Engineering Electromagnetics Essentials

Chapter 9

Waveguides: solution of the wave equation
For a wave in a bounded medium

Objective

To solve wave equation and study propagation of electromagnetic wave through bounded media, namely hollow-pipe waveguides, and predict their characteristics

Topics dealt with

Solution of the wave equation of a rectangular waveguide in transverse electric (TE) mode for all the field components

Characteristic equation or dispersion relation of a rectangular waveguide excited in transverse electric (TE) mode in terms of wave frequency, wave phase propagation constant and waveguide cutoff frequency

Dependence of waveguide cutoff frequency on waveguide dimensions with reference to a waveguide excited in TE mode

Characteristic parameters namely phase velocity, group velocity, guide wavelength and wave impedance and their dependence on wave frequency relative to waveguide cutoff frequency

Evanescent mode in a waveguide

Dimension-wise and mode-wise operating frequency criteria

Excitation of a rectangular waveguide in transverse magnetic (TM) mode

Significance of mode numbers vis-à-vis field pattern

Cylindrical waveguide

Inability of a hollow-pipe waveguide to support a TEM mode

Power flow and power loss in a waveguide

Power loss per unit area, power loss per unit length and attenuation constant there from

Background

Maxwell's equations (Chapter 5), electromagnetic boundary conditions at conductor-dielectric interface (Chapter 7), basic concepts of electromagnetic power flow (Chapter 8) and those of circuit theory

A ‘waveguide’ is a hollow metal pipe used to ‘guide’ the transmission of an electromagnetic ‘wave’ from one point to another.

It is a microwave-frequency counterpart of the lower-frequency connecting wire.

It is thus essentially a type of transmission line, some of the other types being the two-wire line, the coaxial cable, the parallel-plate line, the stripline, the microstrip line, etc.

We have seen that a transverse electromagnetic wave (TEM) mode of propagation is supported by a free-space medium with a phase velocity equal to the velocity of light c.

Can a TEM mode be supported by a hollow-pipe waveguide? Would the phase velocity of a wave propagating through such a waveguide be the same as or different from c?

We can answer to such question and many others concerning the characteristics of a waveguide with the help of electromagnetic analysis of the waveguide.

Such an analysis is based on setting up the wave equation in electric and magnetic fields in the waveguide and solving them with the help of relevant electromagnetic boundary conditions.

We take up here for analysis two types of waveguides, namely rectangular and cylindrical or circular waveguides which have rectangular and circular cross sections respectively perpendicular to the direction of wave propagation.

Rectangular waveguide

Rectangular waveguide can be treated in rectangular system of coordinates in view of the rectangular symmetry of its geometry. The waveguide will be considered for both TE and TM modes of excitation. (We will see later the inability of the hollow-pipe waveguides to support TEM mode of propagation).

The TE mode is also known as the H mode since it is associated with a non-zero value of the axial magnetic field: $H_z \neq 0, E_z = 0.$

The TM mode is also known as the E mode since it is associated with a non-zero value of the axial electric field: $E_z \neq 0, H_z = 0.$

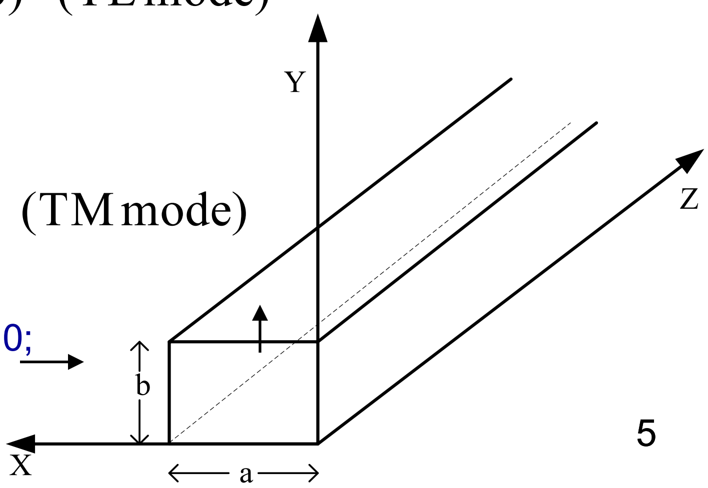
Wave equations

Wave equations have already been introduced in Chapter 6.

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} \quad (H_z \neq 0, E_z = 0) \quad (\text{TE mode})$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2} \quad (E_z \neq 0, H_z = 0) \quad (\text{TM mode})$$

Rectangular waveguide with its right side wall located at *x* = 0; left side at *x* = *a*; bottom wall at *y* = 0 and top wall at *y* = *b*.

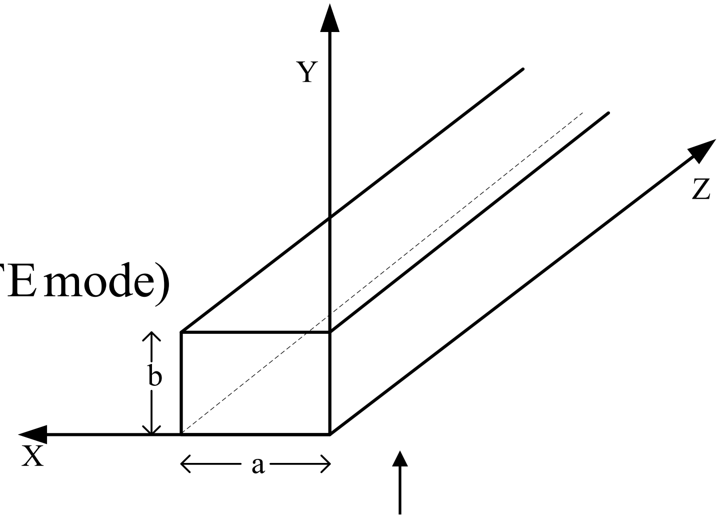


TE mode

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+\frac{\partial^2 H_z}{\partial z^2}=\mu_0\varepsilon_0\frac{\partial^2 H_z}{\partial t^2} \quad (H_z\neq 0, E_z=0) \quad (\text{TE mode})$$

(rewritten)

(wave equation rewritten)



Rectangular waveguide with its right side wall located at x = 0; left side at x = a; bottom wall at y = 0 and top wall at y = b.

Field quantities varying as

$$\exp j(\omega t-\beta z)$$

$$\partial/\partial t = j\omega$$
$$\partial/\partial z = -j\beta$$

$$\frac{\partial^2 H_z}{\partial z^2}=(-j\beta)(-j\beta)H_z=j^2\beta^2 H_z=-\beta^2 H_z$$

$$\frac{\partial^2 H_z}{\partial t^2}=(j\omega)(j\omega)H_z=j^2\omega^2 H_z=-\omega^2 H_z$$

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}-\beta^2 H_z=-\mu_0\varepsilon_0\omega^2 H_z$$

$$k=\omega(\mu_0\varepsilon_0)^{1/2}$$

(free-space propagation constant)

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+(k^2-\beta^2)H_z=0$$

(wave equation)

Field solutions

$H_z = XY$ → Let us use the method of separation of variables for solving wave equation

X is a function of only *x*

$$\frac{\partial H_z}{\partial x} = Y\frac{\partial X}{\partial x} \rightarrow \frac{\partial^2 H_z}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

Y is a function of only *y*

$$\frac{\partial H_z}{\partial y} = X\frac{\partial Y}{\partial y} \rightarrow \frac{\partial^2 H_z}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

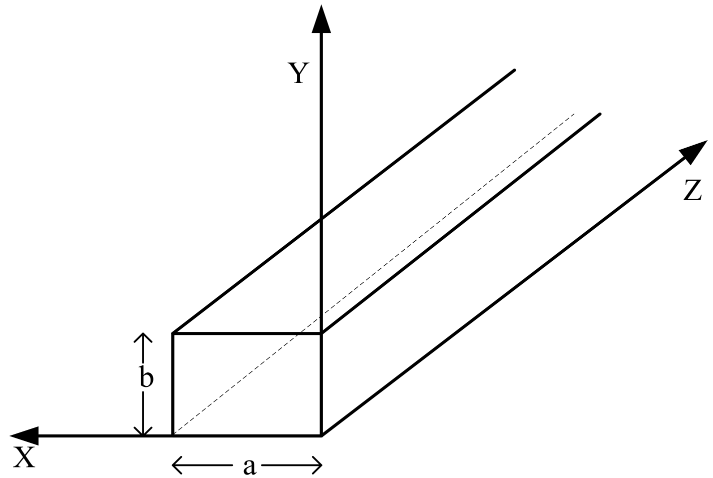
$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (k^2 - \beta^2)H_z = 0$$ (wave equation rewritten)

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + (k^2 - \beta^2)XY = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + (k^2 - \beta^2)XY = 0$$ (rewritten)

↓ ← Dividing by *XY* and rearranging terms

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - (k^2 - \beta^2)$$



← Equates a function of *x* alone in its left-hand side with a function of *y* alone on its right-hand side. This can happen only when the individual functions of *x* and *y* are constants.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -A^2 \qquad -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - (k^2 - \beta^2) = -A^2$$

← A^2 is a constant

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = A^2 - (k^2 - \beta^2) = -B^2$$

← B^2 is a constant

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -A^2 \text{ (rewritten)}$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -B^2 \text{ (rewritten)}$$

Solution

$$X = C_1 \cos Ax + C_2 \sin Ax$$

$$Y = C_3 \cos By + C_4 \sin By$$

C_1 and C_2 are constants
C_3 and C_4 are constants

$$H_z = XY$$

Invoking the factor exp $j(\omega t - \beta z)$
which was otherwise understood

$$H_z = (C_1 \cos Ax + C_2 \sin Ax)(C_3 \cos By + C_4 \sin By) \exp\ j(\omega t - \beta z)$$

Maxwell's equations

$$E_x, E_y \text{ and } H_x, H_y$$

(transverse components of the electric field and magnetic field components)

$$\left.\begin{aligned}\nabla\times\vec{E} &= -\mu_0\frac{\partial\vec{H}}{\partial t}\\ \nabla\times\vec{H} &= \varepsilon_0\frac{\partial\vec{E}}{\partial t}\end{aligned}\right\} \longleftarrow \frac{\partial}{\partial t} = j\omega \longleftarrow$$

Field quantities vary as $\exp j(\omega t - \beta z)$

(Maxwell's equation)

↓ ← Expanding the curl in rectangular system

$$\left.\begin{aligned}&\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\vec{a}_y+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\vec{a}_z\\ &\qquad = -j\omega\mu_0(H_x\vec{a}_x+H_y\vec{a}_y+H_z\vec{a}_z)\\ &\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\vec{a}_y+\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\vec{a}_z\\ &\qquad = j\omega\varepsilon_0(E_x\vec{a}_x+E_y\vec{a}_y+E_z\vec{a}_z)\end{aligned}\right\}$$

$$\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\vec{a}_y+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\vec{a}_z$$
$$=-j\omega\mu_0(H_x\vec{a}_x+H_y\vec{a}_y+H_z\vec{a}_z)$$

$$\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\vec{a}_y+\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\vec{a}_z$$
$$=j\omega\varepsilon_0(E_x\vec{a}_x+E_y\vec{a}_y+E_z\vec{a}_z)$$

← $E_z=0$

(TE mode)

(rewritten)

↓

$$\left(-\frac{\partial E_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial E_x}{\partial z}\right)\vec{a}_y+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\vec{a}_z$$
$$=-j\omega\mu_0(H_x\vec{a}_x+H_y\vec{a}_y+H_z\vec{a}_z)$$

$$\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\vec{a}_y+\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\vec{a}_z$$
$$=j\omega\varepsilon_0(E_x\vec{a}_x+E_y\vec{a}_y)$$

$$\left(-\frac{\partial E_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial E_x}{\partial z}\right)\vec{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\vec{a}_z = -j\omega\mu_0(H_x\vec{a}_x + H_y\vec{a}_y + H_z\vec{a}_z)$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\vec{a}_z = j\omega\varepsilon_0(E_x\vec{a}_x + E_y\vec{a}_y)$$

(rewritten)

x component

y component

$$-\frac{\partial E_y}{\partial z} = -j\omega\mu_0 H_x$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon_0 E_y$$

$$\frac{\partial}{\partial z} = -j\beta$$

Field quantities vary as $\exp j(\omega t - \beta z)$

$$j\beta E_y = -j\omega\mu_0 H_x$$

$$H_x = -\beta E_y/(\omega\mu_0)$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon_0 E_y$$

$$j\frac{\beta^2 E_y}{\omega\mu_0} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon_0 E_y$$

Rearranging terms

$$j\frac{\beta^2 - \omega^2\mu_0\varepsilon_0}{\omega\mu_0}E_y - \frac{\partial H_z}{\partial x} = 0$$

Similarly

$$\left(-\frac{\partial E_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial E_x}{\partial z}\right)\vec{a}_y+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\vec{a}_z = -j\omega\mu_0(H_x\vec{a}_x+H_y\vec{a}_y+H_z\vec{a}_z)$$

(rewritten)

$$\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\vec{a}_x+\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\vec{a}_y+\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\vec{a}_z = j\omega\varepsilon_0(E_x\vec{a}_x+E_y\vec{a}_y)$$

y component

x component

$$\frac{\partial E_x}{\partial z}=-j\omega\mu_0 H_y$$

$$\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=j\omega\varepsilon_0 E_x$$

$$\frac{\partial}{\partial z}=-j\beta$$

Field quantities varyas $\exp j(\omega t-\beta z)$

$$-j\beta E_x=-j\omega\mu_0 H_y$$

$$\frac{\partial H_z}{\partial y}+j\beta H_y=j\omega\varepsilon_0 E_x$$

$$H_y=\beta E_x/(\omega\mu_0)$$

$$j\frac{\beta^2 E_x}{\omega\mu_0}+\frac{\partial H_z}{\partial y}=j\omega\varepsilon_0 E_x$$

Rearranging terms

$$j\frac{\beta^2-\omega^2\mu_0\varepsilon_0}{\omega\mu_0}E_x+\frac{\partial H_z}{\partial y}=0$$

$$j\frac{\beta^2-\omega^2\mu_0\varepsilon_0}{\omega\mu_0}E_y-\frac{\partial H_z}{\partial x}=0$$ (recalled)

$$j\frac{\beta^2-\omega^2\mu_0\varepsilon_0}{\omega\mu_0}E_x+\frac{\partial H_z}{\partial y}=0$$ (recalled)

$$k=\omega(\mu_0\varepsilon_0)^{1/2}$$ (recalled)

$$E_y=\frac{j\omega\mu_0}{k^2-\beta^2}\frac{\partial H_z}{\partial x}$$

$$E_x=\frac{-j\omega\mu_0}{k^2-\beta^2}\frac{\partial H_z}{\partial y}$$

$$\left.\begin{aligned}E_y&\propto\frac{\partial H_z}{\partial x}\\ E_x&\propto\frac{\partial H_z}{\partial y}\end{aligned}\right\}$$

We will recall the above relations of proportionality in the electromagnetic boundary conditions at the walls of the waveguide in the analysis to follow.

<u>Electromagnetic boundary condition at the waveguide wall</u>:

$$\vec{a}_n\times\vec{E}_2=\vec{a}_n\times(E_x\vec{a}_x+E_y\vec{a}_y+E_z\vec{a}_z)=0 \longleftarrow \vec{a}_n\times(\vec{E}_2-\vec{E}_1)=0$$

$$\downarrow \longleftarrow E_z=0\ \text{(TE mode)}$$

$$\vec{a}_n\times\vec{E}_2=\vec{a}_n\times(E_x\vec{a}_x+E_y\vec{a}_y)=0$$

(subscript 1 referring to the conducting waveguide wall region 1 and subscript 2 to the free-space region 2 inside the waveguide with unit vector directed from region 1 to 2)

$$\vec{a}_n \times \vec{E}_2 = \vec{a}_n \times (E_x \vec{a}_x + E_y \vec{a}_y) = 0$$

(electromagnetic boundary condition at the left and right side walls of the rectangular waveguide)

Left side wall:

$$x = a$$

$$\vec{a}_n = -\vec{a}_x$$

$$\downarrow$$

$$-\vec{a}_x \times (E_x \vec{a}_x + E_y \vec{a}_y) = 0$$

$$\downarrow$$

$$-E_y \vec{a}_z = 0$$

$$\downarrow$$

$$E_y = 0 \quad (0 \le y \le b) \text{ at } x = a$$

$$\downarrow$$

$$\frac{\partial H_z}{\partial x} = 0 \quad (0 \le y \le b) \text{ at } x = a$$

(left side wall)

$$E_y \propto \frac{\partial H_z}{\partial x}$$

(recalled)

Right side wall:

$$x = 0$$

$$\vec{a}_n = \vec{a}_x$$

$$\downarrow$$

$$\vec{a}_x \times (E_x \vec{a}_x + E_y \vec{a}_y) = 0$$

$$\downarrow$$

$$E_y \vec{a}_z = 0$$

$$\downarrow$$

$$E_y = 0 \quad (0 \le y \le b) \text{ at } x = 0$$

$$\downarrow$$

$$\frac{\partial H_z}{\partial x} = 0 \quad (0 \le y \le b) \text{ at } x = 0$$

(right side wall)

$$H_z = (C_1 \cos Ax + C_2 \sin Ax)(C_3 \cos By + C_4 \sin By) \exp\, j(\omega t - \beta z)$$

(recalled)

$$\left.\begin{aligned} \frac{\partial H_z}{\partial x} &= (-C_1 A \sin Ax + C_2 A \cos Ax)(C_3 \cos By + C_4 \sin By) \\ \frac{\partial H_z}{\partial y} &= (C_1 \cos Ax + C_2 \sin Ax)(-C_3 B \sin By + C_4 B \cos By) \end{aligned}\right\}$$

$$\frac{\partial H_z}{\partial x} = 0 \quad (0 \le y \le b) \text{ at } x = a$$

(left side wall)

$$\frac{\partial H_z}{\partial x} = 0 \quad (0 \le y \le b) \text{ at } x = 0$$

(right side wall)

$$\left(\frac{\partial H_z}{\partial x} =\right)(C_2 A)(C_3 \cos By + C_4 \sin By) = 0$$

(for all values of y)

This can happen only if

$C_2 = 0, A \neq 0$ — Situation I

$A = 0, C_2 \neq 0$ — Situation II

$C_2 = 0, A \neq 0$ $\longrightarrow$ $\dfrac{\partial H_z}{\partial x} = 0 \quad (0 \leq y \leq b)$ at $x = a$ (left side wall)

Situation I

$\downarrow$

$$(\frac{\partial H_z}{\partial x} =)(-C_1 A \sin Ax + C_2 A \cos Ax)(C_3 \cos By + C_4 \sin By) = 0$$

$\downarrow$

$$-C_1 A \sin Ax (C_3 \cos By + C_4 \sin By) = 0$$

(for all values of y)

$\swarrow$

$C_1 = 0, A \neq 0$ (Situation I) (left side wall at $x = a$)

$C_2 = 0$

$\downarrow$

$$H_z = (C_1 \cos Ax + C_2 \sin Ax)(C_3 \cos By + C_4 \sin By) \exp j(\omega t - \beta z)$$

(Situation I)

$\downarrow$ $\longleftarrow$ $C_1 = 0,\ C_2 = 0$

$H_z = 0$ $\longrightarrow$ The finding contradicts with the TE-mode condition (Situation I): $H_z \neq 0$

$C_1 = 0, A \neq 0$ (Situation I) (left side wall at *x* = *a*)
$C_2 = 0$ $\longrightarrow$ leads to the contradiction with the TE-mode condition

Therefore, in order to avoid this contradiction in Situation I let us put

$$A \neq 0, C_1 \neq 0 \text{ besides } C_2 = 0$$ (left side wall at *x* = *a*)

$$\downarrow$$

$$\left(\frac{\partial H_z}{\partial x} =\right)(-C_1 A \sin Ax + C_2 A \cos Ax)(C_3 \cos By + C_4 \sin By) = 0$$

$$(\text{for all values of } y)$$ (Situation I) (left side wall at *x* = *a*) (rewritten)

$$\downarrow$$

$$\sin Aa = 0 \longrightarrow Aa = m\pi\,(m = 1, 2, 3, \ldots)$$

(*m* = 0 that makes *A* = 0 is excluded to avoid contradiction with A ≠ 0 taken here)

$$\downarrow$$

$$A = \frac{m\pi}{a}(m = 1, 2, 3, \ldots) \text{ (Situation I)}$$

$$\downarrow$$

$$H_z = (C_1 \cos Ax)(C_3 \cos By + C_4 \sin By) \exp j(\omega t - \beta z)$$
$$\text{with } A = \frac{m\pi}{a}(m = 1, 2, 3, \ldots) \text{ (Situation I)}$$

Similarly we can proceed recalling Situation II: $A = 0, C_2 \neq 0$

↓

$$H_z = (C_1 \cos Ax + C_2 \sin Ax)(C_3 \cos By + C_4 \sin By) \exp\ j(\omega t - \beta z)$$

(recalled)

↓

$$H_z = (C_1)(C_3 \cos By + C_4 \sin By) \exp\ j(\omega t - \beta z)$$

(for all values of y) (Situation II)

↓ ← Tacitly expressed as

$$H_z = (C_1 \cos Ax)(C_3 \cos By + C_4 \sin By) \exp\ j(\omega t - \beta z)$$

↖ interpreting $A = \dfrac{m\pi}{a} (m = 0)$ (Situation II)

↓

$$H_z = (C_1 \cos Ax)(C_3 \cos By + C_4 \sin By) \exp\ j(\omega t - \beta z)$$

with $A = \dfrac{m\pi}{a} (m = 0)$ (Situation II)

$$H_z = (C_1 \cos Ax)(C_3 \cos By + C_4 \sin By)\exp\, j(\omega t - \beta z)$$
$$\text{with } A = \frac{m\pi}{a}\,(m=0)\ (\text{Situation II})$$

(rewritten)

(for all values of y)

Combining

$$H_z = (C_1 \cos Ax)(C_3 \cos By + C_4 \sin By)\exp\, j(\omega t - \beta z)$$
$$\text{with } A = \frac{m\pi}{a}\,(m=1,2,3,\ldots)\ (\text{Situation I})$$

(recalled)

$$H_z = (C_1 \cos Ax)(C_3 \cos By + C_4 \sin By)\exp\, j(\omega t - \beta z)$$
$$\text{with } A = \frac{m\pi}{a}\,(m=0,1,2,3,\ldots)\ (\text{for all values of } y)$$

We have so far considered electromagnetic boundary conditions at the left and right sidewalls of the rectangular waveguide. Let us next turn towards electromagnetic boundary conditions at the top and bottom walls of the rectangular waveguide.

$$\vec{a}_n \times \vec{E}_2 = \vec{a}_n \times (E_x \vec{a}_x + E_y \vec{a}_y) = 0$$

(electromagnetic boundary condition at the bottom and top walls of the rectangular waveguide)

Bottom wall:

$$y = 0$$

$$\vec{a}_n = \vec{a}_y$$

$$\downarrow$$

$$\vec{a}_y \times (E_x \vec{a}_x + E_y \vec{a}_y) = 0$$

$$\downarrow$$

$$-E_x \vec{a}_z = 0$$

$$\downarrow$$

$$E_x = 0 \quad (0 \le x \le a) \text{ at } y = 0$$

$$\downarrow$$

$$\frac{\partial H_z}{\partial y} = 0 \quad (0 \le x \le a) \text{ at } y = 0$$

(bottom wall)

Top wall:

$$y = b$$

$$\vec{a}_n = -\vec{a}_y$$

$$\downarrow$$

$$-\vec{a}_y \times (E_x \vec{a}_x + E_y \vec{a}_y) = 0$$

$$\downarrow$$

$$E_x \vec{a}_z = 0$$

$$\downarrow$$

$$E_x = 0 \quad (0 \le x \le a) \text{ at } y = b$$

$$\downarrow$$

$$\frac{\partial H_z}{\partial y} = 0 \quad (0 \le x \le a) \text{ at } y = b$$

(top wall)

$$E_x \propto \frac{\partial H_z}{\partial y}$$

(recalled)

$$\frac{\partial H_z}{\partial y} = 0 \quad (0 \le x \le a) \text{ at } y = 0$$

(bottom wall)

$$\frac{\partial H_z}{\partial y} = 0 \quad (0 \le x \le a) \text{ at } y = b$$

(top wall)

$$\frac{\partial H_z}{\partial y} = (C_1 \cos Ax + C_2 \sin Ax)(-C_3 B \sin By + C_4 B \cos By)$$

Hence using the same procedure as followed earlier, which starts from the boundary conditions at the left and right side walls, we can obtain, now starting from the boundary conditions at the bottom and top walls, the following expression:

$$H_z = (C_3 \cos By)(C_1 \cos Ax + C_2 \sin Ax) \exp j(\omega t - \beta z)$$

$$\text{with } B = \frac{n\pi}{b} \, (n = 0, 1, 2, 3, \ldots) \quad (\text{for all values of } x)$$

We have obtained the following:

$$H_z = (C_1 \cos Ax)(C_3 \cos By + C_4 \sin By) \exp j(\omega t - \beta z)$$

$$\text{with } A = \frac{m\pi}{a} \; (m = 0,1,2,3,\ldots) \quad (\text{for all values of } y)$$

$$H_z = (C_3 \cos By)(C_1 \cos Ax + C_2 \sin Ax) \exp j(\omega t - \beta z)$$

$$\text{with } B = \frac{n\pi}{b} \; (n = 0,1,2,3,\ldots) \quad (\text{for all values of } x)$$

When combined, these two expressions for H_z, while remembering that the former is valid for all values of *y* and that the latter for all values of *x*, prompt us to take the factor cos*Ax* from the former and the factor cos*By* from the latter in the field expression to write it as

$$A = \frac{m\pi}{a}, \; B = \frac{n\pi}{b} \; (m, n = 0,1,2,3,\ldots)$$

$$H_z = H_z = (C_1 \cos Ax)(C_3 \cos By) \exp j(\omega t - \beta z)$$

$$\text{Putting } H_{z0} = C_1 C_3$$

$$H_z = H_{z0} \cos Ax \cos By \exp j(\omega t - \beta z) \text{ with } A = \frac{m\pi}{a}, B = \frac{n\pi}{b} \; (m, n = 0,1,2,3,\ldots)$$

$$H_z = H_{z0} \cos Ax \cos By \exp j(\omega t - \beta z)$$

$$\text{with } A = \frac{m\pi}{a} \ (m = 0,1,2,3,\ldots); B = \frac{n\pi}{b} \ (n = 0,1,2,3,\ldots)$$

$$\downarrow \ \longleftarrow \ A = \frac{m\pi}{a}, B = \frac{n\pi}{b} \ (m, n = 0,1,2,3,\ldots)$$

$$H_z = H_{z0} \cos(\frac{m\pi}{a} x) \cos(\frac{n\pi}{b} y) \exp j(\omega t - \beta z) \quad (m, n = 0,\ 1,\ 2,\ 3,\ \ldots)$$

$$\downarrow$$

$$\frac{\partial H_z}{\partial x} = -\frac{m\pi}{a} H_{z0} \sin(\frac{m\pi}{a} x) \cos(\frac{n\pi}{b} y) \exp j(\omega t - \beta z) \ \ (m, n = 0,\ 1,\ 2,\ 3,\ \ldots)$$

$$\frac{\partial H_z}{\partial y} = -\frac{n\pi}{b} H_{z0} \cos(\frac{m\pi}{a} x) \sin(\frac{n\pi}{b} y) \exp \ j(\omega t - \beta z)$$

$$\frac{\partial H_z}{\partial x} = -\frac{m\pi}{a} H_{z0} \sin(\frac{m\pi}{a} x)\cos(\frac{n\pi}{b} y)\exp j(\omega t - \beta z) \quad (m, n = 0, 1, 2, 3, \ldots)$$

(rewritten)

$$\frac{\partial H_z}{\partial y} = -\frac{n\pi}{b} H_{z0} \cos(\frac{m\pi}{a} x)\sin(\frac{n\pi}{b} y)\exp j(\omega t - \beta z)$$

(rewritten)

$$E_y = \frac{j\omega\mu_0}{k^2 - \beta^2} \frac{\partial H_z}{\partial x}$$ (rewritten)

$$E_x = \frac{-j\omega\mu_0}{k^2 - \beta^2} \frac{\partial H_z}{\partial y}$$ (rewritten)

$$E_y = -\frac{j\omega\mu_0}{k^2 - \beta^2} \frac{m\pi}{a} H_{z0} \sin(\frac{m\pi}{a} x)\cos(\frac{n\pi}{b} y)\exp j(\omega t - \beta z) \quad (m, n = 0, 1, 2, 3, \ldots)$$

$$E_x = \frac{j\omega\mu_0}{k^2 - \beta^2} \frac{n\pi}{b} H_{z0} \cos(\frac{m\pi}{a} x)\sin(\frac{n\pi}{b} y)\exp j(\omega t - \beta z) \quad (m, n = 0, 1, 2, 3, \ldots)$$

$$A^2-(k^2-\beta^2)=-B^2$$ (introduced earlier while solving the wave equation by the method of separation of variables)

$$k^2-\beta^2=A^2+B^2$$

$$E_y=-\frac{j\omega\mu_0}{k^2-\beta^2}\frac{m\pi}{a}H_{z0}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\exp j(\omega t-\beta z)\ \ (m,n=0,1,2,3,...)$$

(rewritten)

$$E_x=\frac{j\omega\mu_0}{k^2-\beta^2}\frac{n\pi}{b}H_{z0}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\exp j(\omega t-\beta z)\ \ (m,n=0,1,2,3,...)$$

(rewritten)

$$E_y=-\frac{j\omega\mu_0}{A^2+B^2}\frac{m\pi}{a}H_{z0}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\exp j(\omega t-\beta z)\ \ (m,n=0,1,2,3,\ldots)$$

$$E_x=\frac{j\omega\mu_0}{A^2+B^2}\frac{n\pi}{b}H_{z0}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\exp j(\omega t-\beta z)\ \ (m,n=0,1,2,3,\ldots)$$

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$$E_x = \frac{j\omega\mu_0}{A^2+B^2}\frac{n\pi}{b}H_{z0}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\exp j(\omega t-\beta z)\ \ (m,n=0,\ 1,\ 2,\ 3,\ \ldots)$$

$$A=\frac{m\pi}{a},\ B=\frac{n\pi}{b}\ (m,n=0,1,2,3,\ldots)$$

$$E_y = -\frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\frac{m\pi}{a}H_{z0}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\exp j(\omega t-\beta z)$$

$$(m,n=0,1,2,3,\ldots)$$

$$E_x = \frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\frac{n\pi}{b}H_{z0}\cos(\frac{m\pi}{a})x\sin(\frac{n\pi}{b})y\exp\ j(\omega t-\beta z)$$

$$(m,n=0,1,2,3,...)$$

$$E_y = -\frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{m\pi}{a} H_{z0} \sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\exp j(\omega t - \beta z) \quad (m,n = 0,1,2,3,...)$$

(rewritten)

↓

$$H_x = -\frac{\beta}{\omega\mu_0} E_y \longrightarrow H_x = \frac{j\beta}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{m\pi}{a} H_{z0} \sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\exp j(\omega t - \beta z)$$

$$(m,n = 0,1,2,3,\ldots)$$

$$E_x = \frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{n\pi}{b} H_{z0} \cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\exp j(\omega t - \beta z) \quad (m,n = 0,1,2,3,\ldots)$$

(rewritten)

↓

$$H_y = \frac{\beta}{\omega\mu_0} E_x \longrightarrow H_y = \frac{j\beta}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{n\pi}{b} H_{z0} \cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\exp j(\omega t - \beta z)$$

$$(m,n = 0,1,2,3,\ldots)$$

Field expressions of a rectangular waveguide excited in the TE$_{mn}$ mode put together

$$E_x = \frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\frac{n\pi}{b}H_{z0}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\exp j(\omega t-\beta z) \quad (m,n=0,1,2,3,\ldots)$$

$$E_y = -\frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\frac{m\pi}{a}H_{z0}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\exp j(\omega t-\beta z) \quad (m,n=0,1,2,3,\ldots)$$

$$E_z = 0$$

$$H_x = \frac{j\beta}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\frac{m\pi}{a}H_{z0}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\exp j(\omega t-\beta z) \quad (m,n=0,1,2,3,\ldots)$$

$$H_y = \frac{j\beta}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\frac{n\pi}{b}H_{z0}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\exp j(\omega t-\beta z) \quad (m,n=0,1,2,3,\ldots)$$

$$H_z = H_{z0}\cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\exp j(\omega t-\beta z) \quad (m,n=0,\ 1,\ 2,\ 3,\ \ldots)$$

(*m* and *n* are the mode numbers of the rectangular waveguide excited in the TE$_{mn}$ mode)

Characteristic equation or dispersion relation of a rectangular waveguide excited in the TE mode

$$k^2 - \beta^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad (m, n = 0, 1, 2, 3, \ldots)$$ (recalled)

$\downarrow$ $\longleftarrow$ Putting $k_c = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2} \quad (m, n = 0, 1, 2, 3, \ldots)$

$k^2 - \beta^2 = k_c{}^2 \longleftarrow$ $k = \omega(\mu_0\varepsilon_0)^{1/2} = \omega / c$, free-space propagation constant,

$c = 1/(\mu_0\varepsilon_0)^{1/2}$ being the velocity of light

$\downarrow$ $\longleftarrow$ $k_c = \omega_c / c$, cutoff wave number, ω_c being the cutoff frequency

$$\frac{\omega^2}{c^2} - \beta^2 = k_c{}^2 \longrightarrow \omega^2 - \beta^2 c^2 - k_c{}^2 c^2 = 0 \longrightarrow \omega^2 - \beta^2 c^2 - \omega_c{}^2 = 0$$

(dispersion relation)

(ω-β relationship with reference to a rectangular waveguide excited in the TE mode)

$$\left.\begin{aligned} k &= \frac{2\pi}{\lambda} \\ \beta &= \frac{2\pi}{\lambda_g} \\ k_c &= \frac{2\pi}{\lambda_c} \end{aligned}\right\}$$

↓

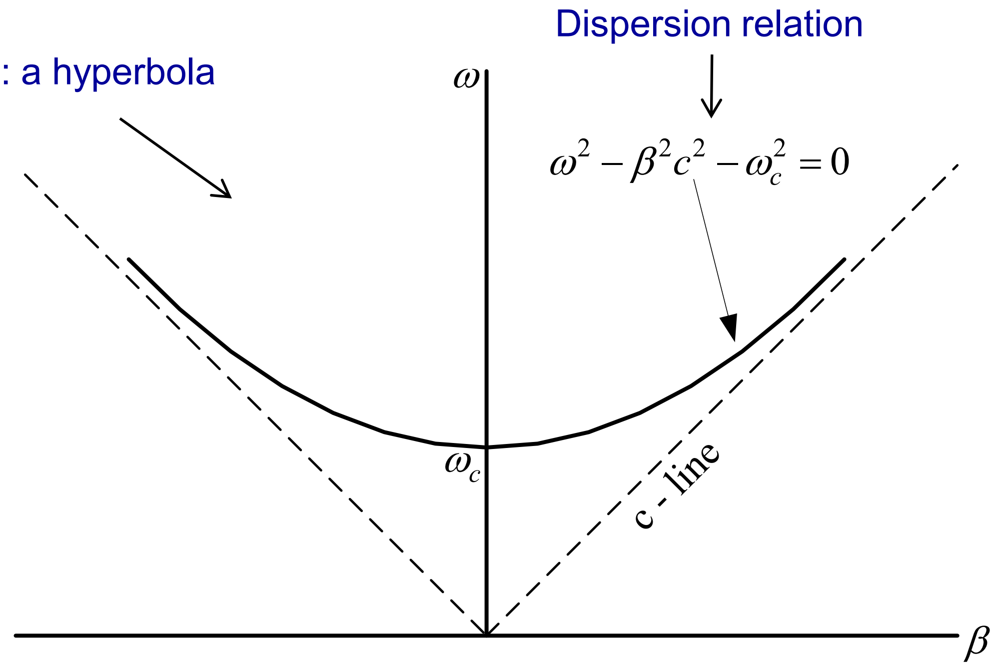
$$k^2 - \beta^2 = k_c^{\ 2}$$ (recalled)

↓

$$\left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{2\pi}{\lambda_g}\right)^2 = \left(\frac{2\pi}{\lambda_c}\right)^2$$

↓

$$\frac{1}{\lambda_c^{\ 2}} = \frac{1}{\lambda^2} - \frac{1}{\lambda_g^{\ 2}}$$ (characteristic equation of the waveguide)



ω versus β dispersion characteristics (dispersion curve) of the waveguide which meets the *c*-line at $\omega \to \infty$, the latter shown as a dotted line having positive and negative slopes of magnitude equal to the velocity of light *c* for the positive and the negative values of β respectively. The intercept of the dispersion curve with the ω axis gives the cutoff frequency ω_c.

$$k_c = \frac{\omega_c}{c}$$ (cutoff wave number)

$$k_c = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

$$f_c = \frac{\omega_c}{2\pi}$$ (cutoff frequency)

$$\lambda_c = \frac{c}{f_c}$$ (cutoff wavelength)

$$\omega_c = c\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$

$$f_c = \frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}$$

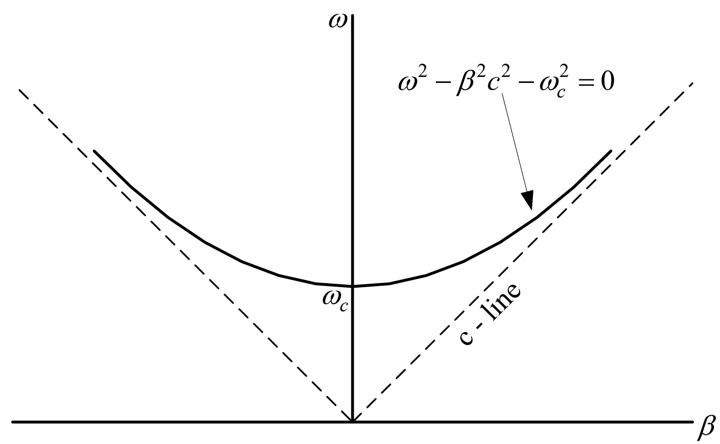
$$\lambda_c \left(= \frac{c}{f_c}\right) = \frac{2}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}}$$

$$(m, n = 0,\ 1,\ 2,\ 3,\ \ldots)$$

$$\omega^2 - \beta^2 c^2 - \omega_c{}^2 = 0 \longrightarrow \beta^2 = \frac{\omega^2 - \omega_c{}^2}{c^2}$$

$$\beta = \pm \frac{(\omega^2 - \omega_c{}^2)^{1/2}}{c}$$

In general, there will be two values of β for a given value of ω –one positive and the other negative.



Typical waveguide dimensions and corresponding cutoff frequencies (f_c) and cutoff wavelengths (λ_c) for the TE_{10} mode for typical commercial rectangular waveguide types →

Waveguide	*a* (inch)	*b* (inch)	λ_c (mm)	$f_c(=c/2a)$ (GHz)
WR-2300	23	11.5	1168.4	0.257
WR-340	3.4	1.7	172.72	1.737
WR-284	2.84	1.34	144.27	2.08
WR-137	1.372	0.62	69.70	4.30
WR-112	1.122	0.497	57.0	5.26
WR-90	0.9	0.4	45.72	6.56
WR-3	0.034	0.017	1.73	173.7

Modes in order of increasing cutoff frequency for typical commercial rectangular waveguide types →

Waveguide	**Waveguide modes in order of increasing cutoff frequency (f_c) shown in parenthesis against the modes**
WR-2300	TE_{10} (0.257 GHz) TE_{20}/ TE_{01} (0.51 GHz) TE_{11} (0.61 GHz)
WR-284	TE_{10} (2.08 GHz) TE_{20} (4.16 GHz) TE_{01} (4.41 GHz) TE_{11} (4.87 GHz)
WR-90	TE_{10} (6.56 GHz) TE_{11} (9.88 GHz) TE_{20} (13.1 GHz) TE_{01} (14.76 GHz)

Characteristic parameters

$$\beta = \frac{(\omega^2 - \omega_c{}^2)^{1/2}}{c}$$

(recalled)

Wave phase velocity: →

$$v_{\text{ph}} = \frac{\omega}{\beta} = \frac{\omega}{\dfrac{(\omega^2 - \omega_c{}^2)^{1/2}}{c}} = \frac{\omega c}{(\omega^2 - \omega_c{}^2)^{1/2}}$$

$$\frac{v_{\text{ph}}}{c} = \frac{\omega}{(\omega^2 - \omega_c{}^2)^{1/2}} = \frac{1}{\left(1 - \dfrac{\omega_c{}^2}{\omega^2}\right)^{1/2}} = \frac{1}{\left(1 - \dfrac{f_c^2}{f^2}\right)^{1/2}}$$

Guide wavelength: →

$$\lambda_g = \frac{2\pi}{\beta}$$

$$k = \frac{2\pi}{\lambda}, \quad \beta = \frac{2\pi}{\lambda_g}$$

$$k = \frac{\omega}{c}, \quad \beta = \frac{\omega}{v_{\text{ph}}}$$

$$\frac{\lambda_g}{\lambda} = \frac{k}{\beta} = \frac{v_{\text{ph}}}{c} = \frac{1}{\left(1 - \dfrac{f_c^2}{f^2}\right)^{1/2}}$$

$$\frac{v_{\text{ph}}}{c} = \frac{\lambda_g}{\lambda} = \frac{1}{\left(1 - \dfrac{f_c^2}{f^2}\right)^{1/2}}$$

(rewritten)

Wave group velocity

We have already defined the phase velocity of a wave v_{ph} that involves the phase propagation constant β, which, in turn, occurs in the representation of a wave associated with a physical quantity, for instance, the electric field:

$$E = E_0 \exp j(\omega t - \beta z) \quad \longleftarrow$$

$$E_0 = \text{field amplitude}$$

$$\beta = \text{phase propagation constant} = \omega / v_{\text{ph}}$$

$$v_{\text{ph}} = \omega / \beta = \text{phase velocity}$$

However, the phase velocity has a meaning only with reference to an infinite monochromatic wave train.

In order to convey information, we have to resort to a group of wave trains at different frequencies the form a finite wave train or a wave packet, say, in the form of a modulated wave such that the modulation envelope contains the information to be conveyed.

The velocity of the wave packet or group of waves is called the group velocity of the wave, which also represents the velocity with which the energy is transported.

We may then find the group velocity as the velocity of the amplitude of the group of waves in the wave packet. Through any medium, which, in general, is dispersive, the wave components of the group travel with different phase velocities.

In what will follow, we are going to show, considering two components of the wave packet, that the wave group velocity v_g can be found from the relation:

$$v_g = \frac{\partial \omega}{\partial \beta}$$

Let us consider two components of the wave packet — one at $\omega - \Delta\omega$ and the other at $\omega + \Delta\omega$, which are separated in frequency by $2\Delta\omega$ with phase propagation constants $\beta - \Delta\beta$ and $\beta + \Delta\beta$ respectively.

Considering the magnitudes of the two electric field components to be the same for the sake of simplicity

$$E = E_0[\exp\ j[(\omega - \Delta\omega)t - (\beta - \Delta\beta)z] + E_0[\exp\ j[(\omega + \Delta\omega)t - (\beta + \Delta\beta)z]$$

$$E = E_0 \exp\ j(\omega t - \beta z)[\exp - j[(\Delta\omega)t - (\Delta\beta)z] + \exp\ j[(\Delta\omega)t - (\Delta\beta)z]$$

In view of the relation $\exp \pm j\phi = \cos\phi \pm j\sin\phi$

$$E = 2E_0 \cos(\Delta\omega t - \Delta\beta z)\exp\ j(\omega t - \beta z)$$

The amplitude of the electric field of the combination of the wave components of the group has the wave-like variation with distance and time appearing in the cosine function.

Putting the argument of the cosine function as constant in order to find the wave group velocity v_g

$$\Delta\omega t - \Delta\beta z = \text{constant}$$

Differentiating

$$\Delta\omega - \Delta\beta \frac{dz}{dt} = 0 \quad \rightarrow \quad \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta}$$

For small values of $\Delta\omega$ in the limit

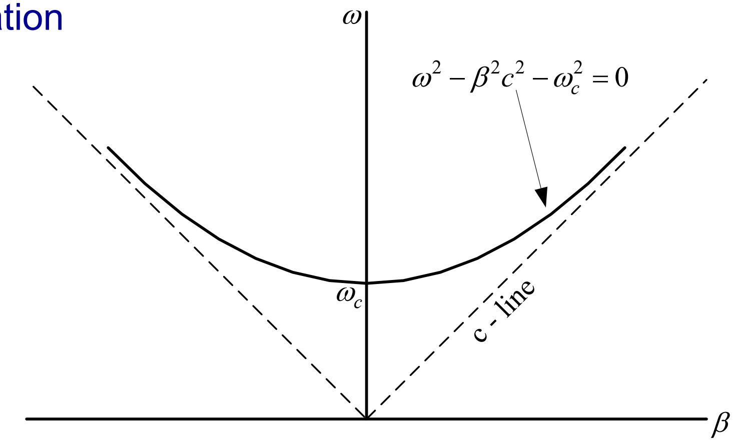
$$v_g = \frac{\partial\omega}{\partial\beta}$$

(wave group velocity)

Group velocity of the waveguide:

The group velocity v_g of the waveguide can be found from the relation

$$v_g = \frac{\partial \omega}{\partial \beta}$$



The slope of the ω-β dispersion curve of the waveguide at any point represents the group velocity of the waveguide at the point while the slope of the line joining the point and the origin of the curve represents the phase velocity of the waveguide.

Taking the partial derivative

$$\omega^2 - \beta^2 c^2 - \omega_c{}^2 = 0 \quad \longrightarrow \quad 2\omega \frac{\partial \omega}{\partial \beta} - c^2 2\beta = 0$$

$$v_{\mathrm{ph}} = \frac{\omega}{\beta}$$

$$v_g = \frac{\partial \omega}{\partial \beta} = \frac{c^2 \beta}{\omega} = \frac{c^2}{\dfrac{\omega}{\beta}} = \frac{c^2}{v_{\mathrm{ph}}} \longrightarrow v_{\mathrm{ph}} v_g = c^2 \longrightarrow \frac{v_g}{c} = \frac{c}{v_{\mathrm{ph}}} = \frac{1}{\dfrac{v_{\mathrm{ph}}}{c}} = \left(1 - \frac{f_c{}^2}{f^2}\right)^{1/2}$$

$$\frac{v_{\mathrm{ph}}}{c} = \frac{1}{\left(1 - \dfrac{f_c^2}{f^2}\right)^{1/2}}$$

You will surely enjoy an illustrative example in which we will find the length of the waveguide of cutoff wavelength 69.70 mm that will ensure that a signal at 8.6 GHz emerging out of the guide is delayed by 1 μs with respect to the signal that propagates outside the waveguide.

$$\lambda_c = 69.7\times10^{-3}\ \text{m (given)}$$

$$f_c = \frac{c}{\lambda_c} = \frac{3\times10^8}{69.7\times10^{-3}} = 4.3\times10^9\ \text{Hz} = 4.3\ \text{GHz}$$

$$\frac{v_g}{c} = \left(1-\frac{f_c^{\,2}}{f^2}\right)^{1/2}$$

$$v_g = \left(1-\frac{f_c^{\,2}}{f^2}\right)^{1/2} \times 3\times10^8$$

$$f = 8.6\ \text{GHz and } f_c = 4.3\ \text{GHz (given); } c = 3\times10^8\ \text{m/s}$$

$$v_g = (1-0.25)^{1/2}\times3\times10^8 = 2.598\times10^8\ \text{m/s}$$

$$c = 3\times10^8\ \text{m/s}$$

$$\tau_{\text{delay}} = 1\ \mu\text{s} = 10^{-6}\ \text{s}$$ (time delay) (given)

$$\tau_{\text{delay}} = l\left(\frac{1}{v_g}-\frac{1}{c}\right) = l(0.3849\times10^{-8} - 0.3333\times10^{-8}) = l(0.0516)\times10^{-8}\ \text{s}$$

(time delay)

l = waveguide length

$$l = \frac{10^{-6}}{(0.0516)\times10^{-8}} = 19.38\times10^2\ \text{m} = 1.938\ \text{km}$$

(required waveguide length)

Wave impedance:

$$Z_W = \frac{E_{\text{transverse}}}{H_{\text{transverse}}} = \frac{(E_x{}^2 + E_y{}^2)^{1/2}}{(H_x{}^2 + H_y{}^2)^{1/2}} \longleftarrow \left.\begin{aligned} H_x &= -\frac{\beta}{\omega\mu_0} E_y \\ H_y &= \frac{\beta}{\omega\mu_0} E_x \end{aligned}\right\} \text{(recalled)}$$

$$\downarrow$$

$$Z_W = \frac{E_{\text{transverse}}}{H_{\text{transverse}}} = \frac{(E_x{}^2 + E_y{}^2)^{1/2}}{[(\frac{\beta^2}{\omega^2\mu_0{}^2})(E_y{}^2 + E_x{}^2)]^{1/2}} = \frac{\omega\mu_0}{\beta} \longrightarrow \frac{Z_W}{\eta_0} = \frac{\omega\mu_0}{\beta\eta_0} = \frac{\omega\mu_0}{\beta\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}} = \frac{\omega}{\beta}(\mu_0\varepsilon_0)^{1/2}$$

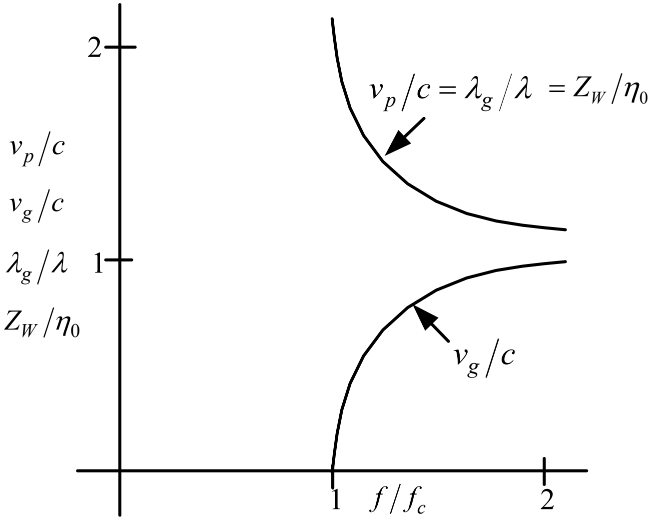
$$\left.\begin{aligned} v_{\text{ph}} &= \frac{\omega}{\beta} \\ c &= \frac{1}{(\mu_0\varepsilon_0)^{1/2}} \end{aligned}\right\} \longrightarrow \quad \downarrow \quad \longleftarrow \frac{v_{\text{ph}}}{c} = \frac{1}{\left(1 - \frac{f_c^2}{f^2}\right)^{1/2}}$$

$$\frac{Z_W}{\eta_0} = \frac{v_{\text{ph}}}{c} = \frac{1}{\left(1 - \frac{f_c^2}{f^2}\right)^{1/2}}$$

Waveguide characteristic parameters put together:

$$\frac{v_{\text{ph}}}{c} = \frac{1}{\left(1 - \frac{f_c^2}{f^2}\right)^{1/2}}$$

$$\frac{\lambda_g}{\lambda} = \frac{1}{\left(1 - \frac{f_c^2}{f^2}\right)^{1/2}}$$



$$\frac{v_g}{c} = \left(1 - \frac{f_c^2}{f^2}\right)^{1/2}$$

$$\frac{Z_W}{\eta_0} = \frac{v_{\text{ph}}}{c} = \frac{1}{\left(1 - \frac{f_c^2}{f^2}\right)^{1/2}}$$

$$\left.\begin{array}{l} v_{\text{ph}} = c \\ \lambda_g = \lambda \\ v_g = c \\ Z_W = \eta_0 \end{array}\right\} (f \to \infty)$$

$$\left.\begin{array}{l} v_{\text{ph}} = \infty \\ \lambda_g = \infty \\ v_g = 0 \\ Z_W = \infty \end{array}\right\} (f = f_c)$$

Evanescent mode

$$\beta = \pm \frac{(\omega^2 - \omega_c{}^2)^{1/2}}{c} \quad \text{(recalled)}$$

Taking the upper sign

$$\beta = \frac{(\omega^2 - \omega_c{}^2)^{1/2}}{c} \longrightarrow \text{Positive real for } \omega > \omega_c$$

For frequencies above the cutoff frequency: $\omega > \omega_c$, the phase propagation constant β becomes positive real, which occurring in the phase factor exp $j(\omega t - \beta z)$ and, in turn, in the expressions for the RF quantities, refers to a '*propagating*' wave through the waveguide along positive z.

Let us now take frequencies less than the cutoff frequency: $\omega < \omega_c$ and take the phase factor as exp $j(\omega t - \gamma z)$:

$$\gamma = j\beta = (j)(\pm \frac{(\omega^2 - \omega_c{}^2)^{1/2}}{c} = \pm(j)\frac{[(-1)(\omega_c{}^2 - \omega^2]^{1/2}}{c}$$

$$= \pm(j)\frac{j(\omega_c{}^2 - \omega^2)^{1/2}}{c} = \mp \frac{(\omega_c{}^2 - \omega^2)^{1/2}}{c} \longrightarrow \text{Positive or negative real for } \omega < \omega_c$$

Phase factor occurring in field expressions taking the positive lower sign in γ:

$$\alpha = \frac{(\omega_c{}^2 - \omega^2)^{1/2}}{c}$$

$$\exp j(\omega t - \beta z) = \exp(j\omega t)\exp(-\gamma z) = \exp(j\omega t)\exp - (\frac{(\omega_c{}^2 - \omega^2)^{1/2}}{c} z)$$

$$= \exp j\omega t \exp(-\alpha z) \longrightarrow \text{Non-propagating evanescent mode } (\omega < \omega_c)$$

Phase factor occurring in field expressions

$$\exp j(\omega t-\beta z)=\exp(j\omega t)\exp(-\gamma z)=\exp(j\omega t)\exp-\left(\frac{(\omega_c{}^2-\omega^2)^{1/2}}{c}z\right)$$ ⟵ Recalled

$$=\exp j\omega t\exp(-\alpha z)$$ (Non-propagating evanescent mode) $(\omega<\omega_c)$

$$\alpha=\frac{(\omega_c{}^2-\omega^2)^{1/2}}{c}$$

$$E_x=\frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\frac{n\pi}{b}H_{z0}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\exp j(\omega t-\beta z)\quad(m,n=0,1,2,3,\ldots)$$

(recalled)

$(\omega<\omega_c)$

⟵ Replacing $\exp j(\omega t-\beta z)$ by $\exp j\omega t\exp(-\alpha z)$

$$E_x=\frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\frac{n\pi}{b}H_{z0}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\exp(-\alpha z)\exp j\omega t\quad(m,n=0,1,2,3,\ldots)$$

$(\omega<\omega_c)$

The field amplitude decays exponentially with the axial distance z as exp(- αz) for $\omega<\omega_c$. α is known as the attenuation constant in this non-propagating mode called the evanescent mode (corresponding to non-propagating mode vanishing).

Power consideration in this mode is taken up later.

$$E_x = \frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{n\pi}{b} H_{z0} \cos(\frac{m\pi}{a} x) \sin(\frac{n\pi}{b} y) \exp(-\alpha z) \exp(j\omega t)$$ (rewritten)

↓ ← Taking complex conjugate $(\omega < \omega_c)$

$$E_x^* = \frac{-j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{n\pi}{b} H_{z0} \cos(\frac{m\pi}{a} x) \sin(\frac{n\pi}{b} y) \exp(-\alpha z) \exp(-j\omega t)$$

Following the same procedure we can write

$$E_y = \frac{-j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{m\pi}{a} H_{z0} \sin\frac{m\pi}{a} x \cos\frac{n\pi}{b} y \exp(-\alpha z) \exp(j\omega t)$$

$(\omega < \omega_c)$

$$E_y^* = \frac{j\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{m\pi}{a} H_{z0} \sin\frac{m\pi}{a} x \cos\frac{n\pi}{b} y \exp(-\alpha z) \exp(-j\omega t)$$

$$\left.\begin{aligned} H_x &= \frac{j\gamma}{\omega\mu_0} E_y \\ H_y &= \frac{-j\gamma}{\omega\mu_0} E_x \end{aligned}\right\}$$

$$\gamma = j\beta$$

$$j\gamma = j^2\beta = -\beta$$

$$(\omega < \omega_c)$$

$$\left.\begin{aligned} H_x &= -\frac{\beta}{\omega\mu_0} E_y \\ H_y &= \frac{\beta}{\omega\mu_0} E_x \end{aligned}\right\}$$ (recalled)

$$\left.\begin{aligned} H_x^* &= \frac{-j\gamma}{\omega\mu_0} E_y^* \\ H_y^* &= \frac{j\gamma}{\omega\mu_0} E_x^* \end{aligned}\right\}$$

$$(\vec{\mathrm{P}}_{\text{average}})_{z-\text{component}} = \mathrm{Re}(\vec{\mathrm{P}}_{\text{complex}})_{z-\text{component}} = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)_z$$

$$= \frac{1}{2}\mathrm{Re}(E_x H_y^* - E_y H_x^*) \qquad (\omega < \omega_c)$$

(z-component of average Poynting vector)

$$(\vec{\mathrm{P}}_{\text{average}})_{z-\text{component}} = \mathrm{Re}(\vec{\mathrm{P}}_{\text{complex}})_{z-\text{component}} = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^{*})_{z}$$

$$= \frac{1}{2}\mathrm{Re}(E_x H_y^{*} - E_y H_x^{*}) \qquad (\omega < \omega_c)$$

(rewritten)

(z-component of average Poynting vector)

↓

$$(\vec{\mathrm{P}}_{\text{average}})_{z-\text{component}} = \frac{1}{2}\mathrm{Re}(E_x H_y^{*} - E_y H_x^{*})$$

$$= \frac{1}{2}\frac{\gamma}{\omega\mu_0}\mathrm{Re}\, j(E_x E_x^{*} + E_y E_y^{*})$$

$$(\omega < \omega_c)$$

←

$$\left.\begin{aligned} H_x^{*} &= \frac{-j\gamma}{\omega\mu_0}E_y^{*} \\ H_y^{*} &= \frac{j\gamma}{\omega\mu_0}E_x^{*} \end{aligned}\right\}$$

$$E_x E_x^* = \left[\frac{\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{n\pi}{b} \right]^2 H_{z0}^2 \cos^2 \frac{m\pi}{a} x \sin^2 \frac{n\pi}{b} y \exp(-2\alpha z)$$

$$E_y E_y^* = \left[\frac{\omega\mu_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{m\pi}{a} \right]^2 H_{z0}^2 \sin^2 \frac{m\pi}{a} x \cos^2 \frac{n\pi}{b} y \exp(-2\alpha z)$$

$\longrightarrow$ Real quantities

$(\omega < \omega_c)$

$$(\vec{\mathrm{P}}_{\mathrm{average}})_{z-\mathrm{component}} = \frac{1}{2} \frac{\gamma}{\omega\mu_0} \mathrm{Re}\, j(E_x E_x^* + E_y E_y^*)$$ $\longleftarrow$ $j(E_x E_x^* + E_y E_y^*)$ is purely imaginary

$$= 0 \qquad (\omega < \omega_c)$$

The average Poynting vector being thus zero, we infer that there would be no power flow in the waveguide in the evanescent mode below cutoff ($\omega<\omega_c$). The waveguide in this mode acts as a reactive load such that the power oscillates back and forth between the source and the waveguide. .

Let us take up a simple numerical example to show that a rectangular waveguide of cross-sectional dimensions 2.5 cm × 1 cm can support typically the propagating mode TE_{10} and the evanescent mode TE_{01} at 10 GHz operating frequency.

TE_{01} mode:

$$m = 0, n = 1, \quad a = 2.5\times10^{-2}\ \text{m}, \quad b = 1.0\times10^{-2}\ \text{m}, \quad c = 3\times10^{8}\ \text{m/s}$$

TE_{10} mode:

$$m = 1, n = 0, \quad a = 2.5\times10^{-2}\ \text{m}, \quad b = 1.0\times10^{-2}\ \text{m}, \quad c = 3\times10^{8}\ \text{m/s}$$

$$f_c = \frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}$$

$$f_c = \frac{3\times10^{8}}{2}\left[\left(\frac{m}{2.5\times10^{-2}}\right)^2 + \left(\frac{n}{1.0\times10^{-2}}\right)^2\right]^{1/2}$$

(TE_{01} mode)

$$f_c = \frac{3\times10^{8}}{2\times1.0\times10^{-2}} = 15\times10^{9}\ \text{Hz} = 15\ \text{GHz}$$

(TE_{10} mode)

$$f_c = \frac{3\times10^{8}}{2\times2.5\times10^{-2}} = 6\times10^{9}\ \text{Hz} = 6\ \text{GHz}$$

$$f < f_c \longleftarrow f = 10\ \text{GHz (given)} \longrightarrow f > f_c$$

$f < f_c$ → (Evanescent mode)

$f > f_c$ → (Propagating mode)

Thus, the waveguide can support non-propagating evanescent mode TE_{01} and the propagating mode TE_{10}

What will be the nature of the attenuation constant versus frequency plot of a waveguide in the evanescent mode?

$$\alpha = \frac{(\omega_c^{\ 2} - \omega^2)^{1/2}}{c} \quad (\omega < \omega_c)$$ ← Attenuation constant in evanescent mode

(recalled)

↓

$$\alpha^2 c^2 + \omega^2 = \omega_c^{\ 2}$$

↓

$$\frac{\omega^2}{\omega_c^{\ 2}} + \frac{\alpha^2}{(\omega_c / c)^2} = 1$$ → $$\omega = x,\ \alpha = y$$ $$\omega_c = a,\ \omega_c / c = b$$ → $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(equation of an ellipse)

↓ ↓ ↓

The plot of α versus ω becomes an ellipse. The intercept of the ellipse on the α axis (ordinate), corresponding to ω = 0, is found to be ω_c/c. Similarly, the intercept of the ellipse on ω axis (abscissa), corresponding to α = 0, is found to be ω_c.

TM mode

The method of analysis of the rectangular waveguide excited in the TM mode $(E_z \neq 0, H_z = 0)$ closely follows that of the waveguide excited in the TE mode $(H_z \neq 0, E_z = 0)$ —the latter already presented. We can then obtain the TM-mode expressions using the same steps as that followed while obtaining the TE-mode expressions as follows.

TE-mode expressions already developed in steps	→	**Corresponding TM-mode expressions that follow**
$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2}$	→	$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2}$
$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (k^2 - \beta^2) H_z = 0$	→	$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (k^2 - \beta^2) E_z = 0$
$H_z = (C_1 \cos Ax + C_2 \sin Ax)$ $(C_3 \cos By + C_4 \sin By) \exp j(\omega t - \beta z)$	→	$E_z = (C_1' \cos A'x + C_2' \sin A'x)$ $(C_3' \cos B'y + C_4' \sin B'y) \exp j(\omega t - \beta z)$

TE-mode expressions already developed	Corresponding TM-mode expressions that follow
$E_y = 0$ (left sidewall $x = a$) for all y $E_y = 0$ (right sidewall $x = 0$) for all y $E_x = 0$ (bottom wall $y = 0$) for all x $E_x = 0$ (top wall $y = b$) for all x	$E_z = 0$ (left sidewall $x = a$) for all y $E_z = 0$ (right sidewall $x = 0$) for all y $E_z = 0$ (bottom wall $y = 0$) for all x $E_z = 0$ (top wall $y = b$) for all x
$H_z = H_{z0} \cos Ax \cos By \exp j(\omega t - \beta z)$	$E_z = E_{z0} \sin A'x \sin B'y \exp j(\omega t - \beta z)$
$\left.\begin{aligned} A &= \frac{m\pi}{a} \\ B &= \frac{n\pi}{b} \end{aligned}\right\} (m, n = 0, 1, 2, 3, \ldots)$	$\left.\begin{aligned} A' &= \frac{m\pi}{a} \\ B' &= \frac{n\pi}{b} \end{aligned}\right\} (m, n = 0, 1, 2, 3, \ldots)$
$H_z = H_{z0} \cos(\frac{m\pi}{a} x) \cos(\frac{n\pi}{b} y) \exp j(\omega t - \beta z)$	$E_z = E_{z0} \cos(\frac{m\pi}{a} x) \cos(\frac{n\pi}{b} y) \exp j(\omega t - \beta z)$

TE-mode expressions already developed		Corresponding TM-mode expressions that follow
$A^2-(k^2-\beta^2)=-B^2$	$\longrightarrow$	$A'^2-(k^2-\beta^2)=-B'^2$
$k^2-\beta^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$	$\longrightarrow$	$k^2-\beta^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$
$k^2-\beta^2=k_c{}^2$	$\longrightarrow$	$k^2-\beta^2=k_c{}^2$
$k_c=\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^{1/2}\quad(m,n=0,1,2,3,\ldots)$	$\longrightarrow$	$k_c=\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^{1/2}\quad(m,n=0,1,2,3,\ldots)$
$\omega^2-\beta^2c^2-k_c{}^2c^2=0$	$\longrightarrow$	$\omega^2-\beta^2c^2-k_c{}^2c^2=0$
$\omega^2-\beta^2c^2-\omega_c{}^2=0$	$\longrightarrow$	$\omega^2-\beta^2c^2-\omega_c{}^2=0$

TE-mode expressions already developed → **Corresponding TM-mode expressions that follow**

TE-mode expressions already developed	Corresponding TM-mode expressions that follow
$\omega_c = c\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$	$\omega_c = c\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$
$f_c = \frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}$	$f_c = \frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}$
$\lambda_c \left(= \frac{c}{f_c}\right) = \frac{2}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}}$	$\lambda_c \left(= \frac{c}{f_c}\right) = \frac{2}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}}$
$(m, n = 0,\ 1,\ 2,\ 3,\ \ldots)$	$(m, n = 0,\ 1,\ 2,\ 3,\ \ldots)$

Dominant mode of a rectangular waveguide

• The cutoff frequency of the TE_{10} mode is the lowest of all other modes.

• If we choose the operating frequency of the waveguide above the cutoff frequency of the TE_{10} mode, then the TE_{10} mode will propagate through the waveguide. Also, any next higher order mode will co-propagate if its cutoff frequency is lower than the operating frequency.

• For instance, let the operating frequency be 4.5 GHz for waveguide WR-284. This is higher than the cutoff frequencies of TE_{10}, TE_{20}, and TE_{01} modes , which are respectively 2.08 GHz, 4.16 GHz, and 4.41 GHz, but lower than the cutoff frequency 4.87 GHz of TE_{11}. (See slide number 33 for dimensions and cutoff frequencies of waveguide WR-284.)

• Therefore, at the operating frequency of 4.5 GHz, waveguide WR-284 will allow propagation of TE_{10}, TE_{20}, and TE_{01} but makes mode TE_{11} evanescent, attenuating it.

• Thus, TE_{10} mode (which has the lowest cutoff frequency of all the modes) will be excited in a rectangular waveguide if the operating frequency is higher than its cutoff frequency irrespective of whether any other modes are exited or not. Hence the TE_{10} mode is called the *dominant mode* of a rectangular waveguide.

• If the operating frequency is chosen at 3 GHz for waveguide WR-284, which is above the cutoff frequency 2.08 GHz of the dominant mode TE_{10} but below the cutoff frequency 4.16 GHz of TE_{20}, then only the mode TE_{10}, which is the dominant mode, will be excited in the rectangular waveguide.

• Moreover, the cutoff frequency of the TE10 mode is the lowest of the cutoff frequencies of all the TE and TM modes taken together and continues to be called the dominant mode with due consideration to the excitation of the waveguide in both the TE and TM mode types.

Starting from the expression for E_z and taking help from Maxwell's equations, we can find the transverse electric and magnetic field components in the TM mode as we have obtained earlier in the TE mode. Thus, we obtain the field expressions for the TM_{mn} mode as follows:

$$E_x = \frac{-j\beta}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{m\pi}{a} E_{z0} \cos\frac{m\pi}{a} x \sin\frac{n\pi}{b} y \exp j(\omega t - \beta z) \quad (m, n = 0, 1, 2, 3, \dots)$$

$$E_y = \frac{-j\beta}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{n\pi}{b} E_{z0} \sin\frac{m\pi}{a} x \cos\frac{n\pi}{b} y \exp j(\omega t - \beta z) \quad (m, n = 0, 1, 2, 3, \dots)$$

$$E_z = E_{z0} \cos(\frac{m\pi}{a} x) \cos(\frac{n\pi}{b} y) \exp j(\omega t - \beta z) \quad (m, n = 0, 1, 2, 3, \dots)$$

$$H_x = \frac{j\omega\varepsilon_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{n\pi}{b} E_{z0} \sin\frac{m\pi}{a} x \cos\frac{n\pi}{b} y \exp j(\omega t - \beta z) \quad (m, n = 0, 1, 2, 3, \dots)$$

$$H_y = -\frac{j\omega\varepsilon_0}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \frac{m\pi}{a} E_{z0} \cos\frac{m\pi}{a} x \sin\frac{n\pi}{b} y \exp j(\omega t - \beta z) \quad (m, n = 0, 1, 2, 3, \dots)$$

$$H_z = 0$$

(TM_{mn}-mode field expressions of a rectangular waveguide)

Field pattern and significance of mode numbers of a rectangular waveguide

Expressions for non-zero field components of a rectangular waveguide for typical lower-order modes TE_{10}, TE_{10}, TE_{20} and TM_{11}:

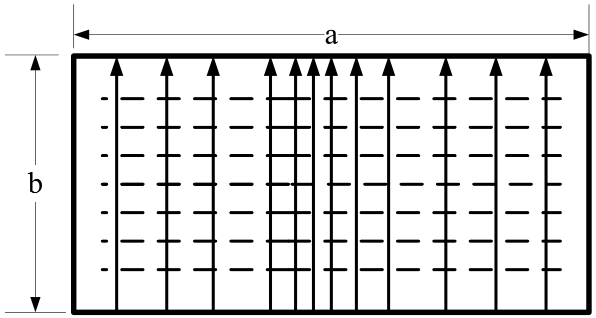
$$\left.\begin{aligned} H_z &= H_{z0}\cos\frac{\pi}{a}x\exp j(\omega t-\beta z) \\ E_y &= -\frac{j\omega\mu_0 a}{\pi}H_{z0}\sin\frac{\pi}{a}x\exp j(\omega t-\beta z) \\ H_x &= \frac{j\beta a}{\pi}H_{z0}\sin\frac{\pi}{a}x\exp j(\omega t-\beta z) \end{aligned}\right\}\ (\mathrm{TE}_{10}\ \text{mode})$$

$$\left.\begin{aligned} H_z &= H_{z0}\cos\frac{\pi}{b}y\exp j(\omega t-\beta z) \\ E_x &= \frac{j\omega\mu_0 b}{\pi}H_{z0}\sin\frac{\pi}{b}y\exp j(\omega t-\beta z) \\ H_y &= \frac{j\beta b}{\pi}H_{z0}\sin\frac{\pi}{b}y\exp j(\omega t-\beta z) \end{aligned}\right\}\ (\mathrm{TE}_{01}\ \text{mode})$$

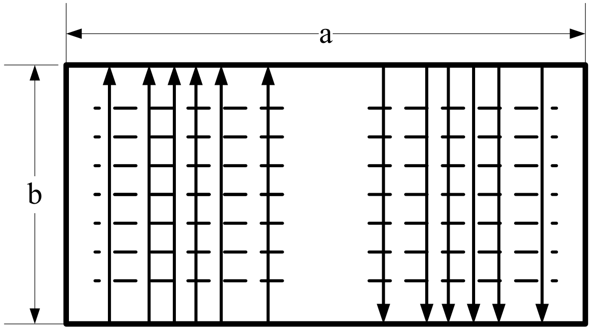
$$
\left.
\begin{aligned}
H_z &= H_{z0}\cos\frac{2\pi}{a}x\exp j(\omega t-\beta z)\\
E_y &= -\frac{j\omega\mu_0 a}{2\pi}H_{z0}\sin\frac{2\pi}{a}x\exp j(\omega t-\beta z)\\
H_x &= \frac{j\beta a}{2\pi}H_{z0}\sin\frac{2\pi}{a}x\exp j(\omega t-\beta z)
\end{aligned}
\right\}\ (\mathrm{TE_{20}\ mode})
$$

$$
\left.
\begin{aligned}
E_x &= \frac{-j\beta}{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\frac{\pi}{a}E_{z0}\cos\frac{\pi}{a}x\sin\frac{\pi}{b}y\exp j(\omega t-\beta z)\\
E_y &= \frac{-j\beta}{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\frac{\pi}{b}E_{z0}\sin\frac{\pi}{a}x\cos\frac{\pi}{b}y\exp j(\omega t-\beta z)\\
E_z &= E_{z0}\sin\frac{\pi}{a}x\sin\frac{\pi}{b}y\exp j(\omega t-\beta z)\\
H_x &= \frac{j\omega\varepsilon_0}{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\frac{\pi}{b}E_{z0}\sin\frac{\pi}{a}x\cos\frac{\pi}{b}y\exp j(\omega t-\beta z)\\
H_y &= -\frac{j\omega\varepsilon_0}{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\frac{\pi}{a}E_{z0}\cos\frac{\pi}{a}x\sin\frac{\pi}{b}y\exp j(\omega t-\beta z)
\end{aligned}
\right\}\ (\mathrm{TM_{11}\ mode})
$$

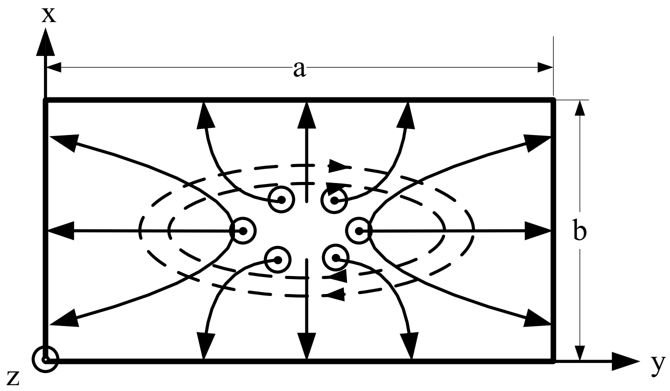
Field pattern of typical lower-order rectangular waveguide modes



TE_{10} mode



TE_{20} mode



TM_{11} mode

The field pattern of the rectangular waveguide has been more elaborately explained in the book. However, in what follows next, let us discuss the significance of the mode number vis-à-vis the field pattern.

Significance of the TE_{mn}-mode or TM_{mn}-mode field patterns of a rectangular waveguide vis-à-vis the mode numbers *m* and *n*

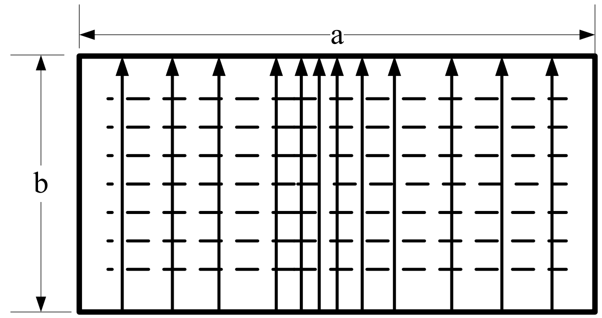
It is easy to infer by examining field patterns that

- mode number *m*, the first suffix of TE_{mn} or TM_{mn}, indicates the number of maxima of any field component, electric or magnetic, along the broad dimension of the waveguide
- mode number *n*, the second suffix of TE_{mn} or TM_{mn}, indicates the number of maxima of any field component, electric or magnetic, along the narrow dimension of the waveguide

Alternatively, *m* and *n* can be interpreted as the numbers of half-wave field patterns across the broad and narrow dimensions of the waveguide respectively.

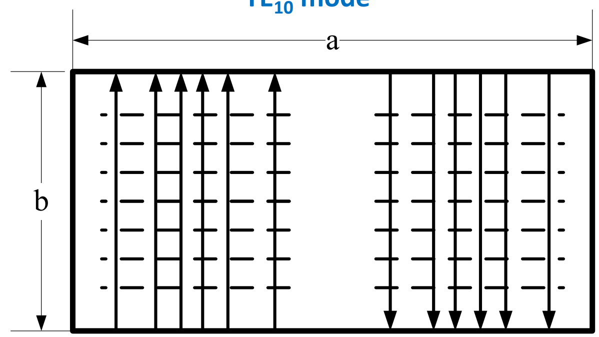
Examples:

For TE_{10} mode (m = 1, n = 0), the first suffix being unity (m = 1), there is a single maximum of the field component along the waveguide broad dimension. The second suffix being zero (n = 0), there is no maximum of the field component along the waveguide narrow dimension.



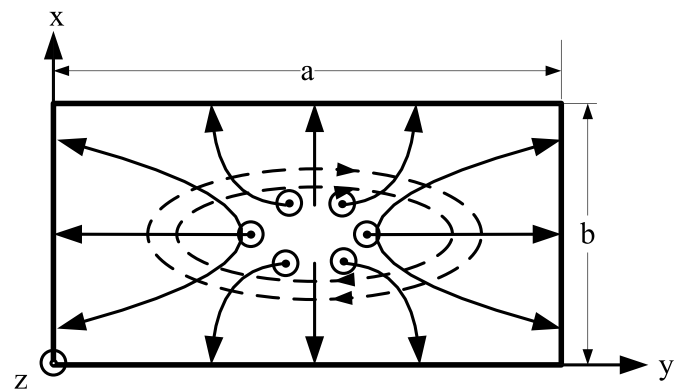
TE_{10} mode

For TE_{20} mode (m =2, n = 0), the first suffix being 2 (m = 2), there exist two maxima of the field component along the waveguide broad dimension. The second suffix being zero (n = 0) there is no maximum of the field component along the waveguide narrow dimension.



TE_{20} mode

Similar observations can be made by examining the field pattern of the TM_{mn} mode, for instance, the TM_{11} mode as well.



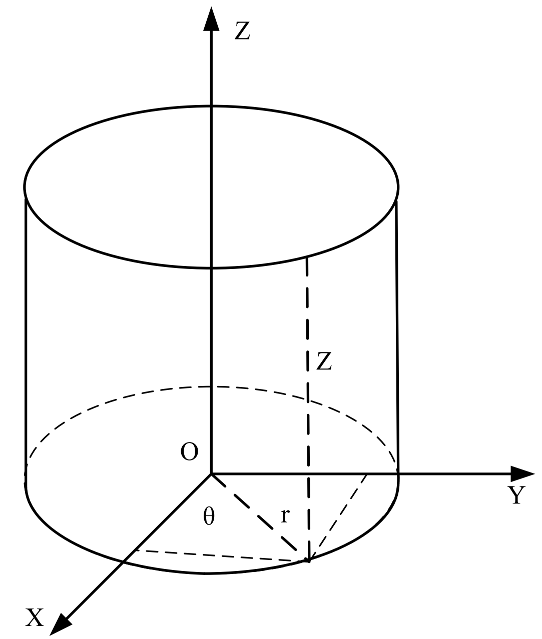
TM_{11} mode

Later on we are going to make similar inferences on the significance of mode numbers vis-à-vis field pattern with respect to a cylindrical waveguide.

Cylindrical waveguide

Cylindrical or circular waveguide can be treated in cylindrical system of coordinates (r,θ,z) in view of the circular symmetry of this type of waveguide.

For a cylindrical waveguide it turns out that the dominant mode is the TE_{11} mode characterized by having the lowest cutoff frequency. Let us emphasize here mainly the TE_{11} mode for analysis, although the analytical approach presented here is rather general and is easily applicable to the analysis of other higher order TE_{mn} and TM_{mn} modes as well.



$$\nabla^2 H_z - \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} = 0$$

← Wave equation for the TE mode $(H_z \neq 0, E_z = 0)$

← Expanding Laplacian in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2} + \frac{\partial^2 H_z}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2}+\frac{\partial^2 H_z}{\partial z^2}-\mu_0\varepsilon_0\frac{\partial^2 H_z}{\partial t^2}=0$$ (rewritten)

↓ ← Carrying out the differentiation of the first term

$$\frac{\partial^2 H_z}{\partial r^2}+\frac{1}{r}\frac{\partial H_z}{\partial r}+\frac{\partial^2 H_z}{\partial z^2}-\mu_0\varepsilon_0\frac{\partial^2 H_z}{\partial t^2}+\frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2}=0$$

↓ ← In view of the field dependence exp $j(\omega t-\beta z)$ which is understood, enabling us to put $\partial/\partial t = j\omega$ and $\partial/\partial z = -j\omega\beta$

$$\frac{\partial^2 H_z}{\partial r^2}+\frac{1}{r}\frac{\partial H_z}{\partial r}+(\omega^2\mu_0\varepsilon_0-\beta^2)H_z+\frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2}=0$$

↓ ← Rearranging terms

$$\frac{\partial^2 H_z}{\partial r^2}+\frac{1}{r}\frac{\partial H_z}{\partial r}+(\omega^2\mu_0\varepsilon_0-\beta^2)H_z=-\frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2}$$

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + (\omega^2 \mu_0 \varepsilon_0 - \beta^2) H_z = -\frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2}$$ (wave equation) (rewritten)

Let us use the method of separation of variables for solving the wave equation and put

$$H_z = R\Theta$$

R is a function of r alone and

Θ is a function of θ alone

$$\frac{\partial H_z}{\partial r} = \frac{\partial (R\Theta)}{\partial r} = \Theta \frac{\partial R}{\partial r}$$

$$\frac{\partial^2 H_z}{\partial r^2} = \Theta \frac{\partial^2 R}{\partial r^2}$$

$$\frac{\partial^2 H_z}{\partial \theta^2} = R \frac{\partial^2 \Theta}{\partial \theta^2}$$

$$\Theta \frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\Theta \frac{\partial R}{\partial r} + (\omega^2 \mu_0 \varepsilon_0 - \beta^2) R\Theta = -\frac{1}{r^2} R \frac{\partial^2 \Theta}{\partial \theta^2}$$

$$\Theta\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\Theta\frac{\partial R}{\partial r}+(\omega^2\mu_0\varepsilon_0-\beta^2)R\Theta=-\frac{1}{r^2}R\frac{\partial^2\Theta}{\partial\theta^2}$$ (rewritten)

↓ ← Dividing by $R\Theta$

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{1}{R}\frac{\partial R}{\partial r}+(\omega^2\mu_0\varepsilon_0-\beta^2)=-\frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2}\frac{1}{\Theta}$$

↓ ← $\gamma=(\omega^2\mu_0\varepsilon_0-\beta^2)^{1/2}=(\omega^2/c^2-\beta^2)^{1/2}=(k^2-\beta^2)^{1/2}$

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2}+\frac{1}{Rr}\frac{\partial R}{\partial r}+\gamma^2=-\frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2}\frac{1}{\Theta}$$

↓ ← Multiplying by r^2

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}+\gamma^2 r^2=-\frac{\partial^2\Theta}{\partial\theta^2}\frac{1}{\Theta}$$

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}+\gamma^2 r^2 = -\frac{\partial^2 \Theta}{\partial \theta^2}\frac{1}{\Theta}$$ (rewritten)

Equates a function of *r* alone on its left hand side with a function of θ alone on its right hand side. This can hold good only when the individual functions each become equal to a constant.

Putting this constant as m^2

$$\left.\begin{aligned} &\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}+\gamma^2 r^2 = m^2 \\ &-\frac{\partial^2 \Theta}{\partial \theta^2}\frac{1}{\Theta} = m^2 \end{aligned}\right\}$$

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}+\gamma^2 r^2 = m^2 \quad \text{(rewritten)}$$

↓ ← Dividng by r^2

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2}+\frac{1}{Rr}\frac{\partial R}{\partial r}+\gamma^2 = \frac{m^2}{r^2}$$

↓ ← Rearranging terms

$$\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}+(\gamma^2-\frac{m^2}{r^2})R = 0$$

↓ ← $\left.\begin{array}{l} x = \gamma r \\ r = \dfrac{x}{\gamma} \end{array}\right\}$

In terms of a dimensionless quantity *x*

$$\gamma^2\frac{\partial^2 R}{\partial x^2}+\gamma^2\frac{1}{x}\frac{\partial R}{\partial x}+\left(\gamma^2-\frac{m^2}{r^2}\right)R = 0$$

We can obtain the expression for the axial component of magnetic field H_z from the solution of its components R and Θ:

$$\left.\begin{aligned} -\frac{\partial^2\Theta}{\partial\theta^2}\frac{1}{\Theta} &= m^2 \\ \gamma^2\frac{\partial^2 R}{\partial x^2}+\gamma^2\frac{1}{x}\frac{\partial R}{\partial x}+\left(\gamma^2-\frac{m^2}{r^2}\right)R &= 0 \end{aligned}\right\} \begin{aligned} &\rightarrow \text{Solve for } \Theta \\ &\rightarrow \text{Solve for } R \end{aligned} \rightarrow H_z = R\Theta$$

(rewritten)

$$-\frac{\partial^2\Theta}{\partial\theta^2}\frac{1}{\Theta} = m^2$$

$$\downarrow$$

$$\frac{\partial^2\Theta}{\partial\theta^2}+m^2\Theta = 0$$

$\downarrow$ ← Well known solution

$$\Theta = C\cos m\theta + D\sin m\theta$$

C and D are constants

Bessel equation

$$\gamma^2\frac{\partial^2 R}{\partial x^2}+\gamma^2\frac{1}{x}\frac{\partial R}{\partial x}+\left(\gamma^2-\frac{m^2}{r^2}\right)R = 0$$

$\downarrow$ ← Well known solution

$$R = PJ_m(x)+QY_m(x)$$

P and Q are constants.

$J_m(x)$ is the m^{th} order ordinary Bessel function of the first kind.

$Y_m(x)$ is the m^{th} order Bessel function of the second kind.

$$\Theta = C\cos m\theta + D\sin m\theta$$ (rewritten)

$$\left.\begin{aligned} C &= M\cos\psi \\ D &= M\sin\psi \end{aligned}\right\}$$

$$\Theta = M(\cos m\theta\cos\psi + \sin m\theta\sin\psi)$$

$$\Theta = M\cos(m\theta - \psi)$$

$$\left.\begin{aligned} M &= (C^2 + D^2)^{1/2} \\ \psi &= \tan^{-1}\frac{D}{C} \end{aligned}\right\}$$

The value of Θ at angle θ would be the same as that at an angle $\theta + 2\pi$ (the radial coordinate r remaining unchanged)

$$\cos(m\theta - \psi) = \cos[m(\theta + 2\pi) - \psi]$$

$$= \cos[(m\theta - \psi) + 2\pi m]$$

a relation that shall hold good only for integral values of m, that is, for m = 0, 1, 2, 3, ….

$$\Theta = M\cos(m\theta - \psi)$$ (rewritten)

Choosing $\psi = 0$ that amounts to taking the reference of θ such that it corresponds to the maximum value $\Theta = M$ at $\theta = 0$

$$\Theta = M\cos m\theta$$

$$R = PJ_m(x) + QY_m(x)$$ (rewritten)

$$Y_m(x) \to \infty \text{ as } x \to 0$$

$$R \to \infty \text{ as } x \to 0$$

which in turn would make $H_z = R\Theta \to \infty$

which is an impossibility since the field cannot shoot up to infinity at x (= γr) $\to$ 0, that is at $r \to 0$ (axis of the waveguide).

In order to prevent this impossibility we must put the constant $Q = 0$ in the expression for R.

$$R = PJ_m(x) + QY_m(x) \longleftarrow Q = 0$$

$$R = PJ_m(x)$$

$$\longleftarrow x = \gamma r$$

$$R = PJ_m(\gamma r)$$

$$H_z = R\Theta \quad \text{(recalled)} \longleftarrow \left.\begin{array}{l} R = PJ_m(\gamma r) \\ \Theta = M\cos m\theta \end{array}\right\}$$

$$\downarrow$$

$$H_z = PMJ_m(\gamma r)\cos m\theta$$

$$\downarrow \quad \longleftarrow \; PM = H_{z0}$$

$$H_z = H_{z0}J_m(\gamma r)\cos m\theta$$

$\downarrow$ ⟵ Invoking the understood factor $\exp j(\omega t - \beta z)$

$$H_z = H_{z0}J_m(\gamma r)\cos m\theta \exp j(\omega t - \beta z)$$

Next we can find the rest of the field expressions with the help of the above axial component of magnetic field and the two of the following Maxwell's equations:

$$\left.\begin{array}{l} \nabla \times \vec{E} = -\mu_0 \dfrac{\partial \vec{H}}{\partial t} \\ \nabla \times \vec{H} = \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \end{array}\right\} \text{(Maxwell's equations)}$$

Maxwell's equations

$E_z = 0$ (TE mode)

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\vec{a}_r + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\vec{a}_\theta + \frac{1}{r}\left(\frac{\partial (rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial \theta}\right)\vec{a}_z = -\mu_0\left(\frac{\partial H_r}{\partial t}\vec{a}_r + \frac{\partial H_\theta}{\partial t}\vec{a}_\theta + \frac{\partial H_z}{\partial t}\vec{a}_z\right)$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\mu_0 \frac{\partial H_r}{\partial t}$$

(r-component)

$\longleftarrow E_z = 0$

$$\beta E_\theta = -\omega\mu_0 H_r$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\mu_0 \frac{\partial H_\theta}{\partial t}$$

(θ-component)

$\longleftarrow E_z = 0$

$$\beta E_r = \omega\mu_0 H_\theta$$

$$\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\left(\frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z}\right)\vec{a}_r + \left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right)\vec{a}_\theta + \frac{1}{r}\left(\frac{\partial (rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\vec{a}_z = \varepsilon_0\left(\frac{\partial E_r}{\partial t}\vec{a}_r + \frac{\partial E_\theta}{\partial t}\vec{a}_\theta + \frac{\partial E_z}{\partial t}\vec{a}_z\right)$$

$$\frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = \varepsilon_0 \frac{\partial E_r}{\partial t}$$

(r-component)

$$\frac{1}{r}\frac{\partial H_z}{\partial \theta} + j\beta H_\theta = j\omega\varepsilon_0 E_r$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \varepsilon_0 \frac{\partial E_\theta}{\partial t}$$

(θ-component)

$$-j\beta H_r - \frac{\partial H_z}{\partial r} = j\omega\varepsilon_0 E_\theta$$

$$\left.\begin{aligned} \beta E_\theta &= -\omega\mu_0 H_r \\ -j\beta H_r - \frac{\partial H_z}{\partial r} &= j\omega\varepsilon_0 E_\theta \end{aligned}\right\}$$

(recalled)

Eliminating H_r and E_θ respectively →

$$\gamma = (\omega^2\mu_0\varepsilon_0 - \beta^2)^{1/2} = (k^2 - \beta^2)^{1/2}$$

$$\left.\begin{aligned} E_\theta &= \frac{j\omega\mu_0}{\gamma^2}\frac{\partial H_z}{\partial r} \\ H_r &= \frac{-j\beta}{\gamma^2}\frac{\partial H_z}{\partial r} \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial H_z}{\partial \theta} + j\beta H_\theta &= j\omega\varepsilon_0 E_r \\ \beta E_r &= \omega\mu_0 H_\theta \end{aligned}\right\}$$

(recalled)

Eliminating E_r and H_θ respectively →

$$\left.\begin{aligned} H_\theta &= \frac{-j\beta}{\gamma^2 r}\frac{\partial H_z}{\partial \theta} \\ E_r &= \frac{-j\omega\mu_0}{\gamma^2 r}\frac{\partial H_z}{\partial \theta} \end{aligned}\right\}$$

$$H_z = H_{z0} J_m(\gamma r)\cos m\theta$$ (recalled)

↓ ← Taking partial derivatives

$$J_m'(\gamma r) = \frac{d}{dr} J_m(\gamma r) \longrightarrow$$

$$\left.\begin{aligned} \frac{\partial H_z}{\partial \theta} &= -H_{z0} m J_m(\gamma r)\sin m\theta \exp j(\omega t - \beta z) \\ \frac{\partial H_z}{\partial r} &= H_{z0}\gamma J_m'(\gamma r)\cos m\theta \exp j(\omega t - \beta z) \end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial H_z}{\partial \theta} &= -H_{z0} m J_m(\gamma r)\sin m\theta \exp j(\omega t - \beta z) \\ \frac{\partial H_z}{\partial r} &= H_{z0}\gamma J'_m(\gamma r)\cos m\theta \exp j(\omega t - \beta z)\end{aligned}\right\}$$

(rewritten)

$$\left.\begin{aligned}E_\theta &= \frac{j\omega\mu_0}{\gamma^2}\frac{\partial H_z}{\partial r} \\ H_r &= \frac{-j\beta}{\gamma^2}\frac{\partial H_z}{\partial r}\end{aligned}\right\}$$

(rewritten)

$$\left.\begin{aligned}H_\theta &= \frac{-j\beta}{\gamma^2 r}\frac{\partial H_z}{\partial \theta} \\ E_r &= \frac{-j\omega\mu_0}{\gamma^2 r}\frac{\partial H_z}{\partial \theta}\end{aligned}\right\}$$

(rewritten)

$$\left.\begin{aligned}E_\theta &= \frac{j\omega\mu_0}{\gamma} H_{z0} J'_m(\gamma r)\cos m\theta \exp j(\omega t - \beta z) \\ H_r &= \frac{-j\beta}{\gamma} H_{z0} J'_m(\gamma r)\cos m\theta \exp j(\omega t - \beta z)\end{aligned}\right\}$$

$$\left.\begin{aligned}H_\theta &= \frac{j\beta}{\gamma^2 r} H_{z0} m J_m(\gamma r)\sin m\theta \exp j(\omega t - \beta z) \\ E_r &= \frac{j\omega\mu_0 m}{\gamma^2 r} H_{z0} J_m(\gamma r)\sin m\theta \exp j(\omega t - \beta z)\end{aligned}\right\}$$

Field expressions of a cylindrical waveguide excited in the TE$_{mn}$ mode put together

$$E_r = \frac{j\omega\mu_0 m}{\gamma^2 r} H_{z0} J_m(\gamma r)\sin m\theta \exp\, j(\omega t - \beta z)$$

$$E_\theta = \frac{j\omega\mu_0}{\gamma} H_{z0} J'_m(\gamma r)\cos m\theta \exp\, j(\omega t - \beta z)$$

$$E_z = 0$$

$$H_r = \frac{-j\beta}{\gamma} H_{z0} J'_m(\gamma r)\cos m\theta \exp\, j(\omega t - \beta z)$$

$$H_\theta = \frac{j\beta}{\gamma^2 r} H_{z0} m J_m(\gamma r)\sin m\theta \exp\, j(\omega t - \beta z)$$

$$H_z = H_{z0} J_m(\gamma r)\cos m\theta \exp\, j(\omega t - \beta z)$$

(*m* and *n* are the mode numbers of the cylindrical waveguide excited in the TE$_{mn}$ mode)

Electromagnetic boundary conditions at the waveguide wall of a cylindrical waveguide

$$\vec{a}_n \times \vec{E}_2 = 0\Big|_{r=a}$$

Boundary condition at the interface between a conductor and a dielectric/free-space introduced in Chapter 7, here the interface being the conducting waveguide wall of radius $r = a$, the subscript 2 referring to the free-space region inside the waveguide

$$\vec{a}_n = -\vec{a}_r$$

Unit vector directed from region 1 (here, conducting waveguide wall) to region 2 (here, free-space region inside the waveguide), thus being radially inward at the waveguide wall

$$\vec{E}_2 = E_r\vec{a}_r + E_\theta\vec{a}_\theta + E_z\vec{a}_z$$

$$-\vec{a}_r \times (E_r\vec{a}_r + E_\theta\vec{a}_\theta) = 0\Big|_{r=a}$$

$$\left.\begin{array}{l}\vec{a}_r \times \vec{a}_r = 0\\ \vec{a}_r \times \vec{a}_\theta = \vec{a}_z\end{array}\right\}$$

$$E_\theta\vec{a}_z = 0\Big|_{r=a} \rightarrow E_\theta = 0\Big|_{r=a} \rightarrow E_\theta = \frac{j\omega\mu_0}{\gamma} H_{z0} J'_m(\gamma r)\cos m\theta \exp j(\omega t - \beta z)$$

(electromagnetic boundary condition at the conducting wall of the waveguide)

$$J'_m(\gamma a) = 0$$

has a number of zeros or roots corresponding to its multiple solutions

$$\gamma a = X_{mn}$$

(corresponding to the n^{th} zero or root, where X_{mn} is called the eigenvalue of the cylindrical waveguide excited in the mode TE_{mn})

$\gamma a = X_{mn}$ (corresponding to the n^{th} zero or root, where X_{mn} is called the eigenvalue of the cylindrical waveguide excited in the mode TE_{mn})

(rewritten)

$$\left.\begin{array}{lll} X_{01} = 3.832, & X_{02} = 7.016, & X_{03} = 10.174 \\ X_{11} = 1.841, & X_{12} = 5.31, & X_{13} = 8.536 \\ X_{21} = 3.054, & X_{22} = 6.706, & X_{23} = 9.970 \end{array}\right\}$$

(a few lower order eigenvalues of the roots taken from easily available text on Bessel functions, the eigenvalue X_{11} = 1.841 corresponding to m =1, n =1 for the TE_{11} mode being the lowest of them)

Dispersion relation and cutoff frequency of a cylindrical waveguide

$$\gamma a = X_{mn} \longleftarrow \gamma = (\omega^2 \mu_0 \varepsilon_0 - \beta^2)^{1/2} = (\omega^2 / c^2 - \beta^2)^{1/2} = (k^2 - \beta^2)^{1/2}$$

$$\downarrow$$

$$(k^2 - \beta^2)^{1/2} a = X_{mn} \longrightarrow k^2 - \beta^2 = \left(\frac{X_{mn}}{a}\right)^2 \longrightarrow k^2 - \beta^2 - k_c{}^2 = 0$$

$$\uparrow \quad k_c = \frac{X_{mn}}{a}$$

$$k = \frac{\omega}{c} \searrow \quad k_c = \frac{\omega_c}{c} \nearrow \quad (\text{into } k^2 - \beta^2 - k_c{}^2 = 0)$$

$$\downarrow$$

$$\omega^2 - \beta^2 c^2 - \omega_c{}^2 = 0$$

(dispersion relation)

$$\downarrow \quad (\text{from } k_c = \frac{X_{mn}}{a} \text{ and } k_c = \frac{\omega_c}{c})$$

$$\omega_c = \frac{X_{mn} c}{a} \longleftarrow \frac{X_{mn}}{a} = \frac{\omega_c}{c}$$

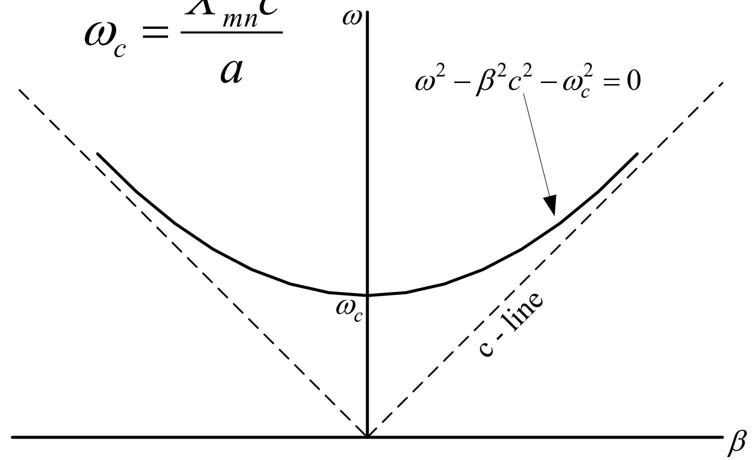
(cutoff frequency)

Dispersion relation of a cylindrical waveguide is the same as that of a rectangular waveguide with appropriate interpretation of the waveguide cutoff frequency.

$$\omega^2 - \beta^2 c^2 - \omega_c{}^2 = 0 \quad \text{(rewritten)}$$

$$\omega_c = \frac{X_{mn} c}{a}$$

Obviously, the nature of the ω-β dispersion curve of a a cylindrical waveguide is identical with that of a rectangular waveguide.



Dominant mode of a cylindrical waveguide

For a cylindrical waveguide, here considered as excited in the TE mode, we have seen that the TE_{11} mode has the lowest eigenvalue X_{11} = 1.841 and correspondingly the lowest cutoff frequency $\omega_c = X_{mn}c/a = 1.841c/a$.

In fact, the analysis in the TM mode (which, though presented earlier for a rectangular waveguide, has not been done here for a cylindrical waveguide) would reveal that, of all the modes of the TE and TM modes of a cylindrical waveguide taken together, the TE_{11} mode has the lowest cutoff frequency.

Therefore,

TE_{11} mode is the dominant mode in a cylindrical waveguide

TE_{10} is the dominant mode is in a rectangular waveguide.

In an illustrative example you will find it of interest to compare the cross-sectional area of a circular or cylindrical waveguide with that of a rectangular waveguide that has its wider dimension twice its narrower dimension and that has its cutoff frequency the same as that of the rectangular waveguide, taking both the waveguides excited in the dominant mode.

Rectangular waveguide

$$f_c = \frac{c}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}$$

(a = wider dimension; b = narrower dimension)

↓ ← Dominant mode: m =1, n = 0

$$f_c = \frac{c}{2a} \longrightarrow a = \frac{c}{2f_c}$$

↓

$$(\text{area})_{\text{rect}} = a \times b = a \times \frac{a}{2} = \frac{a^2}{2} = \frac{c^2}{8f_c^{\,2}}$$

(cross-sectional area of the given rectangular waveguide in terms of the cutoff frequency, the latter being the same as that of the given circular or cylindrical waveguide in the example)

Cylindrical waveguide

$$\omega_c = \frac{X_{mn}c}{a}$$

(recalled)

↓ ← Dominant mode: m =1, n = 1

$$2\pi f_c = \frac{X_{11}c}{a'}$$

(replacing the symbol a with a' to avoid the confusion with the symbol a for the narrow dimension of a rectangular waveguide)

↓ $(X_{11} = 1.841)$

$$f_c = \frac{X_{11}c}{2\pi a'} = \frac{1.841c}{2\pi a'} \longrightarrow a' = \frac{1.841c}{2\pi f_c}$$

↓

$$(\text{area})_{\text{circular}} = \pi a'^2 = \frac{(1.841)^2 c^2}{4\pi f_c^{\,2}}$$

$$\frac{(\text{area})_{\text{circular}}}{(\text{area})_{\text{rect}}} = \frac{(1.841)^2 c^2/(4\pi f_c^{\,2})}{c^2/(8f_c^{\,2})} = \frac{2\times(1.841)^2}{3.143} = 2.16$$

Field pattern and significance of mode numbers of a cylindrical waveguide

Field expressions of a cylindrical waveguide for typical lower-order modes TE_{01}, TE_{02} and TE_{11} (interpreting the TEmn-mode field expressions already deduced):

$$E_r = 0$$

$$E_\theta = \frac{-j\omega\mu_0}{\gamma} H_{z0} J_1\left(\frac{3.832}{a} r\right) \exp j(\omega t - \beta z)$$

$$E_z = 0$$

$$H_r = \frac{j\beta}{\gamma} H_{z0} J_1\left(\frac{3.832}{a} r\right) \exp j(\omega t - \beta z)$$

$$H_z = H_{z0} J_0\left(\frac{3.832}{a} r\right) \exp j(\omega t - \beta z)$$

$$J_0'(x) = -J_1(x)$$

(recurrence relation taking help of)

(TE_{01} mode)

$$E_r = 0$$

$$E_\theta = \frac{-j\omega\mu_0}{\gamma} H_{z0} J_1\left(\frac{7.016}{a} r\right) \exp j(\omega t - \beta z)$$

$$E_z = 0$$

$$H_r = \frac{j\beta}{\gamma} H_{z0} J_1\left(\frac{7.016}{a} r\right) \exp j(\omega t - \beta z)$$

$$H_z = H_{z0} J_0\left(\frac{7.016}{a} r\right) \exp j(\omega t - \beta z)$$

(TE$_{02}$ mode)

$J_1'(1.841r/a) =$
$(a/1.841r)J_1(1.841r/a)$
$-J_2(1.841r/a)$

(relation to be made use of)

$$E_r = \frac{j\omega\mu_0}{\gamma^2 r} H_{z0} J_1\left(\frac{1.841}{a} r\right) \sin\theta \exp j(\omega t - \beta z)$$

$$E_\theta = \frac{j\omega\mu_0}{\gamma} H_{z0} J_1'\left(\frac{1.841}{a} r\right) \cos\theta \exp j(\omega t - \beta z)$$

$$E_z = 0$$

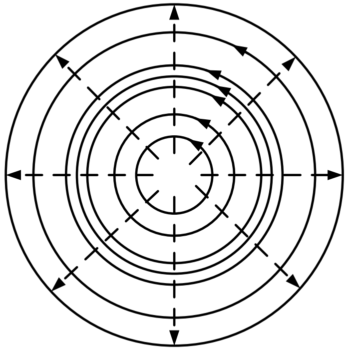
$$H_r = \frac{-j\beta}{\gamma} H_{z0} J_1'\left(\frac{1.841}{a} r\right) \cos\theta \exp j(\omega t - \beta z)$$

$$H_\theta = \frac{j\beta}{\gamma^2 r} H_{z0} J_1\left(\frac{1.841}{a} r\right) \sin\theta \exp j(\omega t - \beta z)$$

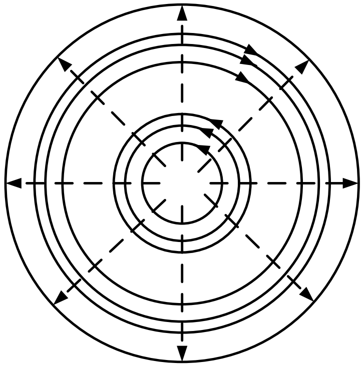
$$H_z = H_{z0} J_1\left(\frac{1.841}{a} r\right) \cos\theta \exp j(\omega t - \beta z)$$

(TE$_{11}$ mode)

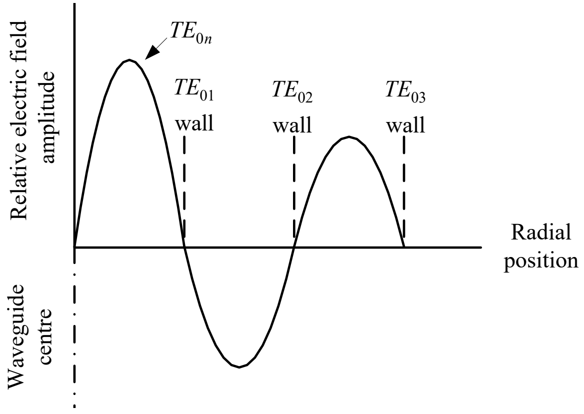
Field pattern of typical lower-order cylindrical waveguide modes



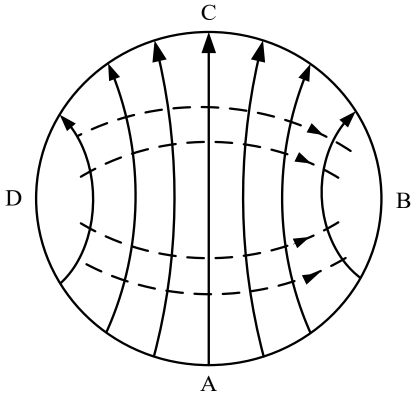
(TE_{01} mode)



(TE_{02} mode)



(TE_{0n} mode; n = 1, 2, 3)



(TE_{11} mode)

The field pattern of the cylindrical waveguide has been more elaborately explained in the book. However, in what follows next let us discuss the significance of the mode number vis-à-vis the field pattern of a cylindrical waveguide.

Significance of mode numbers of a cylindrical waveguide

TE_{01} mode:

mode number $m = 0$ $\Rightarrow$ no half-wave field patterns around the half circumference
mode number $n = 1$ $\Rightarrow$ one field maximum across the waveguide radius approximately mid-way between the axis and the wall of the waveguide

TE_{02} mode:

mode number $m = 0$ $\Rightarrow$ no half-wave field patterns around the half circumference
mode number $n = 2$ $\Rightarrow$ two maxima across the waveguide radius, between the axis and the wall of the waveguide, such that if one of the maxima is positive the other is negative.

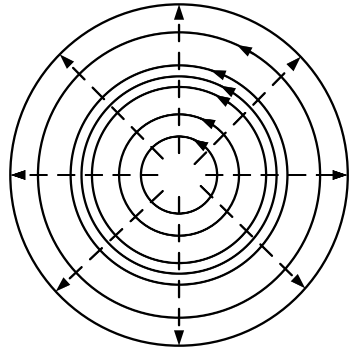
TE_{11} mode:

mode number $m = 1$ $\Rightarrow$ a single half-wave field pattern around the half circumference (or a single full-wave field pattern around the full circumference) of the waveguide
mode number $n = 1$ $\Rightarrow$ across the waveguide radius, a single maximum at the axis of the waveguide

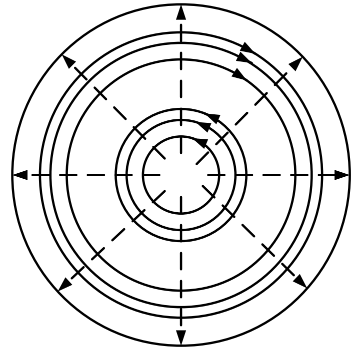
Based on the observation for the TE_{11} mode, TE_{01} and the TE_{02} modes above, the meaning of the mode numbers of a cylindrical waveguide excited in the TE_{mn} mode is

- m (azimuthal mode number) is the number of half-wave field patterns around the half waveguide circumference;
- n (radial mode number) is the number of maxima, positive and negative inclusive, across the waveguide radius.

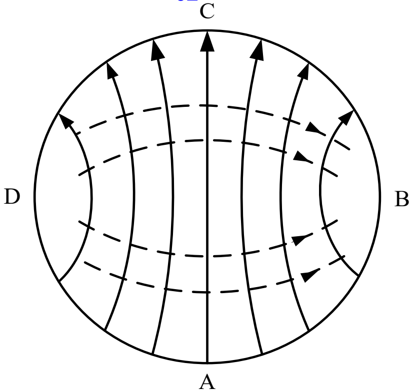
Such interpretation is valid for the TM_{mn} mode as well.



(TE_{01} mode)



(TE_{02} mode)



(TE_{11} mode)

Inability of a hollow-pipe waveguide to support a TEM mode:

The TEM mode is characterized by transverse electric and magnetic fields and no axial electric and magnetic field components ($E_z = H_z = 0$).

However, for continuous magnetic field lines to exist in a hollow-pipe waveguide, there should be an axial current present in the waveguide in the form of either conduction or displacement current in the axial direction.

The absence of a conductor does not allow such conduction current in the axial direction. Further, for the displacement current in the axial direction to exist in the waveguide, there should be an axial electric field, the time variation of which being responsible for such current.

However, the TEM mode does not permit the axial electric field. Therefore, a hollow-pipe waveguide cannot support the TEM mode.

On the other hand, a two-conductor structure comprising a hollow cylindrical waveguide with a conducting coaxial solid circular rod insert, known as a coaxial waveguide, can support the TEM mode.

Power Flow and power loss in a waveguide

We can find power transmitted in the axial direction z of a rectangular waveguide by integrating the axial z-component of the average complex Poynting vector (see Chapter 8):

$$\vec{P}_{\text{average}} = 1/2\,\text{Re}\,\vec{E}\times\vec{H}^*$$

over the waveguide cross-sectional area (= $a\times b$) transverse to the axis z of the waveguide as follows:

$$P = \int_{x=0}^{x=a}\int_{y=0}^{y=b}\frac{1}{2}\text{Re}(\vec{E}\times\vec{H}^*)_z\,dxdy$$

$$(\vec{E}\times\vec{H}^*)_z = E_x H_y^* - E_y H_x^*$$

$$P = \int_{x=0}^{x=a}\int_{y=0}^{y=b}\frac{1}{2}\text{Re}(E_x H_y^* - E_y H_x^*)\,dxdy$$

$$P = \int_{x=0}^{x=a}\int_{y=0}^{y=b} \frac{1}{2}\mathrm{Re}(E_x H_y^* - E_y H_x^*)dxdy$$ (recalled)

(power transmitted in the axial direction *z* of a rectangular waveguide)

Restricting the analysis to the dominant mode TE_{10}

$$\left.\begin{aligned}
E_x &= H_y = 0\\
E_y &= -\frac{j\omega\mu_0}{\pi/a} H_{z0} \sin\frac{\pi}{a}x \exp j(\omega t - \beta z)\\
H_x &= \frac{j\beta}{\pi/a} H_{z0} \sin\frac{\pi}{a}x \exp j(\omega t - \beta z)\\
H_x^* &= \frac{-j\beta}{\pi/a} H_{z0} \sin\frac{\pi}{a}x \exp - j(\omega t - \beta z)
\end{aligned}\right\}$$ (TE_{10} mode)

$$P = \frac{1}{2}\frac{\omega\mu_0 a}{\pi}\frac{\beta a}{\pi} H_{z0}^2 \int_{x=0}^{x=a} \sin^2\frac{\pi x}{a}dx \int_{y=0}^{y=b} dy \quad \longleftarrow \quad \int_{x=0}^{x=a} \sin^2\frac{\pi x}{a}dx = \frac{a}{2}; \quad \int_{y=0}^{y=b} dy = b$$

$$P = \frac{1}{2}\frac{\omega\mu_0 a}{\pi}\frac{\beta a}{\pi} H_{z0}^2 \left(\frac{a}{2}\right)(b) \longrightarrow P = \frac{1}{4\pi^2}\omega\mu_0\beta a^3 b H_{z0}^2$$ (TE_{10} mode)

(power transmitted in the axial direction *z* of a rectangular waveguide)

Power handling capability of a waveguide

Let us find an expression for the maximum permissible P_{maximum}, that is, the power handling capability of the waveguide for a known magnitude of the maximum electric field amplitude $\left|E_y\right|_{\mathrm{maximun}}$ which the atmosphere of the inside of the waveguide can withstand before it breaks down.

Let us take a rectangular waveguide excited in the dominant TE_{10} mode.

$$E_y = -\frac{j\omega\mu_0 a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)$$ (dominant-mode TE_{10} field expressions) (recalled)

↓

$$H_{z0} = \frac{\pi \left|E_y\right|_{\mathrm{maximun}}}{\omega\mu_0 a}$$ (axial magnetic field amplitude in terms of the breakdown electric field)

↓

$$P = \frac{1}{4\pi^2}\omega\mu_0\beta a^3 b H_{z0}^2$$ (power transmitted in the axial direction z of a rectangular waveguide)

↓

$$P_{\mathrm{maximum}} = \frac{1}{4\omega\mu_0}\beta ab\left(\left|E_y\right|_{\mathrm{maximun}}\right)^2$$ (power handling capability of the rectangular waveguide in the dominant TE_{10} mode)

$$P_{\text{maximum}} = \frac{1}{4\omega\mu_0} \beta ab(|E_y|_{\text{maximun}})^2$$ (rewritten)

$$\omega / \beta = v_{\text{ph}} \rightarrow 2\pi f / \beta$$
$$v_{\text{ph}} / c = 1/(1 - f_c^2 / f^2)^{1/2}$$ (recalled)

$$P_{\text{maximum}} = \frac{ab(1 - f_c^2 / f^2)^{1/2}}{4\mu_0 c} (|E_y|_{\text{maximun}})^2$$

In an illustrative example let us calculate the maximum power handling capability of WR-340 rectangular waveguide, excited in the dominant mode, operating at 3 GHz frequency, taking the breakdown limit of air as 29 kV/cm and taking the waveguide dimensions as:

$$a = 3.4'' = 3.4 \times 2.54 \times 10^{-2} \text{ m and } b = 1.7'' = 1.7 \times 2.54 \times 10^{-2} \text{ m.}$$

Recalling the expression for the cutoff frequency of a rectangular waveguide in the dominant mode, we can calculate:

$$f_c = \frac{c}{2a} \leftarrow a = 3.4'' = 3.4 \times 2.54 \times 10^{-2} \text{ m}$$ (dominant mode: *m* =1, *n* = 0)

$$f_c = 1.737 \times 10^6 \text{ Hz} = 1.737 \text{ GHz}$$

$$ab = 3.4\times2.54\times1.7\times2.54\times10^{-4}$$
$$= 37.29\times10^{-4}\,\text{m}^2$$
$$\left(\frac{f_c}{f}\right)^2 = 0.335$$
$$(|E_y|_{\text{maximun}})^2 = (29\times10^5)^2\,(\text{V/m})^2$$

←

$$a = 3.4'' = 3.4\times2.54\times10^{-2}\ \text{m}$$
$$b = 1.7'' = 1.7\times2.54\times10^{-2}\ \text{m}$$
(given)
$$f_c = 1.737\times10^6\,\text{Hz} = 1.737\,\text{GHz}$$
(rewritten)
$$f = 3\,\text{GHz} = 3\times10^6\ \text{Hz}$$
(given)
$$|E_y|_{\text{maximun}} = 29\ \text{kV/cm} = 29\times10^5\ \text{V/m}$$
(given)

↓

$$P_{\text{maximum}} = \frac{ab(1-f_c^{\,2}/f^{\,2})^{1/2}}{4\mu_0 c}(|E_y|_{\text{maximun}})^2$$

←

$$\mu_0 = 4\pi\times10^{-7}\ \text{H/m}$$
$$c = 3\times10^8\ \text{m/s}$$

↓

$$P_{\text{maximum}} = \frac{37.29\times0.8155\times29\times29}{4\times4\times3.143\times\times3}\times10^{-4}\times10^{9}\ \text{W}$$
$$= 16.952\times10^6\ \text{W} = 16.952\ \text{MW}$$

Power loss per unit area and power loss per unit length of a rectangular waveguide

Power loss per unit area at a conducting surface here the interface between the conducting wall and the inside of the rectangular waveguide is given by (see Chapter 8)

$$P_{\mathrm{LA}} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^* \longleftarrow$$

P_{LA} = power loss per unit area

R_S = surface resistance

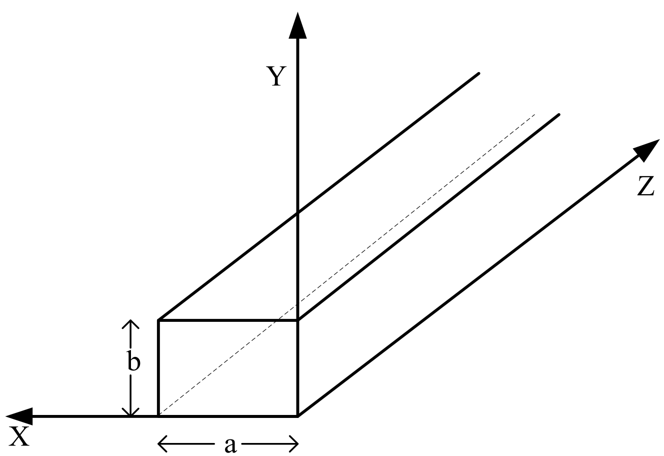
$\vec{J}_s$ = surface current density

Surface current density at any of the four waveguide walls can be found from the following electromagnetic boundary condition at the concerned waveguide wall:

$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s \longleftarrow$$

Recalling (from Chapter 7) the electromagnetic boundary condition at the interface between region 2, here the free-space region inside the waveguide wall, and region 1, here the conducting wall region

Let us next find the surface current densities developed at the right side wall (x = 0), left side wall (x = a), bottom wall (y = 0) and top wall (y = b) respectively.



$\vec{a}_n$ is the unit vector directed from the waveguide region 1 (conducting wall) to region 2 (free-space inside)

$$H_x = \frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)$$

$$H_y = 0$$

$$H_z = H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)$$

(TE_{10} mode)

(recalled)

$$\vec{H}_2 = H_x \vec{a}_x + H_y \vec{a}_y + H_z \vec{a}_z \longleftarrow$$

$$\downarrow$$

$$\vec{H}_2 = \frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z$$

$$\downarrow$$

$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$ (boundary condition) (recalled) $\longleftarrow$ $\vec{a}_n = \vec{a}_x$ (right side wall)

$$\vec{J}_{s\ \text{right sidewall}} = [\vec{a}_x] \times \left[\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z \right]$$

$$\vec{a}_x \times \vec{a}_x = 0$$

$$\vec{a}_x \times \vec{a}_z = -\vec{a}_y$$

$$x = 0 \ \text{(right side wall)}$$

$$\cos 0 = 1$$

$$\downarrow$$

$$\vec{J}_{s\ \text{right sidewall}} = -H_{z0} \exp j(\omega t - \beta z)\vec{a}_y$$

Similarly, we can derive next an expression for the surface current density developed at the left side wall.

$$\vec{H}_2 = \frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z$$

$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$$ (boundary condition) (recalled) ← $\vec{a}_n = -\vec{a}_x$ (left side wall)

$$\vec{J}_{s\ \text{leftsidewall}} = [-\vec{a}_x] \times \left[\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z \right]$$

$$\left.\begin{aligned} &\vec{a}_x \times \vec{a}_x = 0 \\ &\vec{a}_x \times \vec{a}_z = -\vec{a}_y \\ &x = a \ \ \text{(left side wall)} \\ &\cos\pi = -1 \end{aligned}\right\}$$

$$\vec{J}_{s\ \text{leftsidewall}} = -H_{z0} \exp j(\omega t - \beta z)\vec{a}_y$$

Next, let us derive next an expression for the surface current density developed at the bottom wall.

$$\vec{H}_2 = \frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z$$ (TE$_{10}$ mode)

↓

$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$$ (boundary condition) (recalled) ← $\vec{a}_n = \vec{a}_y$ (bottom wall)

↓

$$\vec{J}_{s \text{ bottom wall}} = [\vec{a}_y] \times \left[\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z \right]$$

↑

$$\left. \begin{aligned} &\vec{a}_y \times \vec{a}_x = -\vec{a}_z \\ &\vec{a}_y \times \vec{a}_z = \vec{a}_x \\ &y = 0 \text{ (bottom wall)} \end{aligned} \right\}$$

↓

$$\vec{J}_{s \text{ bottom wall}} = -\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x$$

Next, let us derive next an expression for the surface current density developed at the top wall.

$$\vec{H}_2 = \frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a}x \exp j(\omega t - \beta z)\vec{a}_x + H_{z0}\cos\frac{\pi}{a}x \exp j(\omega t - \beta z)\vec{a}_z$$

$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$$ (boundary condition) (recalled) ← $\vec{a}_n = -\vec{a}_y$ (top wall)

$$\vec{J}_{s \text{ top wall}} = [-\vec{a}_y] \times \left[\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a}x \exp j(\omega t - \beta z)\vec{a}_x + H_{z0}\cos\frac{\pi}{a}x \exp j(\omega t - \beta z)\vec{a}_z\right]$$

$$\left.\begin{array}{l} \vec{a}_y \times \vec{a}_x = -\vec{a}_z \\ \vec{a}_y \times \vec{a}_z = \vec{a}_x \\ y = b \ \ (\text{top wall}) \end{array}\right\}$$

$$\vec{J}_{s \text{ top wall}} = \frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a}x \exp j(\omega t - \beta z)\ \vec{a}_z - H_{z0}\cos\frac{\pi}{a}x \exp j(\omega t - \beta z)\ \vec{a}_x$$

Expressions for surface current densities at the waveguide walls are required to find power loss per unit area P_{LA} in terms of their surface resistance R_S for each of these walls with the help of the expression:

$$P_{LA} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^* \quad \text{(deduced in Chapter 8)}$$

(TE$_{10}$ mode) (rewritten)

$$\vec{J}_{s \text{ right sidewall}} = -H_{z0} \exp j(\omega t - \beta z)\vec{a}_y$$

$$\vec{J}_{s \text{ left sidewall}} = -H_{z0} \exp j(\omega t - \beta z)\vec{a}_y$$

$$\vec{J}_{s \text{ bottom wall}} = -\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a}x \exp j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a}x \exp j(\omega t - \beta z)\vec{a}_x$$

$$\vec{J}_{s \text{ top wall}} = \frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a}x \exp j(\omega t - \beta z)\, \vec{a}_z - H_{z0} \cos\frac{\pi}{a}x \exp j(\omega t - \beta z)\, \vec{a}_x$$

Taking complex conjugate

$$\vec{J}^*_{s \text{ right sidewall}} = -H_{z0} \exp - j(\omega t - \beta z)\vec{a}_y$$

$$\vec{J}^*_{s \text{ left sidewall}} = -H_{z0} \exp - j(\omega t - \beta z)\vec{a}_y$$

$$\vec{J}^*_{s \text{ bottom wall}} = \frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a}x \exp - j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a}x \exp - j(\omega t - \beta z)\vec{a}_x$$

$$\vec{J}^*_{s \text{ top wall}} = -\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a}x \exp - j(\omega t - \beta z)\, \vec{a}_z - H_{z0} \cos\frac{\pi}{a}x \exp - j(\omega t - \beta z)\, \vec{a}_x$$

Using the expressions for the surface current densities and those of their complex conjugates already presented, which are involved in the expression for power loss per unit area P_{LA}, we can write the following expressions with respect to all the four sides of the waveguide wall:

$$\vec{J}_s \cdot \vec{J}_s^* = [-H_{z0} \exp j(\omega t - \beta z)\vec{a}_y] \cdot [-H_{z0} \exp - j(\omega t - \beta z)\vec{a}_y] \quad \text{(right side wall)}$$

$$\vec{J}_s \cdot \vec{J}_s^* = [-H_{z0} \exp j(\omega t - \beta z)\vec{a}_y] \cdot [-H_{z0} \exp - j(\omega t - \beta z)\vec{a}_y] \quad \text{(left side wall)}$$

$$\vec{J}_s \cdot \vec{J}_s^* = [-\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x]$$
$$\cdot [\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp - j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a} x \exp - j(\omega t - \beta z)\vec{a}_x] \quad \text{(bottom wall)}$$

$$\vec{J}_s \cdot \vec{J}_s^* = [\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\ \vec{a}_z - H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\ \vec{a}_x \exp j(\omega t - \beta z)\vec{a}_x]$$
$$\cdot [-\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp - j(\omega t - \beta z)\ \vec{a}_z - H_{z0} \cos\frac{\pi}{a} x \exp - j(\omega t - \beta z)\vec{a}_x] \quad \text{(top wall)}$$

$$\downarrow$$

We can next simplify the above expressions before they can be used in the expression for P_{LA}.

$$P_{\mathrm{LA}} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^* = \frac{1}{2}[-H_{z0} \exp j(\omega t - \beta z)\vec{a}_y] \cdot [-H_{z0} \exp - j(\omega t - \beta z)\vec{a}_y]$$

$$= \frac{1}{2} R_S H_{z0}^{\ 2}$$ (right side wall)

$$P_{\mathrm{LA}} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^* = \frac{1}{2}[-H_{z0} \exp j(\omega t - \beta z)\vec{a}_y] \cdot [-H_{z0} \exp - j(\omega t - \beta z)\vec{a}_y]$$

$$= \frac{1}{2} R_S H_{z0}^{\ 2}$$ (left side wall)

$$P_{\mathrm{LA}} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^* = \frac{1}{2} R_S [-\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x]$$

$$\cdot [\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp - j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a} x \exp - j(\omega t - \beta z)\vec{a}_x]$$

$$= \frac{1}{2} R_S H_{z0}^{\ 2} \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x \right)$$ (bottom wall)

$$P_{\mathrm{LA}} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^* = \frac{1}{2} R_S [-\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)\vec{a}_x]$$

$$\cdot [\frac{j\beta a}{\pi} H_{z0} \sin\frac{\pi}{a} x \exp - j(\omega t - \beta z)\vec{a}_z + H_{z0} \cos\frac{\pi}{a} x \exp - j(\omega t - \beta z)\vec{a}_x]$$

$$= \frac{1}{2} R_S H_{z0}^{\ 2} \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x \right)$$ (top wall)

We can then put together the expressions for power loss per unit area on all the four walls of the waveguide:

$$\left.\begin{aligned} P_{\text{LA,right sidewall}} &= P_{\text{LA,left sidewall}} = \frac{1}{2} R_S {H_{z0}}^2 \\ P_{\text{LA,bottom wall}} &= P_{\text{LA,top wall}} = \frac{1}{2} R_S {H_{z0}}^2 \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x \right) \end{aligned}\right\}$$

From the expression for power loss per unit area, we can also find power loss per unit length of the waveguide as follows.

$$dP_{\text{LL}} = \frac{1}{2} R_S {H_{z0}}^2 dydz$$ (right sidewall) ← Element of power loss dP_{L} over an element of strip of elemental thickness *dz* and elemental area *dydz* across the narrow dimension of the waveguide on its right sidewall (x = 0)

↓

$$P_{\text{LL,right sidewall}} = \int dP_{\text{LL}} = \frac{1}{2} R_S {H_{z0}}^2 \int_{y=0}^{y=b} dy \int_{z=0}^{z=1} dz$$

$$= \frac{1}{2} R_S {H_{z0}}^2 b$$ (right sidewall) ← Power loss per unit length P_{LL} across the narrow dimension of the waveguide on its right sidewall

$$dP_{\text{LL}} = \frac{1}{2} R_S {H_{z0}}^2 dydz$$ (left sidewall) ← Element of power loss dP_{L} over an element of strip of elemental thickness *dz* and elemental area *dydz* across the narrow dimension of the waveguide on its left sidewall (x = a)

↓

$$P_{\text{LL,left sidewall}} = \int dP_{\text{LL}} == \frac{1}{2} R_S {H_{z0}}^2 \int_{y=0}^{y=b} dy \int_{z=0}^{z=1} dz$$

$$= \frac{1}{2} R_S {H_{z0}}^2 b$$ (left sidewall) ← Power loss per unit length P_{LL} across the narrow dimension of the waveguide on its left sidewall

$$P_{\text{LA,bottom wall}} = \frac{1}{2} R_S H_{z0}{}^2 \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x \right)$$

$$dP_{\text{LL}} = \frac{1}{2} R_S H_{z0}{}^2 \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x \right) dxdz$$

Element of power loss dP_{L} over an element of strip of elemental thickness *dz* and elemental area *dxdz* across the broad dimension of the waveguide on its bottom wall (y = 0)

(bottom wall)

$$P_{\text{LL,bottom wall}} = \int dP_{\text{LL}} = \frac{1}{2} R_S H_{z0}{}^2 \int_{y=0}^{y=a} \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x\right) dx \int_{z=0}^{z=1} dz$$ (bottom wall)

(using the relation

$$\sin^2 \frac{\pi}{a} x = \frac{1 - \cos \frac{2\pi}{a} x}{2}$$

$$\cos^2 \frac{\pi}{a} x = \frac{1 + \cos \frac{2\pi}{a} x}{2}$$

and evaluating the integral)

Power loss per unit length P_{LL} across the broad dimension of the waveguide on its bottom wall

$$P_{\text{LL,bottom wall}} = \frac{1}{2} R_S H_{z0}{}^2 \frac{a}{2} \left(\frac{\beta^2 a^2}{\pi^2} + 1 \right)$$

$$P_{\text{LA,top wall}} = \frac{1}{2} R_S H_{z0}^{\;2} \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x \right)$$

$$dP_{\text{LL}} = \frac{1}{2} R_S H_{z0}^{\;2} \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x \right) dxdz$$

(top wall)

Element of power loss dP_{L} over an element of strip of elemental thickness *dz* and elemental area *dxdz* across the broad dimension of the waveguide on its top wall ($y = b$)

$$P_{\text{LL,top wall}} = \int dP_{\text{LL}} = \frac{1}{2} R_S H_{z0}^{\;2} \int_{y=0}^{y=a} \left(\frac{\beta^2 a^2}{\pi^2} \sin^2 \frac{\pi}{a} x + \cos^2 \frac{\pi}{a} x \right) dx \int_{z=0}^{z=1} dz$$

(top wall)

Power loss per unit length P_{LL} across the broad dimension of the waveguide on its top wall

(using the relation

$$\sin^2 \frac{\pi}{a} x = \frac{1 - \cos \frac{2\pi}{a} x}{2}$$

$$\cos^2 \frac{\pi}{a} x = \frac{1 + \cos \frac{2\pi}{a} x}{2}$$

and evaluating the integral)

$$P_{\text{LL,top wall}} = \frac{1}{2} R_S H_{z0}^{\;2} \frac{a}{2} \left(\frac{\beta^2 a^2}{\pi^2} + 1 \right)$$

Power loss per unit length of the rectangular waveguide adding contributions of all the four walls

$$P_{\mathrm{LL}} = \frac{1}{2} R_S {H_{z0}}^2 b + \frac{1}{2} R_S {H_{z0}}^2 b + \frac{1}{2} R_S {H_{z0}}^2 \frac{a}{2}\left(\frac{\beta^2 a^2}{\pi^2} + 1\right) + \frac{1}{2} R_S {H_{z0}}^2 \frac{a}{2}\left(\frac{\beta^2 a^2}{\pi^2} + 1\right)$$

$$= R_S {H_{z0}}^2 \left[b + \frac{a}{2}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right)\right]$$ (TE_{10} mode)

The above expression is useful in finding the expression for the attenuation constant of the waveguide.

Power loss caused by the finite resistivity of the waveguide wall may be accounted for by generalizing the dependence of field components as

$$\exp(j\omega t - \gamma z) = \exp j\omega t \exp(-\alpha z)\exp(-j\beta z) = \exp(-\alpha z)\exp j(\omega t - \beta z)$$

↓

The amplitude of the field component decreases exponentially with *z*, with the factor exp ($-\alpha z$)

↓

Power *P* (*z*) transmitted through the waveguide decreases exponentially with *z*, with the factor exp ($-2\alpha z$) →

↓

$$P(z) = P(0)\exp(-2\alpha z)$$

(power transmitted through the waveguide)

$\gamma = \alpha + j\beta =$ propagation constant

$\beta =$ phase propagation constant

$\alpha =$ attenuation constant

See for instance average complex Poynting or power density vector involving the electric and magnetic field components each decreasing with the factor exp ($-\alpha z$) making their product depending on the factor exp ($-2\alpha z$) → $\vec{\mathrm{P}}_{\mathrm{average}} = 1/2\,\mathrm{Re}\,\vec{E}\times\vec{H}^*$

$$P(z) = P(0)\exp(-2\alpha z)$$ (rewritten)

↓ ← Differentiating

$$\frac{dP(z)}{dz} = -2\alpha P(0)\exp(-2\alpha z) = -2\alpha P(z)$$

↓

$$\alpha = \frac{-\dfrac{dP(z)}{dz}}{2P(z)}$$ ← $$-\frac{dP(z)}{dz} = P_{\mathrm{LL}}$$

↓

$$\alpha = \frac{P_{\mathrm{LL}}}{2P(z)}$$

↓ ← dropping the parenthesis from *P*(*z*)

$$\alpha = \frac{P_{\mathrm{LL}}}{2P}$$

$$P_{\mathrm{LL}} = R_S H_{z0}{}^2\left[b + \frac{a}{2}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right)\right]$$ (recalled) (TE_{10} mode)

$$\alpha = \frac{P_{\mathrm{LL}}}{2P}$$ (rewritten)

$$P = \frac{1}{4\pi^2}\omega\mu_0\beta a^3 b H_{z0}^2$$ (recalled) (TE_{10} mode)

$$\alpha = \frac{2\pi^2 R_S\left[b + \frac{a}{2}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right)\right]}{\omega\mu_0\beta\, a^3 b}$$

$$R_s = \frac{1}{\sigma\delta}$$ (in terms of the conductivity σ and skin depth δ of the material of the waveguide wall; see Chapter 6)

$$\beta = \frac{\omega}{v_{\mathrm{ph}}}$$

$$\frac{v_{\mathrm{ph}}}{c} = \frac{1}{\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{1/2}}$$ (recalled)

$$\alpha = \frac{2\pi^2\left[b + \frac{a}{2}\left(1 + \frac{\beta^2 a^2}{\pi^2}\right)\right]}{\sigma\,\delta\,\omega\mu_0\,\beta\, a^3 b}$$

$$\alpha = \frac{1}{\sigma\,\delta\,\eta_0\, b}\,\frac{\omega_c}{\omega}\,\frac{\frac{2b}{a} + \frac{\omega^2}{\omega_c{}^2}}{\left(\frac{\omega^2}{\omega_c{}^2} - 1\right)^{1/2}} \qquad \left(\omega_c = \frac{c\pi}{a}\right)$$ (TE_{10} mode)

(attenuation constant of the rectangular waveguide in the dominant TE_{10} mode)

In a numerical example, let us calculate the attenuation constant of a rectangular waveguide ($a = 0.9''$ and $b = 0.4''$; cutoff frequency = 6.56 GHz) made of aluminum (σ = 35.4×10^6 mho/m) at the operating frequency 9 GHz in the dominant mode.

$$\alpha = \frac{1}{\sigma\delta\eta_0 b}\frac{\omega_c}{\omega}\frac{\frac{2b}{a}+\frac{\omega^2}{\omega_c^{\,2}}}{\left(\frac{\omega^2}{\omega_c^{\,2}}-1\right)^{1/2}} \quad (\omega_c = \frac{c\pi}{a})$$

(TE_{10} dominant mode) (recalled)

After a simple algebra and recalling $\delta = \sqrt{\frac{1}{\pi f \mu_0 \sigma}}$ (skin depth)

$$\alpha = \frac{1}{\eta_0 b}\sqrt{\frac{\pi f \mu_0}{\sigma}}\frac{1+\frac{2b}{a}\frac{f_c^{\,2}}{f^2}}{\left(1-\frac{f_c^{\,2}}{f^2}\right)^{1/2}}$$

$\eta_0 = 377$ ohm

$\mu_0 = 4\pi\times10^{-7}$ H/m

$a = 0.9'' = 0.9\times2.54\times10^{-2}$ m

$b = 0.4'' = 0.4\times2.54\times10^{-2}$ m

$f = 9\times10^9$ Hz

$f_c = 6.56\times10^9$ Hz

$\sigma = 35.4\times10^6$ mho/m

(given)

$\alpha = 0.0177$ Np/m $= 0.0177 = 8.68\times0.0177 = 0.154$ dB/m

In another interesting problem, let us find the value of the wave frequency relative to the cutoff frequency of a rectangular waveguide excited in the dominant mode that results in the minimum attenuation due to the finite conductivity of the material of the waveguide, in terms of the waveguide dimensions. Numerically appreciate the problem taking $a = 1.8b$ and $f_c = 6.56$ GHz.

$$\alpha = \frac{1}{\sigma\delta\eta_0 b}\frac{f_c}{f}\frac{\frac{2b}{a}+\frac{f^2}{f_c^2}}{\left(\frac{f^2}{f_c^2}-1\right)^{1/2}} \qquad \text{(recalled interpreting } \omega_c/\omega = f_c/f\text{)}$$

$$\frac{1}{\sigma\delta} = \sqrt{\frac{\pi f\mu_0}{\sigma}}$$

$$\xi = \left(\frac{f_c}{f}\right)^2$$

$$G = \frac{1}{\eta_0 b}\sqrt{\frac{\pi\mu_0 f_c}{\sigma}}$$

$$\alpha = G\frac{\xi^{-1/4}+\frac{2b}{a}\xi^{3/4}}{(1-\xi)^{1/2}}$$

$$\alpha = G\frac{\xi^{-1/4} + \frac{2b}{a}\xi^{3/4}}{(1-\xi)^{1/2}}$$ (rewritten)

$$\xi = \left(\frac{f_c}{f}\right)^2$$ (rewritten)

$$\frac{d\alpha}{df} = \frac{d\alpha}{d\xi}\frac{d\xi}{df} = 0$$

← At the frequency where the minimum attenuation of the waveguide takes place

$$\frac{d\alpha}{df} = \frac{-2G}{f_c(1-\xi)^{1/2}}\left(\frac{-\frac{1-\xi}{2}\xi^{1/4} + \frac{3b}{a}(1-\xi)\xi^{5/4} + \xi^{5/4} + \frac{2b}{a}\xi^{9/4}}{2(1-\xi)}\right) = 0$$

← Quantity outside the parenthesis being non-zero for practical values of $\xi(=\omega_c{}^2/\omega^2)$

$$\frac{-\frac{1-\xi}{2}\xi^{1/4} + \frac{3b}{a}(1-\xi)\xi^{5/4} + \xi^{5/4} + \frac{2b}{a}\xi^{9/4}}{2(1-\xi)} = 0$$

$$\frac{-\frac{1-\xi}{2}\xi^{1/4}+\frac{3b}{a}(1-\xi)\xi^{5/4}+\xi^{5/4}+\frac{2b}{a}\xi^{9/4}}{2(1-\xi)}=0$$

(rewritten)

↓

$$\frac{3b}{a}(1-\xi)\xi^{5/4}+\xi^{5/4}+\frac{2b}{a}\xi^{9/4}=\frac{1-\xi}{2}\xi^{1/4}$$

↓ ← Dividing by $\xi^{1/4}$

$$\frac{3b}{a}(1-\xi)\xi+\xi+\frac{2b}{a}\xi^{2}=\frac{1-\xi}{2}$$

Can be rearranged as →

$$2b\xi^{2}-(3a+6b)\xi+a=0$$

↓ Can be solved as

$$\xi=\left(\frac{f_c}{f}\right)^2=\frac{(3a+6b)\pm\sqrt{(3a+6b)^2-8ab}}{4b}$$

↙

$$\frac{f}{f_c}=\sqrt{\frac{4b}{(3a+6b)\pm\sqrt{(3a+6b)^2-8ab}}}$$

$$\frac{f}{f_c} = \sqrt{\frac{4b}{(3a+6b) \pm \sqrt{(3a+6b)^2 - 8ab}}}$$ (rewritten)

$a = 1.8b$ (given)

$$3a + 6b = 3 \times 1.8b + 6b = 11.4b$$

$$8ab = 8 \times 1.8b \times b = 14.4b^2$$

$$\frac{f}{f_c} = \sqrt{\frac{4b}{11.4b \pm \sqrt{(11.4)^2 b^2 - 14.4b^2}}} = 2.46$$

$f_c = 6.56$ GHz

$f = 6.56 \times 2.46 = 16.13$ GHz

Summarising Notes

✓ Waveguide—a hollow pipe made of a conducting material—is extensively used for the transmission of power in the microwave frequency range.

✓ Waveguide can support transverse electric (TE) mode, which is characterized by non-zero axial magnetic field and zero axial electric field

✓ Waveguide can also support transverse magnetic TM mode, which is characterized by non-zero axial electric field and zero axial magnetic field.

✓ Waveguide behaves as a high-pass filter supporting propagating waves above a cutoff frequency that is related to waveguide dimensions.

✓ TE-mode and TM-mode field solutions for both the rectangular and cylindrical waveguides have been obtained.

✓ Characteristic equation or dispersion relation of a waveguide can be found with the help of the field solutions and electromagnetic boundary condition that the tangential component of the electric field is nil at the conducting surface of the waveguide wall.

✓ One and the same dispersion relation is obtained between the wave angular frequency ω and phase propagation constant β of a waveguide for the TE and the TM modes involving their respective cutoff frequencies ω_c.

✓ ω-β dispersion plots of Identical nature are generated for rectangular and cylindrical waveguides excited in TE or TM mode.

✓ Cutoff frequency of the waveguide is the frequency ω corresponding to zero value of phase propagation constant β (which can be identified as the point of intersection between the ω-β dispersion plot and the abscissa, that is, ω-axis of the plot).

✓ Cutoff frequency of the waveguide depends on the waveguide dimensions and waveguide mode chosen.

✓ Characteristic parameters of a waveguide are guide wavelength, phase propagation constant, phase velocity, group velocity and wave impedance, each of them depending on the operating frequency relative to the cutoff frequency of the waveguide.

✓ Evanescent mode is supported by a waveguide below its cutoff frequency associated with no component of the average Poynting vector in the direction of wave propagation, corresponding to no power flow in the waveguide.

✓ Mode numbers m and n are subscripted in the nomenclatures TE_{mn} and the TM_{mn} representing respectively the transverse electric and transverse magnetic modes of the waveguide.

✓ Dominant mode of a waveguide is characterized by the lowest value of the cutoff frequencies of all the TE_{mn} and the TM_{mn} modes of the waveguide.

√Dominant mode of a rectangular waveguide is the mode TE_{10}.

√ Dominant mode of a cylindrical waveguide is the mode TE_{11}.

✓ Mode numbers *m* and *n* of the TE_{mn} and TM_{mn} modes of a rectangular or cylindrical waveguide can be correlated with their respective field patterns across the waveguide cross section.

✓ For a rectangular waveguide, the mode number *m* indicates the number of maxima (of any field component) along the broad dimension of the waveguide, while the mode number *n* indicates the number of maxima (of any field component) along the narrow dimension of the waveguide. Alternatively, you may interpret *m* and *n* as the numbers of half-wave field patterns across the broad and the narrow dimensions of the waveguide respectively.

✓ For a cylindrical waveguide, the mode number *m* indicates the number of half-wave field pattern around the half circumference and *n* indicates the number of positive or negative maxima across the waveguide radius

✓ Why a hollow-pipe waveguide cannot support transverse electromagnetic (TEM) mode, for which the axial electric field and the axial magnetic field are each nil, has been explained.

✓ Expression for the power propagating through a rectangular waveguide above its cutoff frequency has been developed and hence the power handling capability of the waveguide has been found in terms of the breakdown voltage of the medium filling the hollow region of the waveguide for dominant-mode excitation of the waveguide.

✓ Expression for the power loss per unit length of the walls of a rectangular waveguide due to the finite resistivity of the material making the walls has been developed for dominant-mode excitation of the waveguide.

✓ Expression for the attenuation constant of a dominant-mode-excited rectangular waveguide has been developed using the expressions for propagating power and power loss per unit length of the waveguide.

✓ Attenuation constant of a waveguide depends on the operating frequency and the waveguide dimensions which should be taken into consideration while choosing the waveguide mode and frequency for lower waveguide attenuation.

Readers are encouraged to go through Chapter 9 of the book for more topics and more worked-out examples and review questions.