space with different rates in different directions.

The magnitude of the gradient of the scalar quantity, here V, is the maximum rate of variation of the quantity in space and the direction of the gradient is the direction in which this maximum rate of variation takes place.

Gradient of a scalar quantity in curvilinear system

Elemental distance vector $d\vec{R}$

directed from A to H situated
on the two equipotentials *V*
and *V+dV*



$$d\vec{R} = h_1 du_1 \vec{a}_1 + h_2 du_2 \vec{a}_2 + h_3 du_3 \vec{a}_3$$

AH $= dR$

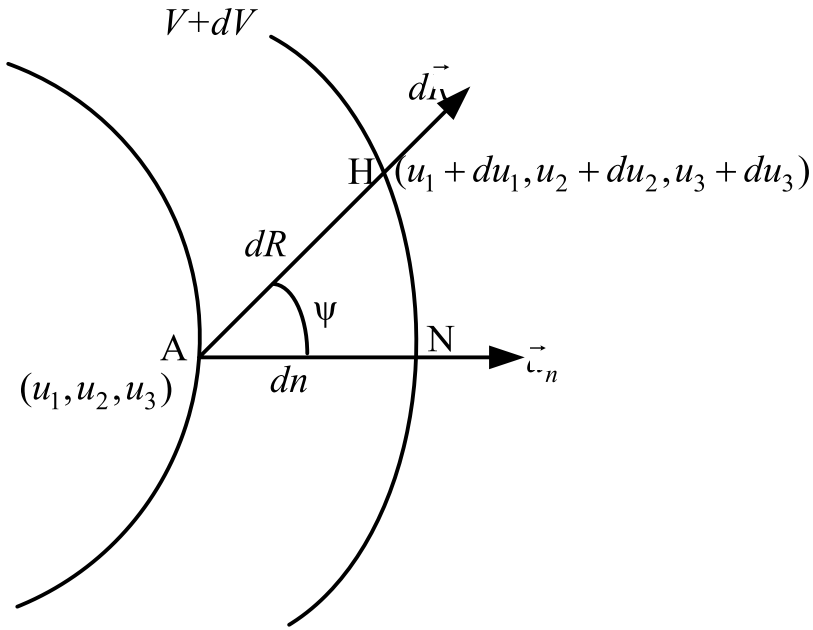
AN $= dn$

AN= AH $\cos\psi$

$$dn = dR\cos\psi$$

$$dn = \vec{a}_{.n} \cdot d\vec{R}$$

$$dn = \vec{a}_{.n} \cdot d\vec{R}$$



$$dV = \frac{\partial V}{\partial n} dn = \frac{\partial V}{\partial n} \vec{a}_n \cdot d\vec{R}$$

$$\text{grad}V = \nabla V = \frac{\partial V}{\partial n} \vec{a}_n$$

$$dV = \nabla V \cdot d\vec{R}$$

$$\nabla V = (\nabla V)_1 \vec{a}_1 + (\nabla V)_2 \vec{a}_2 + (\nabla V)_3 \vec{a}_3$$

$$dV = \nabla V \cdot d\vec{R}$$

$$d\vec{R} = h_1 du_1 \vec{a}_1 + h_2 du_2 \vec{a}_2 + h_3 du_3 \vec{a}_3$$

$$dV = [(\nabla V)_1 \vec{a}_1 + (\nabla V)_2 \vec{a}_2 + (\nabla V)_3 \vec{a}_3] \cdot [h_1 du_1 \vec{a}_1 + h_2 du_2 \vec{a}_2 + h_3 du_3 \vec{a}_3]$$

$$dV = (\nabla V)_1 h_1 du_1 + (\nabla V)_2 h_2 du_2 + (\nabla V)_3 h_3 du_3$$

Also,

$$dV = \frac{\partial V}{\partial u_1} du_1 + \frac{\partial V}{\partial u_2} du_2 + \frac{\partial V}{\partial u_3} du_3$$

$$dV = (\nabla V)_1 h_1 du_1 + (\nabla V)_2 h_2 du_2 + (\nabla V)_3 h_3 du_3$$

Also,

$$dV = \frac{\partial V}{\partial u_1} du_1 + \frac{\partial V}{\partial u_2} du_2 + \frac{\partial V}{\partial u_3} du_3$$

Equating the right hand sides

$$\left.\begin{aligned} \frac{\partial V}{\partial u_1} &= (\nabla V)_1 h_1 \\ \frac{\partial V}{\partial u_2} &= (\nabla V)_2 h_2 \\ \frac{\partial V}{\partial u_3} &= (\nabla V)_3 h_3 \end{aligned}\right\}$$

$$\left.\begin{aligned}(\nabla V)_1 &= \frac{1}{h_1}\frac{\partial V}{\partial u_1}\\ (\nabla V)_2 &= \frac{1}{h_2}\frac{\partial V}{\partial u_2}\\ (\nabla V)_3 &= \frac{1}{h_3}\frac{\partial V}{\partial u_3}\end{aligned}\right\} \longleftarrow \left.\begin{aligned}\frac{\partial V}{\partial u_1} &= (\nabla V)_1 h_1\\ \frac{\partial V}{\partial u_2} &= (\nabla V)_2 h_2\\ \frac{\partial V}{\partial u_3} &= (\nabla V)_3 h_3\end{aligned}\right\}$$

$$\nabla V = (\nabla V)_1 \vec{a}_1 + (\nabla V)_2 \vec{a}_2 + (\nabla V)_3 \vec{a}_3$$

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u_1}\vec{a}_1 + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\vec{a}_2 + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\vec{a}_3$$

(Gradient of scalar *V* in generalized curvilinear system of coordinates)

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u_1}\vec{a}_1 + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\vec{a}_2 + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\vec{a}_3$$ (Curvilinear system of coordinates)

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial y}\vec{a}_y$$ (Rectangular system of coordinates)

$$(h_1 = 1, h_2 = 1, h_3 = 1;\ u_1 = x, u_2 = y, u_3 = z)$$

$$\nabla V = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{\partial V}{\partial z}\vec{a}_z$$ (Cylindrical system of coordinates)

$$(h_1 = 1, h_2 = r, h_3 = 1;\ u_1 = r, u_2 = \theta, u_3 = z)$$

$$\nabla V = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$$ (Spherical system of coordinates)

$$(h_1 = 1, h_2 = r, h_3 = r\sin\theta;\ u_1 = r, u_2 = \theta, u_3 = \phi)$$

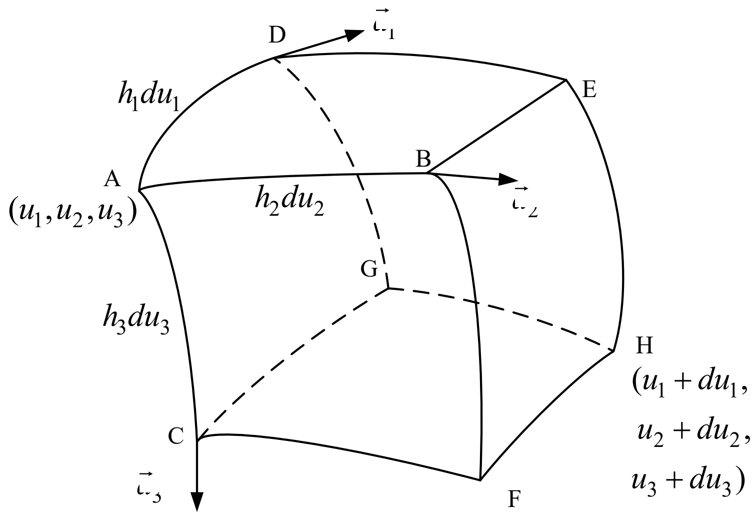
Divergence of a vector quantity

The divergence of a vector quantity such as electric field $\vec{E}$ at a point is a scalar quantity.

The divergence of a vector quantity such as electric field $\vec{E}$ at a point is the outward flux of $\vec{E}$ through the surface of an elemental or differential volume $\Delta\tau$ enclosing the point (closed surface integral of $\vec{E}$ over the surface of the enclosure) divided by the volume element $\Delta\tau$ in the limit the volume element $\Delta\tau$ tending to zero, thereby the volume element shrinking to the point.

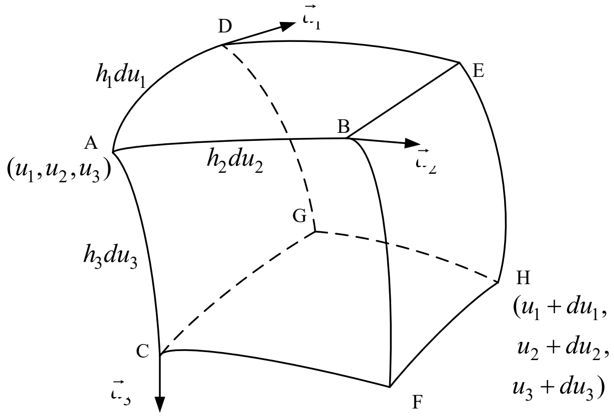
$$\mathrm{div}\vec{E} = \nabla\cdot\vec{E} = \underset{\Delta\tau\to 0}{Lt}\ \frac{\oint_S \vec{E}\cdot\vec{a}_n\, dS}{\Delta\tau}$$

Divergence of a vector quantity in curvilinear system



$$\mathrm{div}\vec{E} = \nabla \cdot \vec{E} = \underset{\Delta\tau \to 0}{Lt} \frac{\oint_S \vec{E} \cdot \vec{a}_n \, dS}{\Delta\tau}$$

Outward flux through the faces normal to the unit vector $\vec{a}_1$



$F_E =$ Flux through the face ABFC in the direction of $\vec{a}_1$

$F_E =$ Inward flux through the face ABFC

$-F_E =$ Outward flux through the face ABFC

$$F_E = (E_1\vec{a}_1 + E_2\vec{a}_2 + E_3\vec{a}_3).(dS_1)\vec{a}_1 = E_1 dS_1$$

$$dS_1 = (\mathrm{AB})(\mathrm{AC}) = (h_2 du_2)(h_3 du_3)$$

$$F_E = E_1 h_2 du_2 h_3 du_3$$

$$F_E = E_1 h_2 du_2 h_3 du_3$$

$F_E + \frac{\partial F_E}{\partial u_1} du_1 =$ Flux through the face DEHG in the direction of $\vec{a}_1$

Outward flux through the face DEHG

$-F_E =$ Outward flux through the face ABFC

Outward flux through the faces ABFC and DEHG each normal to the unit vector $\vec{a}_1$

$$= -F_E + F_E + \frac{\partial F_E}{\partial u_1} du_1 = \frac{\partial F_E}{\partial u_1} du_1 = (du_1 du_2 du_3) \frac{\partial}{\partial u_1} (E_1 h_2 h_3)$$

Outward flux through the faces each normal to the unit vector $\vec{a}_1$

$$= (du_1 du_2 du_3) \frac{\partial}{\partial u_1} (E_1 h_2 h_3)$$

Outward flux through the faces each normal to the unit vector $\vec{a}_2$

$$= (du_1 du_2 du_3) \frac{\partial}{\partial u_2} (E_2 h_3 h_1)$$

Outward flux through the faces each normal to the unit vector $\vec{a}_3$

$$= (du_1 du_2 du_3) \frac{\partial}{\partial u_3} (E_3 h_1 h_2)$$

Outward flux through all the faces of the differential volume

$$= (du_1 du_2 du_3) \left[\frac{\partial}{\partial u_1} (E_1 h_2 h_3) + \frac{\partial}{\partial u_2} (E_2 h_3 h_1) + \frac{\partial}{\partial u_3} (E_3 h_1 h_2) \right]$$

Outward flux through all the faces of the differential volume

$$= (du_1 du_2 du_3)\left[\frac{\partial}{\partial u_1}(E_1 h_2 h_3) + \frac{\partial}{\partial u_2}(E_2 h_3 h_1) + \frac{\partial}{\partial u_3}(E_3 h_1 h_2)\right]$$

$$\mathrm{div}\vec{E} = \nabla \cdot \vec{E} = \underset{\Delta\tau \to 0}{Lt} \frac{\oint_S \vec{E} \cdot \vec{a}_n \, dS}{\Delta\tau}$$

$$\nabla \cdot \vec{E} = \underset{(du_1, du_2, du_3) \to 0}{Lt} \frac{(du_1 du_2 du_3)\left[\frac{\partial}{\partial u_1}(E_1 h_2 h_3) + \frac{\partial}{\partial u_2}(E_2 h_3 h_1) + \frac{\partial}{\partial u_3}(E_3 h_1 h_2)\right]}{h_1 du_1 h_2 du_2 h_3 du_3}.$$

$$\nabla \cdot \vec{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 E_1)}{\partial u_1} + \frac{\partial(h_3 h_1 E_2)}{\partial u_2} + \frac{\partial(h_1 h_2 E_3)}{\partial u_3}\right]$$

$$\nabla \cdot \vec{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 E_1)}{\partial u_1} + \frac{\partial(h_3 h_1 E_2)}{\partial u_2} + \frac{\partial(h_1 h_2 E_3)}{\partial u_3}\right]$$

(Curvilinear system of coordinates)

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

(Rectangular system of coordinates)

$$(h_1 = 1, h_2 = 1, h_3 = 1;\ u_1 = x, u_2 = y, u_3 = z)$$

$$\nabla \cdot \vec{E} = \frac{1}{r}\left[\frac{\partial(rE_r)}{\partial r} + \frac{\partial E_\theta}{\partial \theta} + \frac{\partial(rE_z)}{\partial z}\right] = \frac{1}{r}\frac{\partial(rE_r)}{\partial r} + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z}$$

(Cylindrical system of coordinates)

$$(h_1 = 1, h_2 = r, h_3 = 1;\ u_1 = r, u_2 = \theta, u_3 = z)$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial(r^2 \sin\theta\, E_r)}{\partial r} + \frac{\partial(r \sin\theta\, E_\theta)}{\partial \theta} + \frac{\partial(rE_\phi)}{\partial \phi}\right]$$

$$= \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, E_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(E_\phi)}{\partial \phi}$$

(Spherical system of coordinates)

$$(h_1 = 1, h_2 = r, h_3 = r\sin\theta;\ u_1 = r, u_2 = \theta, u_3 = \phi)$$

Curl of a vector quantity

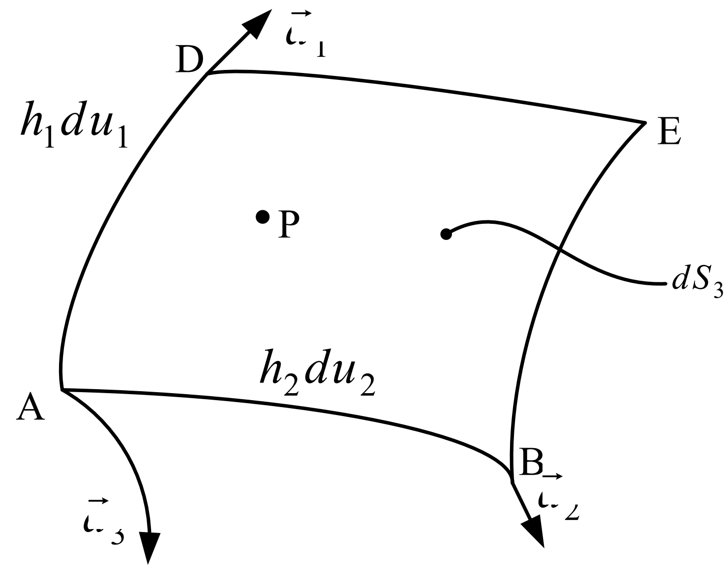
The curl $\nabla\times\vec{E}$ of a vector quantity such as electric field $\vec{E}$ at a point is a vector quantity.

$$\nabla\times\vec{E}=(\nabla\times\vec{E})_1\vec{a}_1+(\nabla\times\vec{E})_2\vec{a}_2+(\nabla\times\vec{E})_3\vec{a}_3$$

$(\nabla\times\vec{E})_1,(\nabla\times\vec{E})_2,(\nabla\times\vec{E})_3$: Components in the directions of $\vec{a}_1,\vec{a}_2,\vec{a}_3$ respectively

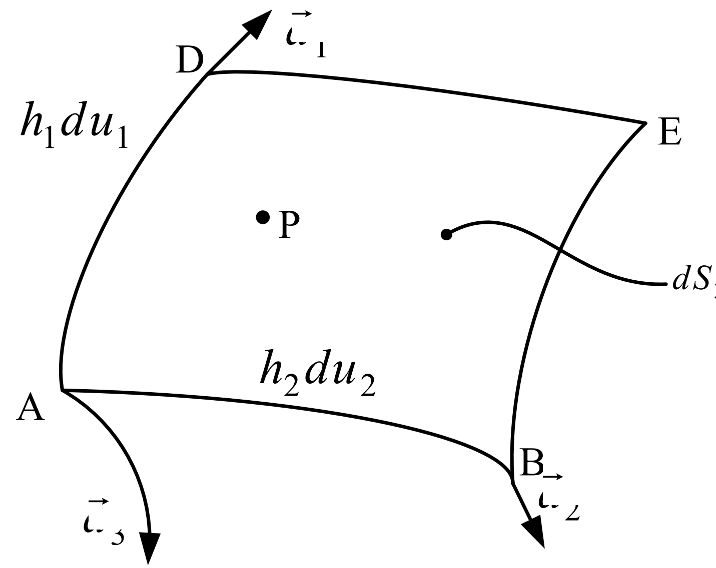
$$(\nabla\times\vec{E})_3=\underset{\Delta S_3\to 0}{Lt}\frac{\oint_l \vec{E}\cdot d\vec{l}}{\Delta S_3}$$

$$\Delta S_3=h_1du_1h_2du_2$$



$$(\nabla \times \vec{E})_3 = \underset{\Delta S_3 \to 0}{Lt} \frac{\oint_l \vec{E} \cdot d\vec{l}}{\Delta S_3}$$

$$\Delta S_3 = h_1 du_1 h_2 du_2$$

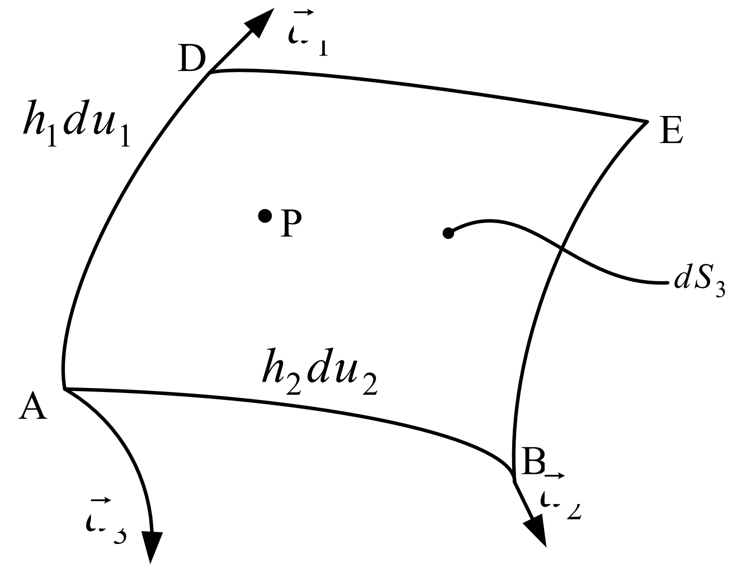


$$(\nabla \times \vec{E})_3 = \underset{dS_3 \to 0}{Lt} \frac{dI_{E(\mathrm{A}\to\mathrm{D})} + dI_{E(\mathrm{D}\to\mathrm{E})} + dI_{E(\mathrm{E}\to\mathrm{B})} + dI_{E(\mathrm{B}\to\mathrm{A})}}{dS_3}$$

dI_E stands for line integral.

$$(\nabla \times \vec{E})_3 = \underset{dS_3 \to 0}{Lt} \frac{dI_{E(\mathrm{A}\to\mathrm{D})} + dI_{E(\mathrm{D}\to\mathrm{E})} + dI_{E(\mathrm{E}\to\mathrm{B})} + dI_{E(\mathrm{B}\to\mathrm{A})}}{dS_3}$$

dI_E stands for line integral.

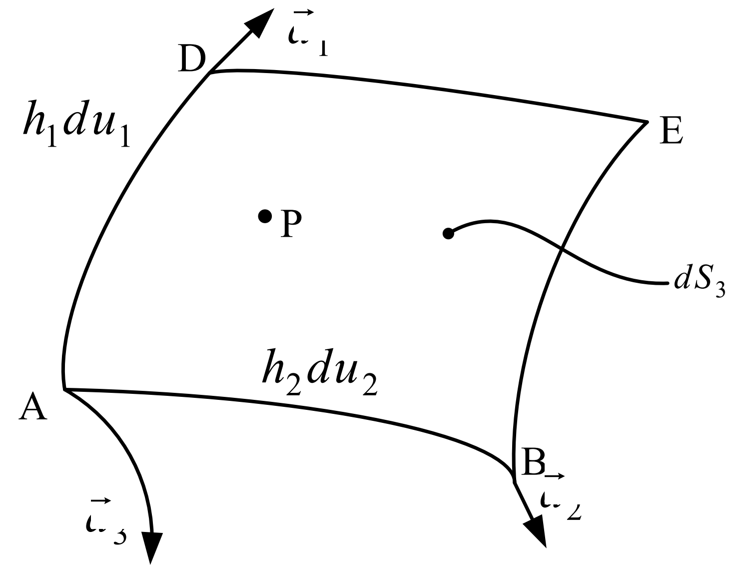


$$dI_{E(\mathrm{A}\to\mathrm{D})} = \vec{E} \cdot (h_1 du_1 \vec{a}_1)$$

$$= (E_1\vec{a}_1 + E_2\vec{a}_2 + E_3\vec{a}_3).(h_1 du_1 \vec{a}_1) = E_1 h_1 du_1 = L$$

$$dI_{E(\mathrm{A}\rightarrow\mathrm{D})} = \vec{E}\cdot(h_1 du_1 \vec{a}_1)$$

$$= (E_1\vec{a}_1 + E_2\vec{a}_2 + E_3\vec{a}_3).(h_1 du_1 \vec{a}_1) = E_1 h_1 du_1 = L$$

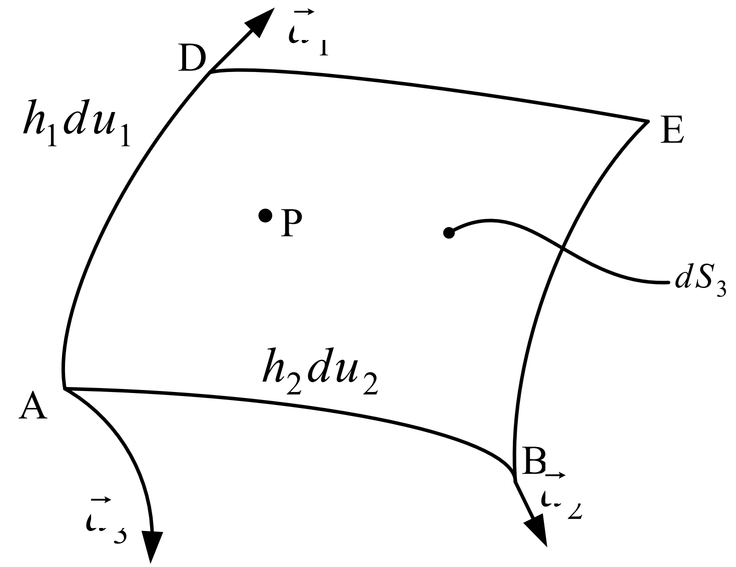


$$dI_{E(\mathrm{B}\rightarrow\mathrm{E})} = L + \frac{\partial L}{\partial u_2} du_2$$

$$dI_{E(\mathrm{E}\rightarrow\mathrm{B})} = -(L + \frac{\partial L}{\partial u_2} du_2)$$

$$dI_{E(\mathrm{A}\to\mathrm{B})} = (E_1\vec{a}_1 + E_2\vec{a}_2 + E_3\vec{a}_3).(h_2 du_2 \vec{a}_2) = E_2 h_2 du_2 = L'$$

$$dI_{E(\mathrm{B}\to\mathrm{A})} = -L'$$



$$dI_{E(\mathrm{D}\to\mathrm{E})} = L' + \frac{\partial L'}{\partial u_1} du_1$$

$$dI_{E(\mathrm{A}\to\mathrm{D})} + dI_{E(\mathrm{D}\to\mathrm{E})} + dI_{E(\mathrm{E}\to\mathrm{B})} + dI_{E(\mathrm{B}\to\mathrm{A})}$$

$$= L + L' + \frac{\partial L'}{\partial u_1} du_1 - (L + \frac{\partial L}{\partial u_2} du_2) - L' = \frac{\partial L'}{\partial u_1} du_1 - \frac{\partial L}{\partial u_2} du_2$$

$$dI_{E(\mathrm{A}\to\mathrm{D})} + dI_{E(\mathrm{D}\to\mathrm{E})} + dI_{E(\mathrm{E}\to\mathrm{B})} + dI_{E(\mathrm{B}\to\mathrm{A})}$$

$$= L + L' + \frac{\partial L'}{\partial u_1} du_1 - (L + \frac{\partial L}{\partial u_2} du_2) - L' = \frac{\partial L'}{\partial u_1} du_1 - \frac{\partial L}{\partial u_2} du_2$$

$$L = E_1 h_1 du_1$$

$$L' = E_2 h_2 du_2$$

$$dI_{E(\mathrm{A}\to\mathrm{D})} + dI_{E(\mathrm{D}\to\mathrm{E})} + dI_{E(\mathrm{E}\to\mathrm{B})} + dI_{E(\mathrm{B}\to\mathrm{A})} = \frac{\partial (h_2 E_2)}{\partial u_1} du_1 du_2 - \frac{\partial (h_1 E_1)}{\partial u_2} du_1 du_2$$

$$(\nabla \times \vec{E})_3 = \underset{dS_3 \to 0}{Lt} \frac{dI_{E(\mathrm{A}\to\mathrm{D})} + dI_{E(\mathrm{D}\to\mathrm{E})} + dI_{E(\mathrm{E}\to\mathrm{B})} + dI_{E(\mathrm{B}\to\mathrm{A})}}{dS_3}$$

$$dI_{E(\mathrm{A}\to\mathrm{D})} + dI_{E(\mathrm{D}\to\mathrm{E})} + dI_{E(\mathrm{E}\to\mathrm{B})} + dI_{E(\mathrm{B}\to\mathrm{A})} = \frac{\partial(h_2 E_2)}{\partial u_1} du_1 du_2 - \frac{\partial(h_1 E_1)}{\partial u_2} du_1 du_2$$

$$\downarrow$$

$$(\nabla \times \vec{E})_3 = \underset{dS_3 \to 0}{Lt} \frac{dI_{E(\mathrm{A}\to\mathrm{D})} + dI_{E(\mathrm{D}\to\mathrm{E})} + dI_{E(\mathrm{E}\to\mathrm{B})} + dI_{E(\mathrm{B}\to\mathrm{A})}}{dS_3}$$

$$(\nabla \times \vec{E})_3 = \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 E_2)}{\partial u_1} - \frac{\partial(h_1 E_1)}{\partial u_2} \right)$$

Similarly,

$$(\nabla \times \vec{E})_1 = \frac{1}{h_1 h_2}\left(\frac{\partial(h_3 E_3)}{\partial u_1} - \frac{\partial(h_2 E_2)}{\partial u_2} \right)$$

$$(\nabla \times \vec{E})_2 = \frac{1}{h_3 h_1}\left(\frac{\partial(h_1 E_1)}{\partial u_3} - \frac{\partial(h_3 E_3)}{\partial u_1} \right)$$

$$\nabla \times \vec{E} = (\nabla \times \vec{E})_1 \vec{a}_1 + (\nabla \times \vec{E})_2 \vec{a}_2 + (\nabla \times \vec{E})_3 \vec{a}_3$$

$$(\nabla \times \vec{E})_1 = \frac{1}{h_1 h_2}\left(\frac{\partial(h_3 E_3)}{\partial u_1} - \frac{\partial(h_2 E_2)}{\partial u_2}\right)$$

$$(\nabla \times \vec{E})_2 = \frac{1}{h_3 h_1}\left(\frac{\partial(h_1 E_1)}{\partial u_3} - \frac{\partial(h_3 E_3)}{\partial u_1}\right)$$

$$(\nabla \times \vec{E})_3 = \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 E_2)}{\partial u_1} - \frac{\partial(h_1 E_1)}{\partial u_2}\right)$$

$$\nabla \times \vec{E} = \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 E_3)}{\partial u_2} - \frac{\partial(h_2 E_2)}{\partial u_3}\right)\vec{a}_1 + \frac{1}{h_3 h_1}\left(\frac{\partial(h_1 E_1)}{\partial u_3} - \frac{\partial(h_3 E_3)}{\partial u_1}\right)\vec{a}_2 + \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 E_2)}{\partial u_1} - \frac{\partial(h_1 E_1)}{\partial u_2}\right)\vec{a}_3$$

$$\nabla \times \vec{E} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \vec{a}_1 & h_2 \vec{a}_2 & h_3 \vec{a}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix}$$

$$\nabla\times\vec{E}=\frac{1}{h_2h_3}\left(\frac{\partial(h_3E_3)}{\partial u_2}-\frac{\partial(h_2E_2)}{\partial u_3}\right)\vec{a}_1+\frac{1}{h_3h_1}\left(\frac{\partial(h_1E_1)}{\partial u_3}-\frac{\partial(h_3E_3)}{\partial u_1}\right)\vec{a}_2+\frac{1}{h_1h_2}\left(\frac{\partial(h_2E_2)}{\partial u_1}-\frac{\partial(h_1E_1)}{\partial u_2}\right)\vec{a}_3$$

(Curvilinear system of coordinates)

$$\nabla\times\vec{E}=\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\vec{a}_y+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\vec{a}_z$$

(Rectangular system of coordinates)

$$(h_1=1,h_2=1,h_3=1;\ u_1=x,u_2=y,u_3=z)$$

$$\nabla\times\vec{E}=\frac{1}{r}\left(\frac{\partial E_z}{\partial\theta}-\frac{\partial(rE_\theta)}{\partial z}\right)\vec{a}_r+\left(\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}\right)\vec{a}_\theta+\frac{1}{r}\left(\frac{\partial(rE_\theta)}{\partial r}-\frac{\partial E_r}{\partial\theta}\right)\vec{a}_z$$

$$=\left(\frac{1}{r}\frac{\partial E_z}{\partial\theta}-\frac{\partial E_\theta}{\partial z}\right)\vec{a}_r+\left(\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}\right)\vec{a}_\theta+\frac{1}{r}\left(\frac{\partial(rE_\theta)}{\partial r}-\frac{\partial E_r}{\partial\theta}\right)\vec{a}_z$$

(Cylindrical system of coordinates)

$$(h_1=1,h_2=r,h_3=1;\ u_1=r,u_2=\theta,u_3=z)$$

$$\nabla\times\vec{E}=\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,E_\phi)}{\partial\theta}-\frac{\partial E_\theta}{\partial\phi}\right)\vec{a}_r+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi}-\frac{\partial(rE_\phi)}{\partial r}\right)\vec{a}_\theta+\frac{1}{r}\left(\frac{\partial(rE_\theta)}{\partial r}-\frac{\partial E_r}{\partial\theta}\right)\vec{a}_\phi$$

$$(h_1=1,h_2=r,h_3=r\sin\theta;\ u_1=r,u_2=\theta,u_3=\phi)$$

(Spherical system of coordinates)

$$\nabla \times \vec{E} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{a}_1 & h_2 \vec{a}_2 & h_3 \vec{a}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix}$$

(Curvilinear system of coordinates)

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

(Rectangular system of coordinates)

$$(h_1 = 1, h_2 = 1, h_3 = 1;\ u_1 = x, u_2 = y, u_3 = z)$$

$$\nabla \times \vec{E} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{a}_1 & h_2 \vec{a}_2 & h_3 \vec{a}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix}$$

(Curvilinear system of coordinates)

$$\nabla \times \vec{E} = \frac{1}{r} \begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & \vec{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ E_r & rE_\theta & E_z \end{vmatrix}$$

(Cylindrical system of coordinates)

$$(h_1 = 1, h_2 = r, h_3 = 1;\ u_1 = r, u_2 = \theta, u_3 = z)$$

$$\nabla \times \vec{E} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{a}_1 & h_2 \vec{a}_2 & h_3 \vec{a}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 E_1 & h_2 E_2 & h_3 E_3 \end{vmatrix}$$

(Curvilinear system of coordinates)

$$\nabla \times \vec{E} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ E_r & rE_\theta & r\sin\theta\,E_\phi \end{vmatrix}$$

(Spherical system of coordinates)

$$(h_1 = 1, h_2 = r, h_3 = r\sin\theta;\ u_1 = r, u_2 = \theta, u_3 = \phi)$$

Laplacian of scalar and vector quantities

Laplacian of a scalar quantity

The gradient of a scalar quantity is a vector quantity.

The divergence of a vector quantity is a scalar quantity.

Then if we take V as the scalar quantity then

$$\text{grad}V = \nabla \vec{V}$$

becomes a vector quantity and its divergence a scalar quantity.

In other words,

$$\nabla \cdot \nabla V = \nabla^2 V$$

is a scalar quantity called the Laplacian of the scalar quantity here V .

Let us take the following two vector quantities:

$$\vec{E} = E_1\vec{a}_1 + E_2\vec{a}_2 + E_3\vec{a}_3$$

$$\nabla V = (\nabla V)_1\,\vec{a}_1 + (\nabla V)_2\,\vec{a}_2 + (\nabla V)_3\,\vec{a}_3$$

Interpreting $\vec{E}$ as ∇V in the following expression

$$\nabla \cdot \vec{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 E_1)}{\partial u_1} + \frac{\partial(h_3 h_1 E_2)}{\partial u_2} + \frac{\partial(h_1 h_2 E_3)}{\partial u_3}\right]$$

we get

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 \nabla V_1)}{\partial u_1} + \frac{\partial(h_3 h_1 \nabla V_2)}{\partial u_2} + \frac{\partial(h_1 h_2 \nabla V_3)}{\partial u_3}\right]$$

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 \nabla V_1)}{\partial u_1} + \frac{\partial(h_3 h_1 \nabla V_2)}{\partial u_2} + \frac{\partial(h_1 h_2 \nabla V_3)}{\partial u_3}\right]$$

$$\left.\begin{aligned}(\nabla V)_1 &= \frac{1}{h_1}\frac{\partial V}{\partial u_1}\\ (\nabla V)_2 &= \frac{1}{h_2}\frac{\partial V}{\partial u_2}\\ (\nabla V)_3 &= \frac{1}{h_3}\frac{\partial V}{\partial u_3}\end{aligned}\right\}$$

$$\nabla^2 V = (\nabla \cdot \nabla V =)\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right].$$

$$\nabla^2 V = (\nabla \cdot \nabla V =) \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right].$$

(Curvilinear system of coordinates)

$$\nabla^2 V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$(h_1 = 1, h_2 = 1, h_3 = 1;\ u_1 = x, u_2 = y, u_3 = z)$$

(Rectangular system of coordinates)

$$\nabla^2 V = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial V}{\partial z}\right)\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}$$

$$(h_1 = 1, h_2 = r, h_3 = 1;\ u_1 = r, u_2 = \theta, u_3 = z)$$

(Cylindrical system of coordinates)

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta\frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial V}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.$$

(Spherical system of coordinates)

$$(h_1 = 1, h_2 = r, h_3 = r\sin\theta;\ u_1 = r, u_2 = \theta, u_3 = \phi)$$

Laplacian of a vector quantity

Laplacian of the vector is the vector sum of the Laplacian of the vector components. For the vector $\vec{E}$

$$\nabla^2 \vec{E} = (\nabla^2 E_1)\vec{a}_1 + (\nabla^2 E_2)\vec{a}_2 + (\nabla^2 E_3)\vec{a}_3$$

where $\nabla^2 E_1, \nabla^2 E_2, \nabla^2 E_3$

are obtained by replacing V by E_1, E_2, E_3 in the expression for $\nabla^2 V$.

Summarising Notes

√ Electromagnetics, a subject that helps in developing the understanding of many concepts in electrical, electronics and communication engineering, is based on the fundamental principles of electricity and magnetism.

√ Vector calculus expressions can concisely describe the concepts of the subject.

√ You can hit the right spot by very easily deriving, in the curvilinear system of coordinates, the expressions for the gradient of a scalar quantity, divergence of a vector quantity, curl of a vector quantity, and Laplacian of a scalar or a vector quantity.

√ Vector symmetry of the terms of the vector calculus expressions must have fascinated you in that from one of the terms of an expression you can permute to write the remaining terms of the expression.

√ Vector calculus expressions in the curvilinear system of coordinates can easily be translated to write the corresponding expressions in the rectangular, cylindrical and spherical systems of coordinates.

√ Vector calculus expressions immensely help us develop concisely the concepts of electromagnetics.

Readers are encouraged to go through Chapter 2 of the book for greater details of vector calculus expressions used extensively in the rest of the book.