# Engineering Electromagnetics Essentials

## Chapter 8

## Electromagnetic Power Flow

## Objective

Development of basic concepts of electromagnetic power flow that can be subsequently used to study power flow through unbounded or bounded medium

## Topics dealt with

Energy and energy density stored in electrostatic and magnetostatic fields

Poynting vector (power density vector)

Poynting theorem (energy balance theorem)

Appreciation of energy density stored in electric field, energy density stored in magnetic field and power loss in a conductor (Joule's law) from the applications of Poynting theorem to the problems of a parallel-plate capacitor of circular cross section, an inductor in the form of a solenoid of circular cross section, a resistive wire of circular cross section carrying a direct current, respectively.

Power loss per unit area in a conductor in terms of the surface resistance and surface current density of the conductor

Complex Poynting vector theorem giving the concept of time-averaged electromagnetic power flow and the associated power loss due to the presence of a lossy conducting medium

Average power going out of a volume enclosure as the outward flux of time averaged complex Poynting vector through the enclosure

Reactive power flowing into a volume enclosure and its relevance to average energies stored in electric and magnetic fields in the volume

Exemplification of the concepts of power flow in conduction current antennas

Hertzian infinitesimal dipole antenna

Antenna directive gain, power gain, radiation resistance, effective length, effective aperture area and Friis transmission equation

Finite-length dipole

Antenna array

## Background

Basic concepts of static electric field, static magnetic field and those of time-varying fields developed in Chapters 3, 4 and 5 respectively as well as basic cocepts of circuit theory

# Energy and energy density stored in electrostatic and magnetostatic fields

Let us find the work done in distributing *n* number of point charges $Q_1$, $Q_2$, $Q_3$, …..$Q_n$ in fee-space one by one in order by finding the work done in placing each of these point charges at their respective positions.

Work done in placing the first point charge $Q_1$ at its position = 0 (since there is no electrostatic field in the region against which the charge has to be moved for placing it at its position)

Work done in placing the second point charge $Q_2$ at its position = $Q_2V_{21}$

Work done in placing the charge $Q_3$ at its position = $Q_3V_{31} + Q_3V_{32}$

Work done in placing the third point charge $Q_4$ at its position = $Q_4V_{41} + Q_4V_{42} + Q_4V_{43}$, and so on

$V_{21}$ is the potential at the location of $Q_2$ due to the point charge $Q_1$, and so on

$$V_{21} = \frac{Q_1}{4\pi\varepsilon_0 r_{21}}, \ldots\ldots,$$

← $r_{21}$ = distance of the location of $Q_2$ from the location of $Q_1$

$$V_{43} = \frac{Q_5}{4\pi\varepsilon_0 r_{43}},$$

← $r_{43}$ = distance of the location of $Q_4$ from the location of $Q_{3,}$, and so no

and so on

$$V_{21} = \frac{Q_1}{4\pi\varepsilon_0 r_{21}} \text{ (recalled)}$$

$V_{21}$ is the potential at the location of $Q_2$ due to the point charge $Q_1$

$$Q_2 V_{21} = Q_2\left(\frac{Q_1}{4\pi\varepsilon_0 r_{21}}\right) = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{21}}$$

$$Q_1 V_{12} = Q_1\left(\frac{Q_2}{4\pi\varepsilon_0 r_{12}}\right) = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}}$$

$r_{21}$ = distance of the location of $Q_2$ from the location of $Q_1$

$r_{12}$ = distance of the location of $Q_1$ from the location of $Q_2$

Right hand sides of the above two expressions are equal since $r_{21} = r_{12}$

Similarly,

$$V_{12} = \frac{Q_1}{4\pi\varepsilon_0 r_{12}}$$

$V_{12}$ is the potential at the location of $Q_1$ due to the point charge $Q_2$

$$Q_2 V_{21} = Q_1 V_{12}$$

Similarly,

$$Q_3 V_{31} = Q_1 V_{13}$$

and so on

We are going to use these identity expressions in the analysis to follow

Work done $W_E$ in distributing $n$ number of point charges $Q_1$, $Q_2$, $Q_3$, …..$Q_n$:

$$W_E = 0 + Q_2V_{21} + Q_3V_{31} + Q_3V_{32} + Q_4V_{41} + Q_4V_{42} + Q_4V_{43} + \ldots.. + Q_nV_{n1} + Q_nV_{n2} + Q_nV_{n3} + \ldots. + Q_nV_{n,n-1}$$

Alternatively

$$W_E = 0 + Q_1V_{12} + Q_1V_{13} + Q_2V_{23} + Q_1V_{14} + Q_2V_{24} + Q_3V_{34} + \ldots.. + Q_1V_{1n} + Q_2V_{2n} + Q_3V_{3n} + \ldots. + Q_{n-1}V_{n-1,n} \ .$$

$$Q_2V_{21} = Q_1V_{12}$$
$$Q_3V_{31} = Q_1V_{13}$$
$$Q_3V_{32} = Q_2V_{23}$$
$$Q_4V_{41} = Q_1V_{14}$$
$$Q_4V_{42} = Q_2V_{24}$$
$$Q_4V_{43} = Q_3V_{34}$$
$$\ldots..$$
$$Q_nV_{n1} = Q_1V_{1n}$$
$$Q_nV_{n2} = Q_2V_{2n}$$
$$Q_nV_{n3} = Q_3V_{3n}$$
$$\ldots..$$
$$Q_nV_{n,n-1} = Q_{n-1}V_{n-1,n}$$

Adding the two alternative expressions for $W_E$ and rearranging terms

$$2W_E = Q_1(V_{12} + V_{13} + V_{14} + \ldots..) + Q_2(V_{21} + V_{23} + V_{24} \ldots.) + Q_3(V_{31} + V_{32} + V_{34} + \ldots..) + Q_4(V_{41} + V_{42} + V_{43} + \ldots..) + \ldots.. + Q_n(V_{n1} + V_{n2} + V_{n3} + \ldots. V_{n,n-1}) \ .$$

$$2W_E = Q_1(V_{12} + V_{13} + V_{14} + \ldots\ldots) + Q_2(V_{21} + V_{23} + V_{24}\ldots\ldots)$$
$$+ Q_3(V_{31} + V_{32} + V_{34} + \ldots\ldots)$$
$$+ Q_4(V_{41} + V_{42} + V_{43} + \ldots\ldots) + \ldots\ldots$$
$$+ Q_n(V_{n1} + V_{n2} + V_{n3} + \ldots\ldots V_{n,n-1})$$

(rewritten)

$$V_1 = V_{12} + V_{13} + V_{14} + \ldots\ldots$$
$$V_2 = V_{21} + V_{23} + V_{24} + \ldots\ldots$$
$$V_3 = V_{31} + V_{32} + V_{34} + \ldots\ldots$$
$$V_4 = V_{41} + V_{42} + V_{43} + \ldots\ldots$$
$$\ldots\ldots$$
$$V_n = V_{n1} + V_{n2} + V_{n3} + \ldots\ldots + V_{n,n-1}$$

$$2W_E = Q_1V_1 + Q_2V_2 + Q_3V_3 + Q_4V_4 + \ldots\ldots + Q_nV_n$$

$$2W_E = \sum_{p=1}^{p=n} Q_pV_p \rightarrow W_E = \frac{1}{2}\sum_{p=1}^{p=n} Q_pV_p$$

$$W_E = \frac{1}{2}\int_\tau V\rho\, d\tau$$

$$\leftarrow \quad \nabla.\vec{D} = \rho$$

$$W_E = \frac{1}{2}\int_\tau V\rho\, d\tau = \frac{1}{2}\int_\tau V\,\nabla.\vec{D}\, d\tau$$

$$W_E = \frac{1}{2}\int_\tau V\rho\, d\tau = \frac{1}{2}\int_\tau V\,\nabla.\vec{D}\, d\tau$$

$$\nabla.(V\vec{D}) = V\nabla.\vec{D} + \nabla V.\vec{D} \quad \text{(vector identity)}$$

$$V\nabla.\vec{D} = \nabla.(V\vec{D}) - \nabla V.\vec{D}$$

$$W_E = \frac{1}{2}\int_\tau (\nabla.(V\vec{D}) - \nabla V.\vec{D})\, d\tau$$

$$\vec{E} = -\nabla V \quad \text{(recalled)}$$

$$= \frac{1}{2}\int_\tau (\nabla.(V\vec{D}) + \vec{E}.\vec{D})\, d\tau$$

$$= \frac{1}{2}[\int_\tau (\nabla.(V\vec{D})\, d\tau + \int_\tau \vec{E}.\vec{D}\, d\tau]$$

$$= \frac{1}{2}[\int_S (V\vec{D}).d\vec{S} + \int_\tau \vec{E}.\vec{D}\, d\tau]$$

We consider a large volume enclosure at a large distance *r* from the charges.

$V \propto 1/r$, $D \propto 1/r^2$, and $dS \propto r^2$ so that the integrand of the first term becomes $\propto 1/r$

The integrand of the first term may be ignored

$$W_E = \frac{1}{2}\oint_\tau \vec{E}.\vec{D}\, d\tau$$

(**electrostatic energy** stored from the work done)

**Electrostatic energy density** $U_E = \frac{1}{2}\vec{E}.\vec{D}$

Analogously,

$$W_H = \frac{1}{2}\oint_\tau \vec{H}.\vec{B}\, d\tau$$

(**magnetostatic energy** stored from the work done)

**Magnetostatic energy density** $U_B = \frac{1}{2}\vec{H}.\vec{B}$

**With particular reference to a simple example of a parallel-plate capacitor, appreciate the expression for electrostatic energy already derived as**

$$U_E = \frac{1}{2}\vec{E}.\vec{D}$$

Let us find from first principles the work done in charging a parallel-plate capacitor by a source of potential. We consider the plate dimensions to be large compared to the distance between the plates of the capacitor.

$$Q_0 = CV_0 \longleftarrow$$

Charge $Q_0$ of the fully charged parallel-plate capacitor of capacitance $C$ with a potential difference $V_0$ between its plates

$$Q = CV \longleftarrow$$

Charge $Q$ of a capacitor of capacitance $C$ charged to a potential difference $V$ between the plates of the capacitor

$$0 < Q < Q_0 \qquad 0 < V < V_0$$

$$dW_{\text{Capacitor}} = VdQ \longleftarrow$$

Element of work $dW_{\text{Capacitor}}$ done in adding an extra element of charge $dQ$ to the capacitor when it is already raised to potential $V$ and has a charge $Q$

$$\downarrow$$

$$dW_{\text{Capacitor}} = \frac{Q}{C}dQ \longrightarrow$$

Integrating $dW_{\text{Capacitor}}$, we can find the work done in charging the capacitor to a given amount

$$dW_{\text{Capacitor}} = \frac{Q}{C}dQ$$ (recalled)

↓ ⟵ Integrating

$$W_{\text{Capacitor}} = \int dW_{\text{Capacitor}} = \int_{Q=O}^{Q=Q_O} \frac{Q}{C}dQ = \frac{1}{C}\left[\frac{Q^2}{2}\right]_{Q=0}^{Q=Q_0} = \frac{1}{2}\frac{Q_0^2}{C}$$

(work done to charge the capacitor to its full amount $Q_0$)

↓ ⟵ $Q_0 = CV_0$

$$W_{\text{Capacitor}} = \frac{1}{2}\frac{Q_0^2}{C} = \frac{1}{2}\frac{C^2 V_0{}^2}{C} = \frac{1}{2}CV_0{}^2$$

(work done to charge the capacitor to its full amount $V_0$)

↓ ⟵ $V_0 = Ed$ ⟵ $E = V_0 / d$ (electric field magnitude *E*, supposedly uniform in a direction perpendicular to plates of the capacitor, with dimensions large compared to the distance *d* between the plates)

$$W_{\text{Capacitor}} = \frac{1}{2}C(Ed)^2$$

$$W_{\text{Capacitor}} = \frac{1}{2} C(Ed)^2$$ (work done in charging the capacitor stored in the form of electrostatic energy of the capacitor)

$\downarrow$ $\leftarrow$ Dividing by the volume *Ad* of the capacitor

$$U_{\text{Capacitor}} = \frac{C(Ed)^2}{2Ad}$$ (electrostatic energy density stored in the electrostatic field of the capacitor)

$\leftarrow$ $C = \dfrac{\varepsilon A}{d}$ (capacitance of a parallel-plate capacitor)

$\leftarrow$ $\vec{D} = \varepsilon \vec{E}$

$\downarrow$

$$U_{\text{Capacitor}} = \frac{1}{2} \varepsilon E^2 = \frac{1}{2} \vec{E}.\vec{D}$$ (electrostatic energy density stored in the electrostatic field of the capacitor)

Interestingly, the above expression with reference to a capacitor agrees with the expression for electrostatic energy density deduced earlier from first principles.

Let us next experience a similar example of agreement with reference to an inductor with respect to magnetostatic energy density. For this purpose we can take the particular problem of finding the energy density stored in the magnetic field of a solenoid.

**Let us then take the particular problem of finding the energy density stored in the magnetic field of a solenoid.**

For this purpose, let us find the work done in establishing a current *i* in a long solenoid of inductance *L* by an induced electromotive force.

Element of work done $dW_{\text{Solenoid}}$ by the induced electromotive force in bringing about an increment of current *di* in the solenoid in an infinitesimal time *dt* thereby adding an element of charge *dq* is

$$dW_{\text{Solenoid}} = \left|\mathsf{E}_{\text{induced}}\right| dq$$

$$\leftarrow \quad \left|\mathsf{E}_{\text{induced}}\right| = \left|\frac{d\phi_B}{dt}\right| = \frac{d\phi_B}{dt}$$

(magnitude of emf induced in the solenoid interpreted as positive for a charge building up with time )

$$\leftarrow \quad dq = idt$$

$$dW_{\text{Solenoid}} = \left|\mathsf{E}_{\text{induced}}\right| dq = \frac{d\phi_B}{dt} idt \quad \leftarrow \quad \phi_B = Li$$

(magnetic flux linked with the solenoid)

$$dW_{\text{Solenoid}} = L\frac{di}{dt} idt = Lidi$$

$$dW_{\text{Solenoid}} = L\frac{di}{dt}idt = Lidi$$

Integrating from the limit $i = 0$ to $i = I_0$

$$W_{\text{Solenoid}} = \int dW_{\text{Solenoid}} = \int_{i=0}^{i=I_0} Lidi = L\left[\frac{i^2}{2}\right]_{i=0}^{i=I_0} = \frac{1}{2}LI_0^2$$ (work done in building up current in the solenoid from $i = 0$ to $i = I_0$)

$$W_{\text{Solenoid}} = \frac{1}{2}LI_0^2$$

(energy stored from the work done in the solenoid when the current is $i = I_0$ in the solenoid)

$L = \mu_0 n^2 \alpha l$ (inductance of the solenoid of length $l$ and cross-sectional area $\alpha$, $n$ being the number of turns per unit length )

(recalled from Chapter 4)

$$W_{\text{Solenoid}} = \frac{1}{2}\mu_0 n^2 \alpha l I_0^2 = \frac{1}{2}\mu_0 (nI_0)^2(\alpha l)$$

(energy stored in the solenoid when the value of the current is $i = I_0$ in the solenoid)

Dividing by the volume $\alpha l$ of the solenoid of length $l$ and cross-sectional area $\alpha$

$$U_{\text{Solenoid}} = \frac{1}{2}\mu_0 (nI_0)^2$$

(energy density stored in the solenoid)

$$U_{\text{Solenoid}} = \frac{1}{2}\mu_0 (nI_0)^2$$

(energy density stored in the solenoid)

$\downarrow \longleftarrow$ $H = nI_0$ (recalled from Chapter 4)

$$U_{\text{Solenoid}} = \frac{1}{2}\mu_0 H^2$$

(magnetostatic energy density stored in the magnetostatic field of the inductor)

Interestingly, the above expression with reference to an inductor agrees with the expression for magnetostatic energy density deduced earlier analogously from the expression for electrostatic energy density deduced earlier from first principles .

**Obtain the following expression for the inductance per unit length of a coaxial cable starting from the expression for energy density stored in a magnetic field, where *a* is the radius of the inner conductor *b* is the inner radius of the outer conductor of the cable:**

$$L = \frac{\mu_0}{2\pi}\ln\frac{b}{a} \text{ (per unit length).}$$

In order to find the required expression let us start with

$$U_B = \frac{1}{2}\vec{H}.\vec{B}$$ (magnetostatic energy density) (recalled)

$$U_B = \frac{1}{2}\vec{H}.\vec{B}$$ (magnetostatic energy density) (recalled)

$\longleftarrow$ $\vec{B} = \mu_0 \vec{H}$

$$U_B = \frac{1}{2}\mu_0 H^2$$ (magnetostatic energy density)

$\longleftarrow$ Multiply by the element of volume *2πrdrl* of the cylindrical shell of length *l* and infinitesimal thickness *dr* of radius *r* lying between the radius *a* of the inner conductor and the inner radius *b* of the outer conductor of the coaxial cable

$$\Delta W_B = \frac{1}{2}\mu_0 H^2 (2\pi\, r\, dr\, dl)$$ (element of magnetic energy stored in the volume element)

$\longleftarrow$ $H = \dfrac{I}{2\pi\, r}$ (recalled from Chapter 4)

$$\Delta W_B = \frac{1}{2}\mu_0 (\frac{I}{2\pi\, r})^2 (2\pi\, r\, dr\, dl)$$ (element of magnetic energy stored in the volume element)

$$\Delta W_B = \frac{1}{2}\mu_0(\frac{I}{2\pi r})^2(2\pi r\, dr\, dl)$$ (element of magnetic energy stored in the volume element)

↓ ← Integrating between the limits *r* = *a* (inner conductor) and *b* (outer conductor) and *l* = 0 and *l*

$$W_B = [\int_{r=a}^{r=b} \frac{1}{2}\mu_0(\frac{I}{2\pi r})^2 2\pi r dr][\int_0^l dl]$$ (energy stored over the length l of the coaxial cable)

↓

$$W_B = \frac{\mu_0 I^2}{4\pi}[\int_{r=a}^{r=b}\frac{1}{r}dr][\int_0^l dl] = \frac{\mu_0 I^2}{4\pi}[\ln r]_a^b[l]_0^l = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}l$$ (energy stored over the length *l* of the coaxial cable)

$$W_B = \frac{1}{2}L'I^2$$ (alternative expression for the energy stored over the length *l* of the coaxial cable from the work done in building the current *I* in the cable of inductance L/) (derived earlier)

Comparing the right hand sides

↓

$$\frac{1}{2}L'I^2 = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}l \longrightarrow L' = \frac{\mu_0}{2\pi}\ln\frac{b}{a}l$$

(inductance of length *l* of the cable)

Dividing by *l*

$$\longrightarrow L = \frac{\mu_0}{2\pi}\ln\frac{b}{a}$$

(inductance per unit length of the cable)

# Poynting vector and Poynting theorem

The phenomenon of the storage, loss and flow of electromagnetic energy follows the basic energy balance principle well stated by the Poynting theorem. We can formulate the theorem in terms of a vector quantity called Poynting vector defined as:

Poynting vector $\vec{\mathrm{P}} = \vec{E} \times \vec{H}$ [having unit of (V/m)×(A/m) = W/m$^2$] ⟵ named after the British physicist John Henry Poynting.

$$W_E = \frac{1}{2}\oint_\tau \vec{E}.\vec{D}\, d\tau \text{ (energy stored in electrostatic field)}$$ (recalled)

$$\vec{D} = \varepsilon \vec{E}$$

$$= \frac{1}{2}\oint_\tau \varepsilon E^2 d\tau$$

Electrostatic energy density

$$U_E = \frac{1}{2}\vec{E}.\vec{D}$$ (recalled)

$$W_B = \frac{1}{2}\oint_\tau \vec{H}.\vec{B}\, d\tau \text{ (energy stored in magnetostatic field)}$$ (recalled)

$$\vec{B} = \mu \vec{H}$$

$$= \frac{1}{2}\oint_\tau \mu H^2 d\tau$$

Magnetostatic energy density

$$U_B = \frac{1}{2}\vec{H}.\vec{B}$$ (recalled)

Vector identity → $\nabla.(\vec{E}\times\vec{H}) = \vec{H}.(\nabla\times\vec{E}) - \vec{E}.(\nabla\times\vec{H})$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

(Maxwell's equation)

$$\nabla\times\vec{H} = \sigma\vec{E} + \frac{\partial\vec{D}}{\partial t}$$

(Maxwell's equation)

$$\vec{D} = \varepsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

$$= \vec{H}.(-\frac{\partial\vec{B}}{\partial t}) - \vec{E}.(\sigma\vec{E} + \frac{\partial\vec{D}}{\partial t})$$

$$= -\mu\vec{H}.\frac{\partial\vec{H}}{\partial t} - \sigma E^2 - \varepsilon\vec{E}.\frac{\partial\vec{E}}{\partial t}$$

$$= -\mu H\frac{\partial H}{\partial t} - \sigma E^2 - \varepsilon E\frac{\partial E}{\partial t}$$

$$= -\frac{\partial}{\partial t}(\frac{1}{2}\mu H^2) - \sigma E^2 - \frac{\partial}{\partial t}(\frac{1}{2}\varepsilon E^2)$$

$$= -\frac{\partial}{\partial t}(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2) - \sigma E^2$$

$$\nabla.(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2) - \sigma E^2$$

(rewritten)

← Integrating over the volume $\tau$

$$\int_\tau \nabla.(\vec{E}\times\vec{H})d\tau = \int_\tau -\frac{\partial}{\partial t}(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2)d\tau - \int_\tau \sigma E^2 d\tau$$

$$\int_\tau \nabla\cdot(\vec{E}\times\vec{H})d\tau = \oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}$$

$$\int_\tau \nabla\cdot\vec{\mathrm{P}}d\tau = \oint_s \vec{\mathrm{P}}\cdot d\vec{S}$$

Vector divergence theorem as applied to the vector:

$$\vec{\mathrm{P}} = \vec{E}\times\vec{H}$$

$$\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\frac{\partial}{\partial t}\int_\tau(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2)d\tau - \int_\tau \sigma E^2 d\tau$$

$$\oint_s (\vec{E}\times\vec{H}).d\vec{S} = \oint_s \vec{\mathrm{P}}.d\vec{S} = -\frac{\partial}{\partial t}\int_\tau (\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2)d\tau - \int_\tau \sigma E^2 d\tau$$

With a change in sign

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S} = \frac{\partial}{\partial t}\int_\tau (\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2)d\tau + \int_\tau \sigma E^2 d\tau$$

**Poynting theorem involving instantaneous Poynting vector:**

$$\vec{\mathrm{P}} = \vec{E}\times\vec{H}$$

The right hand side represents the time rate of change of energy (electrostatic plus magnetostatic), that is, power in the volume plus the time rate of energy, that is, power lost in the volume.

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S}$$

Represents the power going into the volume, one part of which stands for the time rate of increase in the energy stored in the volume, and the other part stands for the power loss (ohmic loss)

Therefore, with a change in sign

$$\oint_s (\vec{E}\times\vec{H}).d\vec{S} = \oint_s \vec{\mathrm{P}}.d\vec{S}$$

represents the power transmitted out of the volume.

(flux of Poynting vector $\vec{E}\times\vec{H}$ going out of the volume).

**Let us now appreciate Joule's law for power loss in a straight wire carrying a direct current**

Power entering a volume enclosure is given by

$$-\oint_S (\vec{E}\times\vec{H}).d\vec{S} = \frac{\partial}{\partial t}\int_\tau (\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2)d\tau + \int_\tau \sigma E^2 d\tau$$ (recalled)

(Poynting theorem involving instantaneous Poynting vector)

direct current: $\partial/\partial t = 0$

$$-\oint_S (\vec{E}\times\vec{H}).d\vec{S} = \int_\tau \sigma E^2 d\tau$$

There being no electric field existing outside the wire, the volume over which the integration of the right-hand side has to be taken is restricted to the region occupied by the wire.

$$\tau = \alpha l$$

$$E = \frac{V}{l}$$ in terms of the potential difference *V* between the ends of the length *l* of the straight wire

$$-\oint_S (\vec{E}\times\vec{H}).d\vec{S} = \sigma E^2 \int_\tau d\tau = \sigma E^2 \tau$$

$$= \sigma(\frac{V}{l})^2(\alpha l) = \frac{V^2}{\frac{1}{\sigma}\frac{l}{\alpha}} = \frac{V^2}{R} = I^2 R$$

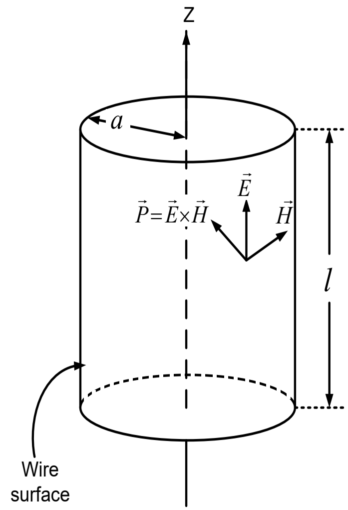
$V = IR$ (Ohm's law)

$$R = \frac{1}{\sigma}\frac{l}{\alpha}$$

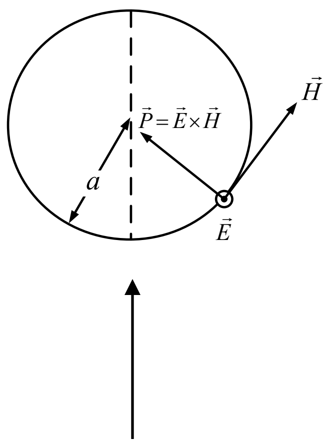
$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = I^2 R$$ (rewritten)

(left hand side representing the power going into the wire that gets lost in the form of the so-called Ohmic loss)

(Joule's law) Ohmic loss




A straight wire of length *l* and radius *a* carrying a direct current showing the electric and magnetic field vectors and the Poynting vector that is directed inward to the wire, all on the surface of the wire →




Cross-sectional view of the wire

If the direction of the direct current is made to reverse, which amounts to reversing the direction of the electric field as well, then the direction of the magnetic field also reverses, which consequently does not cause a change in the direction of the Poynting vector $\vec{P} = \vec{E}\times\vec{H}$.

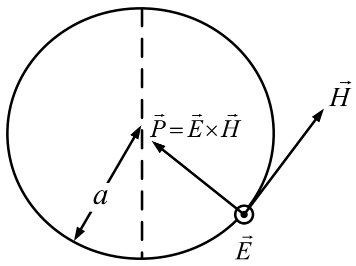
**Let us have a re-look at the problem of power loss in a wire carrying a direct current starting from the power density or Poynting vector at the surface of the wire of circular cross section.**

Poynting vector

$$\vec{P} = \vec{E} \times \vec{H}$$

$$\vec{H} = \frac{I}{2\pi a}\vec{a}_\theta$$

(azimuthal magnetic field at the surface of the wire of radius *a* due to a direct current *I* along z obtainable with the help of Ampere's circuital law)




Cross-sectional view of the wire

$$\vec{P} = (E\vec{a}_z) \times (\frac{I}{2\pi a}\vec{a}_\theta)$$

$$\vec{E} = E\vec{a}_z$$

Electric field directed along z due to a potential difference across the wire that sends the current through the wire also in the z direction

$$\vec{P} = \vec{E} \times \vec{H} = E\frac{I}{2\pi a}\vec{a}_z \times \vec{a}_\theta = -E\frac{I}{2\pi a}\vec{a}_r$$ (directed radially inward)

(in view of the relation $\vec{a}_z \times \vec{a}_\theta = -\vec{a}_r$)

$$\vec{\mathrm{P}} = \vec{E} \times \vec{H} = E \frac{I}{2\pi a} \vec{a}_z \times \vec{a}_\theta = -E \frac{I}{2\pi a} \vec{a}_r$$ (rewritten)

$$E = \frac{I}{(\sigma)(\pi a^2)} \longleftarrow I = \sigma E \pi a^2 \longleftarrow I = J_c \pi a^2 \longleftarrow \vec{J}_c = \sigma \vec{E} \text{ (Ohm's law)}$$

(conduction current density)

$$\vec{\mathrm{P}} = \vec{E} \times \vec{H} = -E \frac{I}{2\pi a} \vec{a}_r = -\frac{I}{(\sigma)(\pi a^2)} \frac{I}{2\pi a} \vec{a}_r$$

$$-\oint_s (\vec{E} \times \vec{H}).d\vec{S} =$$ Power going into the wire over a length *l* through the area $2\pi al$ of its surface

$$d\vec{S} = dS\vec{a}_r$$

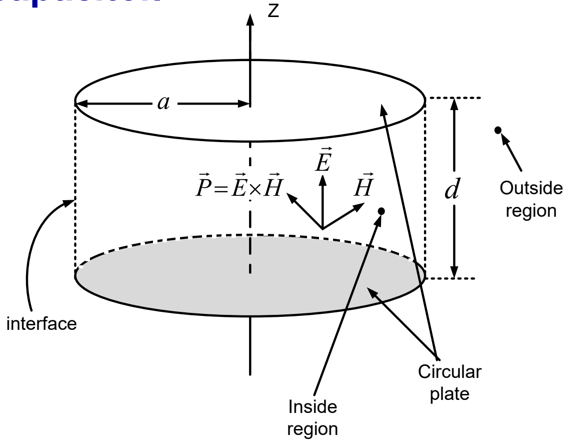
$$-\oint_s (\vec{E} \times \vec{H}).d\vec{S} = -\oint_s -\frac{I}{(\sigma)(\pi a^2)} \frac{I}{2\pi a} \vec{a}_r.d\vec{S} = \oint_s \frac{I}{(\sigma)(\pi a^2)} \frac{I}{2\pi a} \vec{a}_r.dS\vec{a}_r$$

$$\oint_s dS = 2\pi al \qquad R = \frac{1}{\sigma} \frac{l}{\alpha} = \frac{1}{\sigma} \frac{l}{\pi a^2}$$

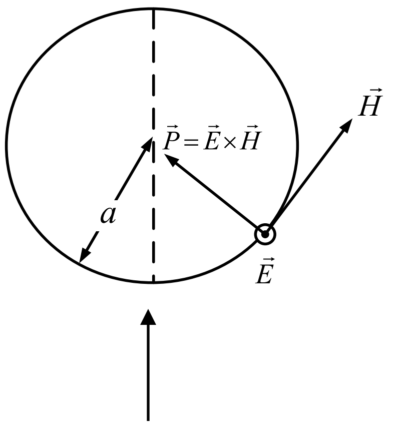
$$-\oint_\tau (\vec{E} \times \vec{H}).d\vec{S} = \frac{I^2}{(\sigma)(\pi a^2)} \frac{1}{2\pi a} \oint_s dS = \frac{I^2}{(\sigma)(\pi a^2)} \frac{1}{2\pi a} (2\pi al) = I^2 R$$

(power going into the wire which is the ohmic loss, agreeing to what was obtained earlier as Joule's law)

**In an interesting illustration, let us apply Poynting theorem to the problem of a parallel-plate capacitor with circular plates of large dimensions compared to the distance between the plates to find the expression for energy density stored in electric field of the capacitor.**




A parallel-plate capacitor with circular plates of radius *a* showing the electric and magnetic field vectors and the Poynting vector that is directed inward to the wire, all at the interface between the inside and outside regions of the capacitor




Cross-sectional view of the capacitor

Poynting vector

$$\vec{\mathrm{P}} = \vec{E}\times\vec{H}$$

$$\vec{E} = E\vec{a}_z$$

$$\vec{H} = \varepsilon\frac{dE}{dt}\frac{a}{2}\vec{a}_\theta$$

(deduced in Chapter 5 using Maxwell’s equation)

$$\vec{\mathrm{P}} = \vec{E}\times\vec{H} = E\,\vec{a}_z\times\varepsilon\frac{dE}{dt}\frac{a}{2}\vec{a}_\theta = \varepsilon\,E\frac{dE}{dt}\frac{a}{2}\vec{a}_z\times\vec{a}_\theta = -\varepsilon\,E\frac{dE}{dt}\frac{a}{2}\vec{a}_r$$

$$\vec{\mathrm{P}} = \vec{E}\times\vec{H} = -\varepsilon\, E\,\frac{dE}{dt}\,\frac{a}{2}\,\vec{a}_r$$ (rewritten)

Power going into the capacitor as indicated by the radially inward direction of the Poynting vector given by the above expression will cause a storage of energy in the capacitor with time. Consequently, only the first term will be significant in the following expression stating the Poynting theorem:

↓

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = \frac{\partial}{\partial t}\int_\tau \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)d\tau + \int_\tau \sigma E^2 d\tau$$ (Poynting theorem) (recalled)

↓ ← Only the first term bring significant

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S} = \frac{\partial}{\partial t}\int_\tau \frac{1}{2}\varepsilon E^2 d\tau$$

↓ ← $$-\oint_S \vec{\mathrm{P}}\cdot d\vec{S} = \frac{\partial W_E}{\partial t}$$

$$\frac{\partial W_E}{\partial t} = \frac{\partial}{\partial t}\int_\tau \frac{1}{2}\varepsilon E^2 d\tau$$

Integrating with respect to *t* ⟶

$$W_E = \int_\tau \frac{1}{2}\varepsilon E^2 d\tau \longrightarrow U_E = \frac{1}{2}\varepsilon E^2$$

**(energy density stored in electric field)**

# Poynting vector in time-varying situations

$$\vec{E}\times\vec{H}$$

$\vec{E} = E_0 \cos\omega t\, \vec{a}_E$ (time-varying field)

$\vec{H} = H_0 \cos(\omega t + \theta)\vec{a}_H$ (time-varying field)

$\vec{a}_E$ and $\vec{a}_H$ (unit vectors in the directions of electric and magnetic fields respectively)

$$\vec{E}\times\vec{H} = E_0 H_0 \cos\omega t \cos(\omega t + \theta)\vec{a}_E \times \vec{a}_H$$

$\theta$ = phase difference between electric and magnetic fields

$$\vec{E}\times\vec{H} = E_0 H_0 \cos\omega t(\cos\omega t \cos\theta - \sin\omega t \sin\theta)\vec{a}_E \times \vec{a}_H$$

$$= E_0 H_0 (\cos^2\omega t \cos\theta - \cos\omega t \sin\omega t \sin\theta)\vec{a}_E \times \vec{a}_H$$ (time-varying fields)

Time-averaged Poynting vector

$$\vec{\mathrm{P}}_{\mathrm{average}} = \frac{\int_0^{2\pi} E_0 H_0 (\cos^2\omega t \cos\theta - \cos\omega t \sin\theta) d(\omega t)}{\int_0^{2\pi} d(\omega t)}\ \vec{a}_E \times \vec{a}_H$$

$$= E_0 H_0 \frac{\cos\theta \int_0^{2\pi} \cos^2\omega t\, d(\omega t) - \sin\theta \int_0^{2\pi} \cos\omega t\, d(\omega t)}{\int_0^{2\pi} d(\omega t)}\ \vec{a}_E \times \vec{a}_H$$

$$= E_0 H_0 \frac{\cos\theta(\frac{1}{2}2\pi)}{2\pi}\vec{a}_E \times \vec{a}_H = \frac{1}{2}E_0 H_0 \cos\theta\ \ \vec{a}_E \times \vec{a}_H$$

Time-averaged Poynting vector $\vec{\mathrm{P}}_{\text{average}} = \frac{1}{2} E_0 H_0 \cos\theta \quad \vec{a}_E \times \vec{a}_H$ (rewritten)

$$\vec{E} = E_0 \cos\omega t \, \vec{a}_E$$
$$\vec{H} = H_0 \cos(\omega t + \theta) \vec{a}_H$$

(magnetic field typically leading electric field by a phase angle $\theta$).

In phasor notation

$$\vec{E} = E_0 \exp j\omega t \, \vec{a}_E$$
$$\vec{H} = H_0 \exp j(\omega t + \theta) \vec{a}_H$$
$$\vec{H}^* = H_0 \exp - j(\omega t + \theta) \vec{a}_H$$

$$\vec{E} \times \vec{H}^* = E_0 H_0 [\exp(j\omega t)][\exp(-j(\omega t + \theta))] \vec{a}_E \times \vec{a}_H$$
$$= E_0 H_0 \exp(-j\theta) \vec{a}_E \times \vec{a}_H$$
$$= E_0 H_0 (\cos\theta - j \sin\theta) \vec{a}_E \times \vec{a}_H$$

$$\mathrm{Re}\, \vec{E} \times \vec{H}^* = E_0 H_0 \cos\theta \, \vec{a}_E \times \vec{a}_H$$

$$\frac{1}{2} \mathrm{Re}\, \vec{E} \times \vec{H}^* = \frac{1}{2} E_0 H_0 \cos\theta \, \vec{a}_E \times \vec{a}_H$$

Time-averaged Poynting vector $\vec{\mathrm{P}}_{\text{average}} = \frac{1}{2} \mathrm{Re}\, \vec{E} \times \vec{H}^*$

Time-averaged Poynting vector $\vec{P}_{\text{average}} = \frac{1}{2}\mathrm{Re}\,\vec{E}\times\vec{H}^{*}$ (for time-varying fields)

$\oint_S (\vec{E}\times\vec{H}).d\vec{S}$ (representing power transmitted out of a volume enclosure (as obtained earlier interpreting Poynting theorem involving instantaneous Poynting vector)

Obviously, with reference to time-varying fields,

$$\oint_S \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^{*}).d\vec{S} = \oint_S \vec{P}_{\text{average}}.d\vec{S}$$

represents the time-averaged power transmitted out of a volume as the interpretation of Poynting theorem for time-varying fields.

The concept of time-averaged power transmitted out of a volume developed here finds practical applications, for instance, in the study of power loss in a conductor and radiated power from an antenna, to be taken up later here as well as in the study of transmission of power through a waveguide in the chapter to follow (in Chapter 9).

Time-averaged Poynting vector $\vec{\mathrm{P}}_{\mathrm{average}} = \frac{1}{2} E_0 H_0 \cos\theta \quad \vec{a}_E \times \vec{a}_H$ (recalled)

↑

$$\frac{1}{2}\mathrm{Re}\,\vec{E}\times\vec{H}^* = \frac{1}{2} E_0 H_0 \cos\theta\, \vec{a}_E \times \vec{a}_H$$ (recalled)

↓

Time-averaged Poynting vector $\vec{\mathrm{P}}_{\mathrm{average}} = \frac{1}{2}\mathrm{Re}\,\vec{E}\times\vec{H}^*$ (recalled)

We define complex Poynting vector $\vec{\mathrm{P}}_{\mathrm{complex}}$ as follows:

$$\vec{\mathrm{P}}_{\mathrm{complex}} = \frac{1}{2}\vec{E}\times\vec{H}^*$$ (Definition of complex Poynting vector)

(Complex Poynting vector) (defined as)

$$\mathrm{Re}\,\vec{\mathrm{P}}_{\mathrm{complex}} = \mathrm{Re}(\frac{1}{2}\vec{E}\times\vec{H}^*) = \frac{1}{2}\mathrm{Re}\,\vec{E}\times\vec{H}^* = \vec{\mathrm{P}}_{\mathrm{average}}$$

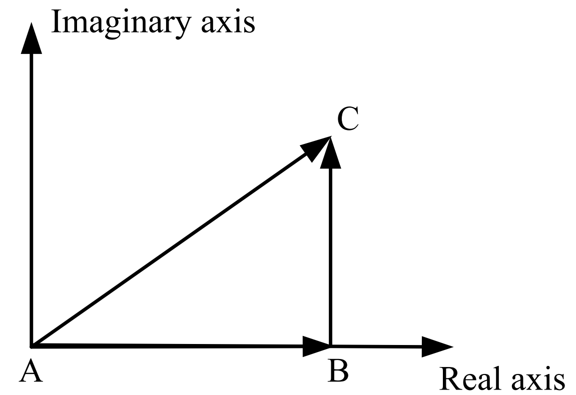
↓

$$\mathrm{Re}\,\vec{\mathrm{P}}_{\mathrm{complex}} = \vec{\mathrm{P}}_{\mathrm{average}}$$ (rewritten)

**Power density triangle**

We can depict the power density vectors in a vector triangle on a complex plane obeying the following relation

$$\overrightarrow{\mathrm{AC}} = \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}}$$




Power density triangle

Hypotenuse of the triangle represents the complex power density vector, also referred to as the virtual power density vector, the magnitude of which represents the apparent power density

$$\overrightarrow{\mathrm{AC}}$$

$$\vec{\mathrm{P}}_{\mathrm{complex}} = \frac{1}{2}\vec{E}\times\vec{H}^{*} = \frac{1}{2}E_0 H_0(\cos\theta - j\sin\theta)\vec{a}_E\times\vec{a}_H$$

(magnetic field typically leading electric field by a phase angle $\theta$).

Side of the triangle on the real axis represents the average power density vector or the real power density vector

$$\overrightarrow{\mathrm{AB}}$$

$$\vec{\mathrm{P}}_{\mathrm{average}} = \frac{1}{2}E_0 H_0\cos\theta \quad \vec{a}_E\times\vec{a}_H$$

Side of the triangle on the imaginary axis represents the reactive power density vector

$$\overrightarrow{\mathrm{BC}}$$

$$\vec{\mathrm{P}}_{\mathrm{reactive}} = \mathrm{Im}\,\vec{\mathrm{P}}_{\mathrm{complex}} = \frac{1}{2}\mathrm{Im}\,\vec{E}\times\vec{H}^{*} = -\frac{1}{2}E_0 H_0\sin\theta\,\vec{a}_E\times\vec{a}_H$$

**In an illustrative example let us obtain an expression for time-averaged power lost in a current-carrying straight wire in terms of the peak current and wire resistance.**

We have already obtained earlier the expression for power going into the wire that gets lost in the form of the so-called Ohmic loss:

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S} = I^2 R$$

(recalled) $\leftarrow I = I_0 \sin \omega t$

$R$ being the wire resistance and $I_0$ the peak current

Instantaneous power $P$ going into the wire and getting lost

$$P = -\oint_S \vec{\mathrm{P}}\cdot d\vec{S} = I_0{}^2 \sin^2 \omega t\, R$$

Taking the average over a complete cycle

$$P_{\text{average}} = \left\langle I_0{}^2 \sin^2 \omega t\, R \right\rangle_{\text{average}} = \frac{\int_{\omega t=0}^{2\pi} I_0{}^2 \sin^2 \omega t\, R d(\omega t)}{\int_{\omega t=0}^{2\pi} d(\omega t)} = \frac{I_0{}^2 R \int_{\omega t=0}^{2\pi} \sin^2 \omega t\, d(\omega t)}{\int_{\omega t=0}^{\omega t=2\pi} d(\omega t)}$$

$$P_{\text{average}} = \frac{I_0{}^2 R \int_{\omega t=0}^{2\pi} \sin^2 \omega t \, d(\omega t)}{\int_{\omega t=0}^{\omega t=2\pi} d(\omega t)}$$ (rewritten)

$$= \frac{I_0{}^2 R \int_{\omega t=0}^{2\pi} \frac{1-\cos 2\omega t}{2} d(\omega t)}{\int_{\omega t=0}^{2\pi} d(\omega t)}$$

$$= \frac{I_0{}^2 R \left( \frac{1}{2} \int_{\omega t=0}^{2\pi} d(\omega t) - \frac{1}{2} \int_{\omega t=0}^{2\pi} \cos 2\omega t \, d(\omega t) \right)}{\int_{\omega t=0}^{2\pi} d(\omega t)}$$

$$= \frac{I_0{}^2 R \left[ \frac{1}{2} [\omega t]_0^{2\pi} - \frac{1}{2} \left[ \frac{\sin 2\omega t}{2\omega} \right]_0^{2\pi} \right]}{[\omega t]_0^{2\pi}}$$

$$= \frac{I_0{}^2 R \left[ \frac{1}{2} (2\pi - 0) - \frac{1}{4\omega} (\sin 4\pi - \sin 0) \right]}{2\pi - 0}$$

$$P_{\text{average}} = \frac{I_0{}^2 R\left[\frac{1}{2}(2\pi - 0) - \frac{1}{4\omega}(\sin 4\pi - \sin 0)\right]}{2\pi - 0}$$ (rewritten)

$$= \frac{I_0{}^2 R\left[\frac{1}{2}2\pi - \frac{1}{4\omega}(0 - 0)\right]}{2\pi - 0} = \frac{1}{2} I_0{}^2 R$$

(average power $P_{\text{average}}$ lost in the wire in the form of Ohmic loss)

**Similarly, in another illustrative example let us find the average power going into a parallel-plate capacitor with circular plates.**

$$\vec{E} = E\vec{a}_z$$

$$\vec{H} = \varepsilon \frac{dE}{dt}\frac{a}{2}\vec{a}_\theta$$

(recalled) → Electric and magnetic fields at the cylindrical interface between the inside and outside regions of the parallel-plate capacitor

$$\varepsilon\omega E_0 = H_0, \text{say}$$

$$\vec{E} = E_0 \sin\omega t\, \vec{a}_z = E_0 \sin\omega t\, \vec{a}_E$$

$$\vec{H} = \varepsilon \frac{dE}{dt}\frac{a}{2}\vec{a}_\theta = \varepsilon\omega E_0 \cos\omega t \frac{a}{2}\vec{a}_\theta = H_0 \cos\omega t\, \vec{a}_\theta = H_0 \cos\omega t\, \vec{a}_H$$

$$\vec{\text{P}} = \vec{E} \times \vec{H} = (E_0 \sin\omega t\, H_0 \cos\omega t)(\vec{a}_E \times \vec{a}_H)$$ (Poynting vector) (instantaneous)

$$\vec{\mathrm{P}} = \vec{E}\times\vec{H} = (E_0 \sin\omega t\, H_0 \cos\omega t)(\vec{a}_E \times \vec{a}_H)$$ (Poynting vector) (instantaneous)

Averaging over a cycle

$$\vec{\mathrm{P}}_{\text{average}} = \left\langle \vec{E}\times\vec{H} \right\rangle_{\text{average}} = \left\langle E_0 H_0 \sin\omega t \cos\omega t\, \vec{a}_E \times \vec{a}_H \right\rangle_{\text{average}}$$

$$= \frac{\int_{\omega t=0}^{2\pi} E_0 H_0 \sin\omega t \cos\omega t\, d(\omega t)\, \vec{a}_E \times \vec{a}_H}{\int_{\omega t=0}^{2\pi} d(\omega t)} \longleftarrow d(\sin\omega t) = \cos\omega t\, d(\omega t)$$

$$= \frac{E_0 H_0 \int_{\omega t=0}^{2\pi} (\sin\omega t)\, d(\sin\omega t)\, \vec{a}_E \times \vec{a}_H}{\int_{\omega t=0}^{2\pi} d(\omega t)} = \frac{E_0 H_0 \frac{1}{2}(0-0)\, \vec{a}_E \times \vec{a}_H}{2\pi - 0} = 0$$

$$-\oint_s \vec{\mathrm{P}}_{\text{average}} . d\vec{S} = 0 \longleftarrow -\oint_s \vec{\mathrm{P}} . d\vec{S} = -\oint_s (\vec{E}\times\vec{H}) . d\vec{S}$$

Average power going into the capacitor found to be nil

Power going into the capacitor

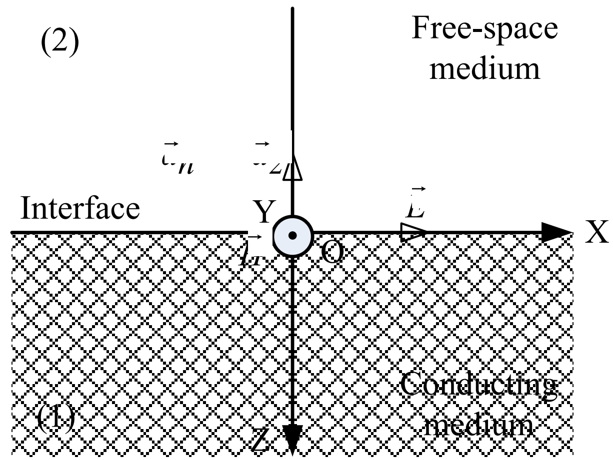
Power alternately goes in and out of the capacitor since the direction of the Poynting vector becomes radially inward and outward at the cylindrical interface between the inside and outside regions of the capacitor at consecutive quarter cycles, which makes the average power going into the capacitor nil. This is as also found analytically above.

# Power loss per unit area in a conductor

Consider a uniform plane wave propagating along z which is incident on the surface of a conducting medium (1) of conductivity $\sigma$ from a free-space medium (2).

$$\sigma >> j\omega\varepsilon \,(\text{good conductor})$$

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \eta = \sqrt{\frac{j\omega\mu_0}{\sigma + j\omega\varepsilon_0}} \cong \sqrt{\frac{j\omega\mu_0}{\sigma}}$$



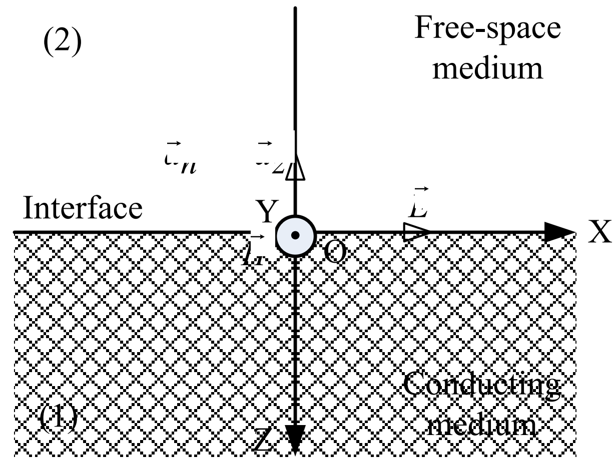

(relation between electric and magnetic field components of a conducting medium in terms of its intrinsic impedance $\eta$; see Chapter 6)

$$\eta = \sqrt{\frac{j\omega\mu_0}{\sigma}} = \sqrt{\frac{\omega\mu_0}{2\sigma}} + j\sqrt{\frac{\omega\mu_0}{2\sigma}} = R_s + jX_s = Z_s$$

$$\frac{E_x}{H_y} = \eta = R_s + jX_s = Z_s$$

(separating the real and imaginary parts by the method explained in Chapter 6)

(intrinsic impedance $\eta$ being equal to surface impedance $Z_s$ comprising surface resistance and surface reactance of the conducting medium)




$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$ (surface current density)

(electromagnetic boundary condition at the interface between the conducting medium (1) and the free-space medium (2) recalled from Chapter 7)

$\vec{a}_n = -\vec{a}_z$ (unit normal vector being directed from medium 1 to 2 in the negative z direction)

$-\vec{a}_z \times H_y \vec{a}_y = \vec{J}_s \longrightarrow \vec{J}_s = -\vec{a}_z \times H_y \vec{a}_y = H_y \vec{a}_x$ (surface current density)

$$\frac{E_x}{H_y} = \eta = Z_s = R_s + jX_s \text{ (recalled)}$$

Time-averaged Poynting vector

$$\vec{\mathrm{P}}_{\text{average}} = \frac{1}{2}\mathrm{Re}\,\vec{E} \times \vec{H}^* = \frac{1}{2}\mathrm{Re}(E_x \vec{a}_x) \times (H_y^* \vec{a}_y) = \frac{1}{2}\mathrm{Re}\,E_x H_y^* (\vec{a}_x \times \vec{a}_y) = \frac{1}{2}\mathrm{Re}\,E_x H_y^* \vec{a}_z$$

$$= \frac{1}{2}\mathrm{Re}(\eta H_y H_y^*) = \frac{1}{2}\mathrm{Re}(Z_s H_y H_y^*) = \frac{1}{2}\mathrm{Re}(R_s + jX_s) H_y H_y^*$$

$$= \frac{1}{2} R_s H_y H_y^*$$ (average power density propagating through the conductor along z)

Power loss per unit area in the conductor

$$\vec{\mathrm{P}}_{\mathrm{average}} = \frac{1}{2} R_s H_y H_y^*$$ (power loss per unit area in the conductor) (rewritten)

$$\vec{J}_s = -\vec{a}_z \times H_y \vec{a}_y = H_y \vec{a}_x$$ (recalled)

$$\vec{J}_s \cdot J_s^* = H_y H_y^*$$

Power loss per unit area in the conductor $= \frac{1}{2} R_s H_y H_y^* = \frac{1}{2} R_s \vec{J}_s \cdot J_s^*$

The expression finds extensive application in finding resistive or Ohmic loss in electromagnetic structures such as waveguides (in estimating attenuation constant) and resonators (in estimating quality factor) (taken up in Chapters 9 and 10 respectively to follow).

In a simple illustrative example let us calculate the incident power density and the power absorbed per unit area in a sheet of brass of conductivity $\sigma = 1.5\text{x}10^7$ mho/m on which a uniform plane wave is incident with a peak electric field of 1 V/cm at 10 GHz.

$$E_{x0} = 1\,\text{V/m} = 100\ \text{V/m (given)} \rightarrow H_{y0} = \frac{E_{x0}}{\eta_0}$$

(peak incident electric field)

$$\eta_0 = 120\pi = 377\ \text{ohm}$$

(Free-space intrinsic impedance)

(peak incident magnetic field)

$$H_{y0} = \frac{100}{377}\ \text{A/m}$$

Given

$$\sigma = 1.5\times10^7\ \text{mho/m}$$

$$f = 10\,\text{GHz} = 10\times10^9 = 10^{10}\ \text{Hz}$$

Incident power density $= \dfrac{1}{2}E_{x0}H_{y0} = \dfrac{1}{2}\dfrac{(100)^2}{377} = 13.26\ \text{W/m}^2$

Power absorbed per unit area $= \dfrac{1}{2}R_S(H_{y0})^2$

$$R_S = \sqrt{\frac{\pi f \mu_0}{\sigma}}$$

$$\mu_0 = 4\pi\times10^{-7}\ \text{H/m}$$

$$= \frac{1}{2}\sqrt{\frac{\pi f\mu_0}{\sigma}}(H_{y0})^2 = \frac{1}{2}\left(\sqrt{\frac{\pi\times10^{10}\times4\pi\times10^{-7}}{1.5\times10^7}}\right)\left(\frac{100}{377}\right)^2 = 1.8\times10^{-3}\ \text{W/m}^2 = 1.8\,\text{mW/m}^2$$

# Complex Poynting vector theorem

Let us recall Poynting theorem involving instantaneous Poynting vector:

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S} = \frac{\partial}{\partial t}\int_\tau (\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2)d\tau + \int_\tau \sigma E^2 d\tau$$

$$\vec{E}\cdot\vec{E} = E^2;\ \vec{H}\cdot\vec{H} = H^2$$

$$\vec{J}_c = \sigma\vec{E}$$

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S} = \frac{\partial}{\partial t}\int_\tau \frac{1}{2}(\varepsilon\vec{E}\cdot\vec{E} + \mu\vec{H}\cdot\vec{H})d\tau + \int_\tau \vec{E}\cdot\vec{J}_c d\tau$$

$$\vec{D} = \varepsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S} = \frac{\partial}{\partial t}\int_\tau \frac{1}{2}(\vec{E}.\vec{D} + \vec{B}.\vec{H})d\tau + \int_\tau \vec{E}\cdot\vec{J}_c d\tau$$

(Poynting theorem involving instantaneous Poynting vector expressed in different forms)

## Poynting theorem involving complex Poynting vector

Starting from

$$\nabla.(\vec{E}\times\vec{H}) = \vec{H}.(\nabla\times\vec{E}) - \vec{E}.(\nabla\times\vec{H})$$

(vector identity)

$$\frac{1}{2}\vec{E}\times\vec{H}^* = \vec{\mathrm{P}}_{\text{complex}}$$

we have already deduced

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S} = \frac{\partial}{\partial t}\int_\tau \frac{1}{2}(\vec{E}.\vec{D} + \vec{B}.\vec{H})d\tau + \int_\tau \vec{E}\cdot\vec{J}_c d\tau$$

Poynting theorem involving instantaneous Poynting vector:

$$\vec{E}\times\vec{H} = \vec{\mathrm{P}}$$

**Similarly, starting from**

$$\nabla.(\vec{E}\times\vec{H}^*) = \vec{H}^*.\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{H}^*$$ **(vector identity)**

**let us now proceed to deduce Poynting theorem involving complex Poynting vector:**

$$\frac{1}{2}\vec{E}\times\vec{H}^* = \vec{\mathrm{P}}_{\text{complex}}$$

$$\nabla.(\vec{E}\times\vec{H}^*) = \vec{H}^*.\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{H}^*$$

**(vector identity)**

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

(Maxwell's equation)

$$\nabla\times\vec{H} = \vec{J}_c + \frac{\partial\vec{D}}{\partial t}$$

$$\nabla\times\vec{H}^* = \vec{J}_c^{\;*} + \frac{\partial\vec{D}^*}{\partial t}$$

$$\vec{D} = \varepsilon\vec{E}$$

$$\nabla\times\vec{H}^* = \vec{J}_c^{\;*} + \varepsilon\frac{\partial\vec{E}^*}{\partial t}$$

$$\nabla.(\vec{E}\times\vec{H}^*) = -\vec{H}^*.\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\left(\vec{J}_c^{\;*} + \varepsilon\frac{\partial\vec{E}^*}{\partial t}\right)$$

$$\vec{B} = \mu\vec{H}$$

$$\nabla.(\vec{E}\times\vec{H}^*) = -\mu\,\vec{H}^*.\frac{\partial\vec{H}}{\partial t} - \varepsilon\,\vec{E}\cdot\frac{\partial\vec{E}^*}{\partial t} - \vec{E}\cdot\vec{J}_c^{\;*}$$

$$\vec{E} = E_0\exp(j\omega t)\vec{a}_E$$
$$\vec{E}^* = E_0\exp(-j\omega t)\vec{a}_E$$
$$\frac{\partial\vec{E}^*}{\partial t} = -j\omega E_0\exp(-j\omega t)\vec{a}_E = -j\omega\vec{E}^*$$
$$\vec{H} = H_0\exp j(\omega t\pm\theta)\vec{a}_H$$
$$\vec{H}^* = H_0\exp - j(\omega t\pm\theta)\vec{a}_H$$
$$\frac{\partial\vec{H}}{\partial t} = j\omega H_0\exp j(\omega t\pm\theta)\vec{a}_H = j\omega\vec{H}$$

$$\int_\tau\nabla.(\vec{E}\times\vec{H}^*)d\tau = -j\omega\int_\tau(\mu\,\vec{H}^*.\vec{H} - \varepsilon\,\vec{E}\cdot\vec{E}^*)d\tau - \int_\tau\vec{E}\cdot\vec{J}_c^{\;*}d\tau$$

$$\int_\tau \nabla.(\vec{E}\times\vec{H}^*)d\tau = -j\omega\int_\tau (\mu\vec{H}^*.\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*)d\tau - \int_\tau \vec{E}\cdot\vec{J}_c^{\,*}d\tau$$ (rewritten)

$$\int_\tau \nabla\cdot(\vec{E}\times\vec{H}^*)d\tau = \oint_s (\vec{E}\times\vec{H}^*)\cdot d\vec{S}$$

Vector divergence theorem as applied to the vector: $\vec{E}\times\vec{H}^*$

$$\oint_S (\vec{E}\times\vec{H}^*)\cdot d\vec{S} = -j\omega\int_\tau (\mu\vec{H}^*\cdot\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*)d\tau - \int_\tau \vec{E}\cdot\vec{J}_c^{\,*}d\tau$$

Dividing by 2

$$\oint_S \frac{1}{2}(\vec{E}\times\vec{H}^*)\cdot d\vec{S} = \oint_S \vec{\mathrm{P}}_{\mathrm{complex}}\cdot d\vec{S} = -j\omega\int_\tau \frac{1}{2}(\mu\vec{H}^*\cdot\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*)d\tau - \int_\tau \frac{1}{2}\vec{E}\cdot\vec{J}_c^{\,*}d\tau$$

With a change in sign

$$-\oint_S \frac{1}{2}(\vec{E}\times\vec{H}^*)\cdot d\vec{S} = -\oint_S \vec{\mathrm{P}}_{\mathrm{complex}}\cdot d\vec{S} = j\omega\int_\tau \frac{1}{2}(\mu\vec{H}^*\cdot\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*)d\tau + \int_\tau \frac{1}{2}\vec{E}\cdot\vec{J}_c^{\,*}d\tau$$

(Poynting theorem involving complex Poynting vector)

Thus, starting from

$$\nabla.(\vec{E}\times\vec{H}) = \vec{H}.(\nabla\times\vec{E}) - \vec{E}.(\nabla\times\vec{H})$$

(vector identity)

we have deduced

$$-\oint_s (\vec{E}\times\vec{H}).d\vec{S} = -\oint_s \vec{\mathrm{P}}.d\vec{S} = \frac{\partial}{\partial t}\int_\tau (\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2) d\tau + \int_\tau \sigma E^2 d\tau$$

Poynting theorem involving instantaneous Poynting vector:

$$\vec{E}\times\vec{H} = \vec{\mathrm{P}}$$

Similarly, starting from

$$\nabla.(\vec{E}\times\vec{H}^*) = \vec{H}^*.\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{H}^*$$

(vector identity)

we have also deduced

$$-\oint_S \frac{1}{2}(\vec{E}\times\vec{H}^*)\cdot d\vec{S} = -\oint_S \vec{\mathrm{P}}_{\mathrm{complex}}\cdot d\vec{S} = j\omega\int_\tau \frac{1}{2}(\mu\vec{H}^*\cdot\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*) d\tau + \int_\tau \frac{1}{2}\vec{E}\cdot\vec{J}_c^{\,*} d\tau$$

Poynting theorem involving complex Poynting vector: $\frac{1}{2}\vec{E}\times\vec{H}^* = \vec{\mathrm{P}}_{\mathrm{complex}}$

$$-\oint_S \frac{1}{2}(\vec{E}\times\vec{H}^*)\cdot d\vec{S} = -\oint_S \vec{\mathrm{P}}_{\mathrm{complex}}\cdot d\vec{S} = j\omega\int_\tau \frac{1}{2}(\mu\vec{H}^*\cdot\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*)d\tau + \int_\tau \frac{1}{2}\vec{E}\cdot\vec{J}_c^{\,*}d\tau$$

(Complex vector theorem)

Real part ↓ | Imaginary part ↓

**Real part:**

$$-\oint_S \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)\cdot d\vec{S} = -\oint_S \mathrm{Re}\,\vec{\mathrm{P}}_{\mathrm{complex}}\cdot d\vec{S} = \int_\tau \frac{1}{2}\vec{E}\cdot\vec{J}_c^{\,*}d\tau$$

$$\vec{\mathrm{P}}_{\mathrm{average}} = \mathrm{Re}\,\vec{\mathrm{P}}_{\mathrm{complex}} = \frac{1}{2}\mathrm{Re}\,\vec{E}\times\vec{H}^*$$

$$-\oint_S \vec{\mathrm{P}}_{\mathrm{average}}\cdot d\vec{S} = \int_\tau \frac{1}{2}\vec{E}\cdot\vec{J}_c^{\,*}d\tau$$

(real part of the expression stating the Complex vector theorem and the balance of real power)

**Imaginary part:**

$$-\oint_S \frac{1}{2}\mathrm{Im}(\vec{E}\times\vec{H}^*)\cdot d\vec{S} = -\oint_S \mathrm{Im}\,\vec{\mathrm{P}}_{\mathrm{complex}}\cdot d\vec{S} = \omega\int_\tau \frac{1}{2}(\mu\vec{H}^*\cdot\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*)d\tau$$

$$\vec{\mathrm{P}}_{\mathrm{reactive}} = \mathrm{Im}\,\vec{\mathrm{P}}_{\mathrm{complex}} = \frac{1}{2}\mathrm{Im}\,\vec{E}\times\vec{H}^*$$

$$-\oint_S \vec{\mathrm{P}}_{\mathrm{reactive}}\cdot d\vec{S} = \omega\int_\tau \frac{1}{2}(\mu\vec{H}^*\cdot\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*)d\tau$$

(imaginary part of the expression stating the Complex vector theorem and the balance of reactive power)

$$-\oint_S \vec{P}_{\text{average}} \cdot d\vec{S} = \int_\tau \frac{1}{2}\vec{E}\cdot\vec{J}_c^{\,*}\,d\tau$$

(real part of the expression stating Complex vector theorem and the balance of real power)

Left-hand side represents the average power entering a volume enclosure while the right-hand side represents the average Ohmic loss of power in the volume enclosure

$$-\oint_S \vec{P}_{\text{reactive}} \cdot d\vec{S} = \omega\int_\tau \frac{1}{2}(\mu\vec{H}^*\cdot\vec{H} - \varepsilon\vec{E}\cdot\vec{E}^*)d\tau$$

(imaginary part of the expression stating the Complex vector theorem and the balance of reactive power)

After a simple algebra

$$-\oint_S \vec{P}_{\text{reactive}} \cdot d\vec{S} =$$

$$2\omega\int_\tau\left[\frac{1}{2}\left(\frac{1}{2}\mu\vec{H}^*\cdot\vec{H}\right) - \frac{1}{2}\left(\frac{1}{2}\varepsilon\vec{E}\cdot\vec{E}^*\right)\right]d\tau$$

Left-hand side represents the reactive power flowing into a volume enclosure while the right-hand side is equal to 2ω times the difference between the average energies stored in electric and magnetic fields in the volume.

# Power flow in conduction current antennas

A conducting element through which an oscillating current passes can radiate electromagnetic energy.

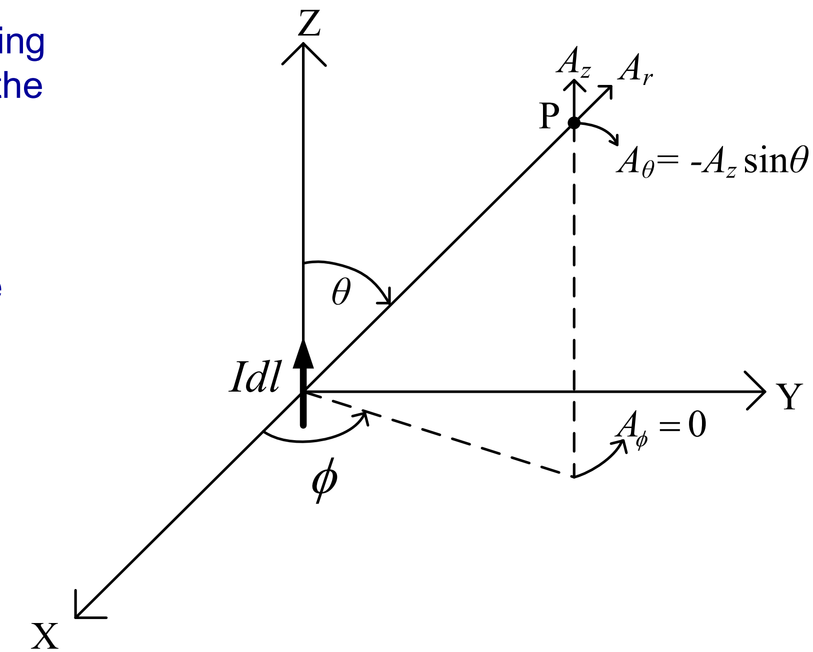
We may be interested to radiate out electromagnetic energy into space either in many directions or in a specific direction at the terminating end of the transmission line. In order to implement the radiation of electromagnetic energy, a transmission line is terminated in a radiating system called the antenna. Further, at the receiving end we need to have a receiving antenna to receive electromagnetic energy from space and subsequently make it propagate through a transmission line in a receiving system. We can apply the concept of vector potential to study some of the fundamentals of conduction current antennas such as a dipole antenna.

# Infinitesimal Hertzian dipole

The infinitesimal dipole, also called the Hertzian dipole, is an oscillating filamentary current element of infinitesimal length over which the amplitude of the current remains uniform. Although the infinitesimal dipole is not a practical antenna, the results of its analysis are of immense significance in establishing a number of important concepts of practical antennas. Further, a finite-length dipole can be considered as constituted by a number of individual infinitesimal dipoles, and therefore the results of analysis of an individual infinitesimal dipole can be integrated to obtain the results of a finite-length dipole antenna.

Let us analyse the infinitesimal dipole in spherical coordinate system of coordinates ($r,\theta,\phi$), considering the element of length *dl* of the current element at the origin of coordinates.

For the sake of convenience, let us consider the current element as aligned along z in free space. Over the infinitesimal length *dl*, the current *I* of the element is considered as constant.




Vector potential component along z at the point P ($r,\theta,\phi$) due to the current element at the origin (0,0,0)

$$A_z = \frac{\mu_0}{4\pi}\frac{Idl}{r}\exp(-j\beta r)$$

(recalled from Chapter 5)

(with the factor exp($j\omega t$) understood)

From the geometry of the problem

$$\left.\begin{aligned} A_r &= A_z\cos\theta = \frac{\mu_0}{4\pi}\frac{Idl}{r}\cos\theta\exp(-j\beta r) \\ A_\theta &= -A_z\sin\theta = -\frac{\mu_0}{4\pi}\frac{Idl}{r}\sin\theta\exp(-j\beta r) \\ A_\phi &= 0 \end{aligned}\right\}$$

$$\vec{A} = A_r\vec{a}_r + A_\theta\vec{a}_\theta + A_\phi\vec{a}_\phi$$
$$= \frac{\mu_0}{4\pi}\frac{Idl}{r}(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta)\exp(-j\beta r)$$

$$\vec{A} = A_r\vec{a}_r + A_\theta\vec{a}_\theta + A_\phi\vec{a}_\phi = \frac{\mu_0}{4\pi}\frac{Idl}{r}(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta)\exp(-j\beta r)$$ (rewritten)

(with the factor exp($j\omega t$) understood)

$$\vec{E} = -j\omega\left(\vec{A} + \frac{\nabla\nabla\cdot\vec{A}}{\omega^2\mu_0\varepsilon_0}\right)$$

$$\vec{H} = \frac{\nabla\times\vec{A}}{\mu_0}$$

(both recalled from Chapter 5)

Therefore, for this purpose we need to find

$\nabla\nabla\cdot\vec{A}$ and $\nabla\times\vec{A}$.

$$A_r = A_z\cos\theta = \frac{\mu_0}{4\pi}\frac{Idl}{r}\cos\theta\exp(-j\beta r)$$

$$A_\theta = -A_z\sin\theta = -\frac{\mu_0}{4\pi}\frac{Idl}{r}\sin\theta\exp(-j\beta r)$$

$$A_\phi = 0$$ (recalled)

↓

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(A_\phi)}{\partial\phi}$$

↓ (derived in Chapter 2)

$$\nabla\cdot\vec{A} = \frac{\mu_0 Idl}{4\pi r^2}\left(\cos\theta\frac{\partial(r(\exp-j\beta r))}{\partial r} - \frac{\exp(-j\beta r)}{\sin\theta}\frac{\partial(\sin^2\theta)}{\partial\theta}\right).$$

$$\nabla \cdot \vec{A} = \frac{\mu_0 Idl}{4\pi r^2}\left( \cos\theta \frac{\partial(r(\exp - j\beta r))}{\partial r} - \frac{\exp(-j\beta r)}{\sin\theta}\frac{\partial(\sin^2\theta)}{\partial\theta} \right)$$

(rewritten)

↓ ← Carrying out the differentiations

$$\nabla \cdot \vec{A} = \frac{\mu_0 Idl}{4\pi r^2}\left( \cos\theta\{\exp(-j\beta r) + (r)(-j\beta)\exp(-j\beta r)\} - \frac{\exp - j\beta r}{\sin\theta} 2\sin\theta\cos\theta \right)$$

↓ ← Simplifies to

$$\nabla \cdot \vec{A} = \frac{-\mu_0 Idl}{4\pi}\left(\frac{1}{r^2} + \frac{j\beta}{r}\right)\cos\theta\exp(-j\beta r)$$

$$\nabla \cdot \vec{A} = \frac{-\mu_0 I dl}{4\pi}\left(\frac{1}{r^2} + \frac{j\beta}{r}\right)\cos\theta \exp(-j\beta r)$$ (rewritten)

Similarly using the expression for the gradient of a scalar given in Chapter 2, we an find the gradient of the scalar, here being the divergence of the vector potential, the latter given by the above expression, as follows:

$$\nabla\nabla \cdot \vec{A} = \frac{\mu_0 I dl}{4\pi}\left[\left(\frac{2}{r^3} + \frac{2j\beta}{r^2} - \frac{\beta^2}{r}\right)\cos\theta\, \vec{a}_r + \left(\frac{1}{r^3} + \frac{j\beta}{r^2}\right)\sin\theta\, \vec{a}_\theta\right]\exp(-j\beta r)$$

Further, similarly using the expression for the curl of a vector given in Chapter 2, we can write the expression for curl of the vector potential as follows:

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\, A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\vec{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\vec{a}_\theta$$
$$+ \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\vec{a}_\phi$$

$$\frac{\partial}{\partial\phi} = 0$$

$$A_\phi = 0$$

$$\nabla \times \vec{A} = \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\vec{a}_\phi$$

$$\left.\begin{aligned}
\vec{A} &= A_r\vec{a}_r + A_\theta\vec{a}_\theta + A_\phi\vec{a}_\phi = \frac{\mu_0}{4\pi}\frac{Idl}{r}(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta)\exp(-j\beta r) \\
\nabla\nabla\cdot\vec{A} &= \frac{\mu_0 Idl}{4\pi}\left[\left(\frac{2}{r^3} + \frac{2j\beta}{r^2} - \frac{\beta^2}{r}\right)\cos\theta\,\vec{a}_r + \left(\frac{1}{r^3} + \frac{j\beta}{r^2}\right)\sin\theta\,\vec{a}_\theta\right]\exp(-j\beta r)
\end{aligned}\right\}$$

$$\downarrow$$

$$\vec{E} = -j\omega\left(\vec{A} + \frac{\nabla\nabla\cdot\vec{A}}{\omega^2\mu_0\varepsilon_0}\right)$$

$$\downarrow$$

$$\vec{E} = -j\omega\frac{\mu_0}{4\pi}Idl\left(\frac{1}{r}(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta)\exp(-j\beta r)\right.$$

$$\left. + \left( \left(\frac{2}{\beta^2 r^3} + \frac{2j\beta}{\beta^2 r^2} - \frac{\beta^2}{\beta^2 r}\right)\cos\theta\,\vec{a}_r + \left(\frac{1}{\beta^2 r^3} + \frac{j\beta}{\beta^2 r^2}\right)\sin\theta\,\vec{a}_\theta\right)\right)\exp(-j\beta r)$$

$$\vec{E} = E_r\vec{a}_r + E_\theta\vec{a}_\theta + E_\phi\vec{a}_\phi$$

$$\downarrow$$

$$\left.\begin{aligned}
E_r &= -\frac{j\,I\,dl}{2\pi}\frac{\eta_0}{\beta}\cos\theta\left(\frac{1}{r^3} + \frac{j\beta}{r^2}\right)\exp(-j\beta r) \\
E_\theta &= -\frac{jI\,dl}{4\pi}\frac{\eta_0}{\beta}\sin\theta\left(\frac{-\beta^2}{r} + \frac{1}{r^3} + \frac{j\beta}{r^2}\right)\exp(-j\beta r) \\
E_\phi &= 0
\end{aligned}\right\}$$

$$\nabla \times \vec{A} = \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\vec{a}_\phi$$

$$\vec{H} = \frac{\nabla \times \vec{A}}{\mu_0}$$

$$\left.\begin{aligned} A_r &= A_z \cos\theta = \frac{\mu_0}{4\pi}\frac{Idl}{r}\cos\theta \exp(-j\beta r) \\ A_\theta &= -A_z \sin\theta = -\frac{\mu_0}{4\pi}\frac{Idl}{r}\sin\theta \exp(-j\beta r) \end{aligned}\right\}$$

$$\vec{H} = \frac{1}{\mu_0 r}\frac{\mu_0}{4\pi} Idl \left(\frac{-\partial(\sin\theta \exp(-j\beta r))}{\partial r} - \frac{\partial(\frac{1}{r}\cos\theta \exp(-j\beta r))}{\partial \theta}\right)\vec{a}_\phi$$

Carrying out differentiation

$$H_\phi = \frac{I\,dl}{4\pi}\sin\theta\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)\exp(-j\beta r)$$

$$\vec{H} = H_r\vec{a}_r + H_\theta\vec{a}_\theta + H_\phi\vec{a}_\phi$$

$$\left.\begin{aligned} H_r &= 0 \\ H_\theta &= 0 \\ H_\phi &= \frac{I\,dl}{4\pi}\sin\theta\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)\exp(-j\beta r) \end{aligned}\right\}$$

$$E_r = -\frac{j\,I\,dl}{2\pi}\frac{\eta_0}{\beta}\cos\theta\left(\frac{1}{r^3}+\frac{j\beta}{r^2}\right)\exp(-j\beta\,r)$$

$$E_\theta = -\frac{jI\,dl}{4\pi}\frac{\eta_0}{\beta}\sin\theta\left(\frac{-\beta^2}{r}+\frac{1}{r^3}+\frac{j\beta}{r^2}\right)\exp(-j\beta\,r)$$

$$E_\phi = 0$$

$$H_r = 0$$

$$H_\theta = 0$$

$$H_\phi = \frac{I\,dl}{4\pi}\sin\theta\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)\exp(-j\beta\,r)$$

← Infinitesimal dipole field components put together

Time dependence exp($j\omega t$) is understood in field expressions here and in the analysis to follow

↓ ← At large distances from the infinitesimal dipole, we can ignore the terms containing higher powers of $r$ in the denominators of the field expressions

$$E_\theta = \frac{j\eta_0\beta}{4\pi\,r}Idl\sin\theta\exp(-j\beta\,r)$$

$$H_\phi = \frac{j\beta}{4\pi\,r}Idl\sin\theta\exp(-j\beta\,r)$$

← Infinitesimal dipole far-field components

↓

$$\frac{E_\theta}{H_\phi} = \eta_0$$

(far-field wave impedance of an infinitesimal dipole becoming equal to free-space intrinsic impedance)

## Power radiated at large distances from an infinitesimal dipole

$$\vec{P}_{\text{complex}} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\begin{vmatrix} \vec{a}_r & \vec{a}_\theta & \vec{a}_\phi \\ 0 & E_\theta & 0 \\ 0 & 0 & H_\phi^* \end{vmatrix} = \frac{1}{2}E_\theta H_\phi^* \vec{a}_r$$

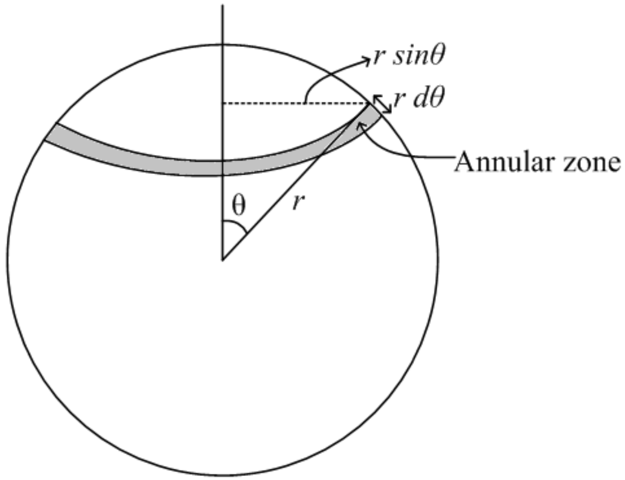
$$E_\theta = \frac{j\eta_0\beta}{4\pi r} Idl \sin\theta \exp(-j\beta r)$$

$$H_\phi = \frac{j\beta}{4\pi r} Idl \sin\theta \exp(-j\beta r)$$

$$= \frac{1}{2}\frac{\eta_0\beta^2}{(4\pi r)^2}(Idl)^2 \sin^2\theta \vec{a}_r$$

Taking the real part

$$\vec{P}_{\text{average}} = \frac{1}{2}\text{Re}\,\vec{E}\times\vec{H}^* = \frac{1}{2}\frac{\eta_0\beta^2}{(4\pi r)^2}(Idl)^2 \sin^2\theta \vec{a}_r$$




Let us now find the power radiated out from the infinitesimal dipole in all directions.

Element of power *dP* propagating through an annular disc of element of area $dS = 2\pi r \sin\theta r d\theta$ at an angle $\theta$ on the surface of a sphere of radius *r*

$$dP = \vec{P}_{\text{average}} \cdot dS\vec{a}_r = \frac{1}{2}\text{Re}(\vec{E}\times\vec{H}^*)\cdot dS\vec{a}_r = \frac{1}{2}\text{Re}(\vec{E}\times\vec{H}^*)\cdot\vec{a}_r 2\pi r \sin\theta \, r d\theta$$

$$dP = \vec{\mathrm{P}}_{\text{average}} \cdot dS\vec{a}_r = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)\cdot dS\vec{a}_r = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)\cdot\vec{a}_r 2\pi r\sin\theta\, rd\theta$$ (rewritten)

$$\vec{\mathrm{P}}_{\text{average}} = \frac{1}{2}\mathrm{Re}\,\vec{E}\times\vec{H}^* = \frac{1}{2}\frac{\eta_0\beta^2}{(4\pi r)^2}(Idl)^2\sin^2\theta\vec{a}_r$$

$$dP = \frac{1}{2}\frac{\eta_0\beta^2}{(4\pi r)^2}(Idl)^2\sin^2\theta\,\vec{a}_r\cdot\vec{a}_r 2\pi r\sin\theta\, rd\theta = \frac{\eta_0\beta^2}{16\pi}(Idl)^2\sin^3\theta\, d\theta$$

Integrating

$$P = \int dP = \int_0^\pi \frac{\eta_0\beta^2}{16\pi}(Idl)^2\sin^3\theta d\theta = \frac{\eta_0\beta^2}{16\pi}(Idl)^2\int_0^\pi \sin^3\theta d\theta$$

$$\int_0^\pi \sin^3\theta d\theta = \frac{4}{3}$$

$$P = \frac{\eta_0\beta^2}{16\pi}(Idl)^2\frac{4}{3}$$

$$\int_0^\pi \sin^3\theta d\theta = \int_0^\pi \sin^2\theta\sin\theta d\theta d\theta = \int_0^\pi -\sin^2\theta d(\cos\theta)$$

$$= -\int_0^\pi (1-\cos^2\theta)d(\cos\theta) = -\int_0^\pi [d(\cos\theta) - \cos^2\theta d(\cos\theta)]$$

$$= -[\int_0^\pi d(\cos\theta) - \int_0^\pi \cos^2\theta d(\cos\theta)] = -[\cos\theta - \frac{\cos^3\theta}{3}]_0^\pi = \frac{4}{3}$$

$$P = \frac{\eta_0 \beta^2}{16\pi} (Idl)^2 \frac{4}{3} \longleftarrow$$

$$\eta_0 = \sqrt{\mu_0 / \varepsilon_0} = 120\pi$$

$$\beta = 2\pi / \lambda$$

$$\downarrow$$

$$P = 40\pi^2 I^2 \left( \frac{dl}{\lambda} \right)^2$$

(power radiated out at large distances from an infinitesimal dipole in all directions)

We are going to use later the above expression for radiated power in the further study of the property of an infinitesimal dipole

**Power (reactive power) associated with near-field quantities of an infinitesimal dipole**

$$E_r = -\frac{j\,I\,dl}{2\pi}\frac{\eta_0}{\beta}\cos\theta\left(\frac{1}{r^3}+\frac{j\beta}{r^2}\right)\exp(-j\beta\,r)$$

$$E_\theta = -\frac{jI\,dl}{4\pi}\frac{\eta_0}{\beta}\sin\theta\left(\frac{-\beta^2}{r}+\frac{1}{r^3}+\frac{j\beta}{r^2}\right)\exp(-j\beta\,r)$$

$$E_\phi = 0$$

$$H_r = 0$$

$$H_\theta = 0$$

$$H_\phi = \frac{I\,dl}{4\pi}\sin\theta\left(\frac{j\beta}{r}+\frac{1}{r^2}\right)\exp(-j\beta\,r)$$

(infinitesimal dipole field components) (recalled)

Significant field quantities are those involving higher powers of *r* in in the denominators for distances close to the infinitesimal dipole

$$E_r = -\frac{j\eta_0}{2\pi\beta r^3}Idl\cos\theta\exp(-j\beta\,r)$$

$$E_\theta = -\frac{j\eta_0}{4\pi\beta r^3}Idl\sin\theta\exp(-j\beta\,r)$$

$$H_\phi = \frac{1}{4\pi r^2}I\,dl\sin\theta\exp(-j\beta\,r)$$

(near-field components of an infinitesimal dipole)

$$E_\theta H_\phi^* = -j\left(\frac{\eta_0}{4\pi\beta r^3}\right)\left(\frac{1}{4\pi r^2}\right)(Idl)^2 \sin^2\theta$$

$$E_r H_\phi^* = -j\left(\frac{\eta_0}{2\pi\beta r^3}\right)\left(\frac{1}{4\pi r^2}\right)(Idl)^2 \sin\theta\cos\theta$$

$\leftarrow$

$$E_r = -\frac{j\eta_0}{2\pi\beta r^3} Idl\cos\theta\exp(-j\beta r)$$

$$E_\theta = -\frac{j\eta_0}{4\pi\beta r^3} Idl\sin\theta\exp(-j\beta r)$$

$$H_\phi = \frac{1}{4\pi r^2} I\,dl\sin\theta\exp(-j\beta r)$$

(near-field components of an infinitesimal dipole)
(rewritten)

$\downarrow$

$E_\theta H_\phi^*$ and $E_r H_\phi^*$ are each imaginary quantities

$\downarrow$

$$\vec{P}_{\text{complex}} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\begin{vmatrix} \vec{a}_r & \vec{a}_\theta & \vec{a}_\phi \\ E_r & E_\theta & 0 \\ 0 & 0 & H_\phi^* \end{vmatrix} = \frac{1}{2}(E_\theta H_\phi^* \vec{a}_r - E_r H_\phi^* \vec{a}_\theta)$$

(recalled)

$\downarrow$

$\vec{P}_{\text{complex}} = \frac{1}{2}\vec{E}\times\vec{H}^*$ becomes an imaginary quantity

$\downarrow$

$$\vec{P}_{\text{average}} = \frac{1}{2}\text{Re}\,\vec{E}\times\vec{H}^* = 0 \longrightarrow$$

Near-field quantities do not correspond to the propagation of power. The power associated with these quantities is of non-radiative type, which can also be referred to as the reactive power. It is implied that the non-radiative power associated with the near-field quantities would be retured to source, feeding this power to the infinitesimal dipole.

**Directive gain and directivity of an infinitesimal dipole**

$$P = 40\pi^2 I^2 \left(\frac{dl}{\lambda}\right)^2$$ (power radiated out at large distances from an infinitesimal dipole in all directions) (recalled)

The average power density, that is, the average power per unit area radiated out of an infinitesimal dipole, $W_0$, is obtained by dividing $P$ by the area of the sphere $4\pi r^2$ of radius $r$:

$$W_0 = \frac{P}{4\pi r^2} = \frac{10\pi I^2}{r^2}\left(\frac{dl}{\lambda}\right)^2$$ (average power density being also the power density radiated out by an equivalent isotropic radiator that radiates equally in all directions, whose power $P$ equals to that of the infinitesimal dipole)

$$W = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*) = \frac{1}{2}\frac{\eta_0\beta^2}{(4\pi r)^2}(Idl)^2 \sin^2\theta$$ (power density in the direction $\theta$)

$$\eta_0 = \sqrt{\mu_0/\varepsilon_0} = 120\pi \text{ and } \beta = 2\pi/\lambda$$

$$W = \frac{15\pi I^2}{r^2}\left(\frac{dl}{\lambda^2}\right)^2 \sin^2\theta$$ (power density in the direction $\theta$)

$$D_g = \frac{W}{W_0} = \frac{3}{2}\sin^2\theta$$ ← Directive gain of an infinitesimal dipole

$$\theta = \frac{\pi}{2}$$

$$D_0 = \frac{3}{2}\sin^2\theta = \frac{3}{2}\sin^2(\pi/2) = \frac{3}{2} = 1.5$$ ← Directivity (maximum directive gain) of an infinitesimal dipole

## Radiation resistance of an infinitesimal dipole

The radiation resistance $R_r$ of an infinitesimal dipole is the resistance of an equivalent resistor that consumes the same power as that radiated out at large distances from an infinitesimal dipole in all directions, namely $P$:

$$P = \frac{1}{2} I^2 R_r \longleftarrow P = 40\pi^2 I^2 \left(\frac{dl}{\lambda}\right)^2$$

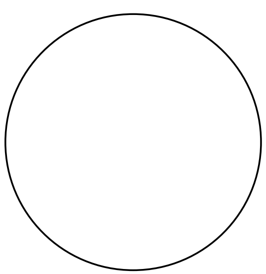
(Time-averaged power for time-harmonic current with an amplitude $I$ radiated out at large distances from an infinitesimal dipole in all directions) (recalled)

$$40\pi^2 I^2 \left(\frac{dl}{\lambda}\right)^2 = \frac{1}{2} I^2 R_r \longrightarrow R_r = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2$$

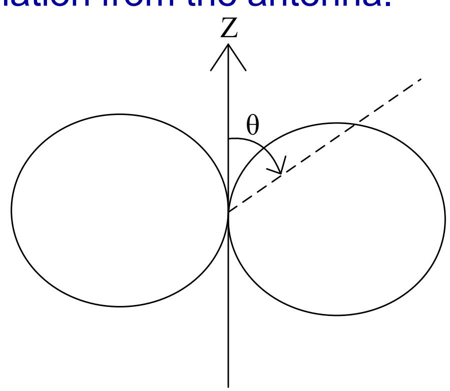
(radiation resistance of an infinitesimal dipole)

## Radiation pattern of an infinitesimal dipole

The radiation pattern of an antenna is the graphical representation of the radiation properties of the antenna with respect to related variables such as the field strength and power density as a function of space. Although the radiation pattern is three-dimensional, it is common to describe it in two planar patterns (obtained by making two slices through the three-dimensional pattern): E-plane and H-plane patterns, where the E-plane is the plane containing the electric field vector and the direction of maximum radiation from the antenna and the H-plane is the plane containing the magnetic field vector and the direction of maximum radiation from the antenna.

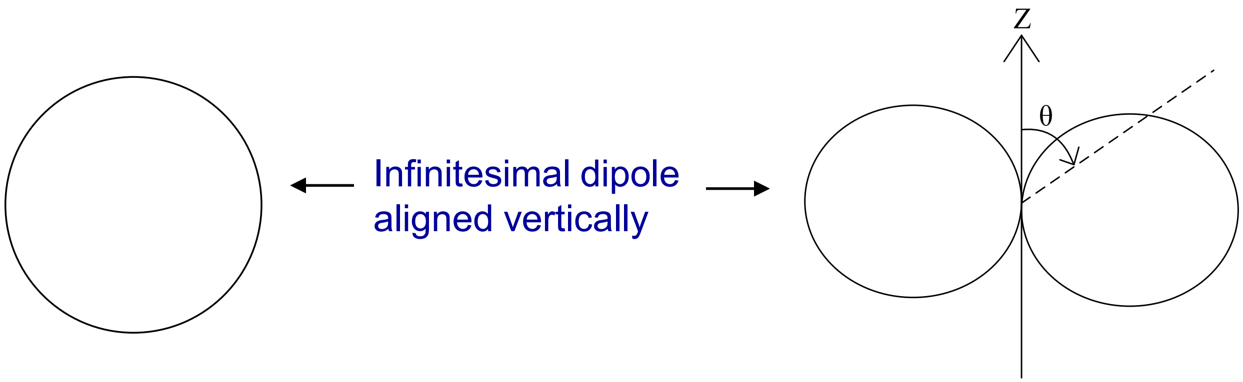


← Infinitesimal dipole aligned vertically →




Horizontal (H-plane) pattern

vertical (E-plane) pattern




Horizontal (H-plane) pattern

vertical (E-plane) pattern

The radiation pattern may can be plotted either as the horizontal pattern or the vertical pattern depending on whether the pattern is plotted on

- the horizontal plane ($\theta = \pi/2$) on which the value of $\theta$ does not vary and remains constant at the value $\theta = \pi/2$ and the value of $\phi$ varies (H-plane pattern),

or

- the vertical plane on which the value of $\phi$ remains constant ($\phi$ = constant) and the value of $\theta$ varies (E-plane pattern).

In the light of this nomenclature of the radiation pattern, the horizontal pattern (H-plane pattern) ($\theta = \pi/2$) is a circle corresponding to the constant amplitude of $E_\theta$ . On the other hand, the vertical pattern (E-plane pattern) ($\phi$ = constant) depends on the angle $\theta$, the amplitude of $E_\theta$ becoming zero and maximum for $\theta = 0$ and $\theta = \pi/2$, respectively. Thus, the shape of the vertical pattern (E-plane pattern) becomes the figure of infinity ($\infty$) and remains azimuthally symmetric ($\partial/\partial\phi = 0$) about the axis of the dipole.

**Half-power bandwidth (HPBW) of an infinitesimal dipole**

$$W = \frac{15\pi I^2}{r^2}\left(\frac{dl}{\lambda^2}\right)^2 \sin^2\theta$$ (power density in the direction $\theta$)

Takes on maximum values at $\theta = \pi/2$ corresponding to $\sin^2\theta = 1$

Takes on half the maximum values at $\theta = \pi/4$ and $\theta = 3\pi/4$ corresponding to $\sin^2\theta = ½$, which in turn leads to the half power bandwidth (HPBW)

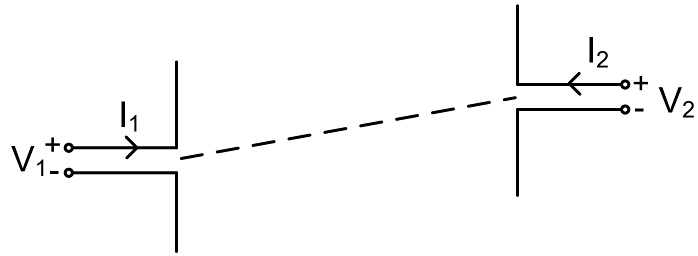
$$\text{HPBW} = (3\pi/4) - (\pi/4) = \pi/2$$

(infinitesimal dipole)

**Effective aperture area of an infinitesimal dipole**

The antenna is used not only to radiate power in the transmitting mode, but also to receive power and deliver it to a load in the receiving mode. Interestingly, an antenna enjoys identical radiation and circuit characteristics in transmitting and receiving modes according to the reciprocity theorem of circuit theory:

In any linear network containing bilateral linear impedances and energy sources, the ratio of the voltage on one mesh to the current in another mesh would remain unaltered if the voltage and the current were interchanged, the other sources being removed.




Transmitting and receiving antennas #1 and #2 separated by a distance showing the voltages and currents at their respective terminals


Network equivalent of the system of transmitting and receiving antennas #1 and #2 showing the voltages and currents at the input and output terminals

$$\left.\begin{aligned} V_1 &= Z_{11} I_1 + Z_{12} I_2 \\ V_2 &= Z_{21} I_1 + Z_{22} I_2 \end{aligned}\right\}$$

← Circuit voltages and currents related by circuit impedances in the equivalent network

↓ ← $Z_{12} = 0$ (Transmitting mode of antenna #1)

$$V_1 = Z_{11} I_1$$

(transfer impedance taken as nil for antenna #2 considered far from antenna #1)

↓

$$Z_{11} = \frac{V_1}{I_1}$$

← Self impedance of antenna #1

$Z_{12} \neq 0$ (providing coupling between antennas #1 and #2)

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

(receiving mode of antenna #1)

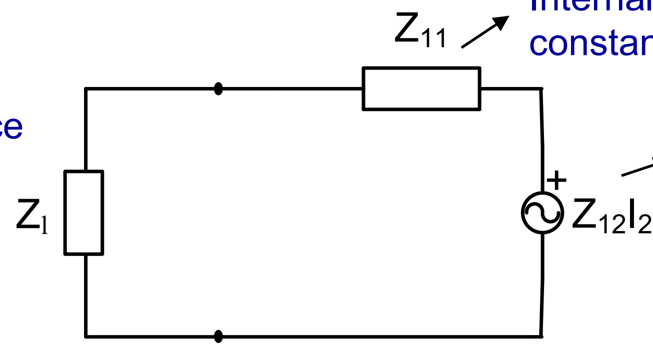
$$V_1 = Z_{11} I_1 + Z_{12} I_2$$ (recalled)

(receiving mode of antenna #1)

Internal impedance of constant voltage source

Load impedance

emf of constant voltage source




emf of the constant voltage source $Z_{12}\, I_2$ taken on the infinitesimal dipole of length $l$ as the voltage induced $El$, with electric field $E$ assumed over the length $l$

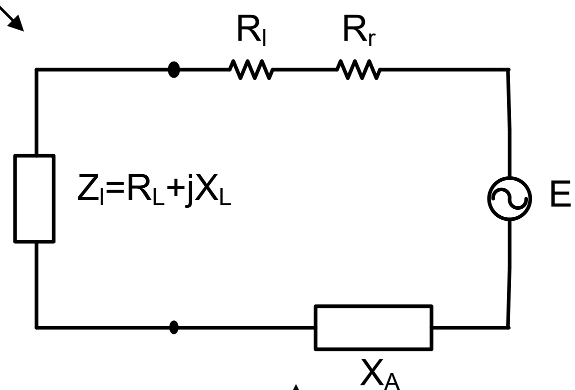
Equivalent network of antenna #1 in the receiving mode

(receiving mode)

$$Z_{11} = R_{11} + jX_{11} = R_r + jX_A$$

$$R_{11} = R_r$$ (radiation resistance)

$$X_{11} = X_A$$ (antenna reactance)

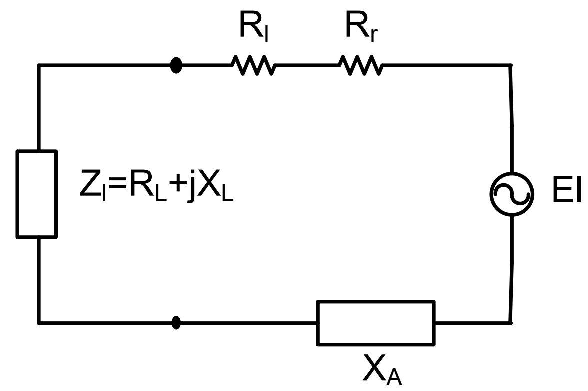



$$I = \frac{El}{\sqrt{(R_l + R_r)^2 + (X_l + X_A)^2}}$$

(amplitude of current through the equivalent network)

Equivalent network of the infinitesimal dipole in the receiving mode

$$I = \frac{El}{\sqrt{(R_l + R_r)^2 + (X_l + X_A)^2}}$$ (rewritten)




$$P_{\text{load}} = \frac{1}{2} I^2 R_l$$

$$= \frac{1}{2}\left(\frac{El}{\sqrt{(R_l + R_r)^2 + (X_l + X_A)^2}}\right)^2 R_l$$ ← Power delivered to the load

← $R_l = R_r$ and $X_l = X_A$ (under maximum power transfer condition)

$$P_{l\text{oad,max}} = \frac{1}{2}\left(\frac{El}{\sqrt{(2R_r)^2}}\right)^2 R_l = \frac{1}{2}\left(\frac{El}{2R_r}\right)^2 R_r = \frac{E^2 l^2}{8R_r}$$ (maximum power transferred to load)

$$A_{e,\max} = \frac{P_{\text{load,max}}}{W}$$ (maximum effective aperture area) ← $A_e = \frac{P_{\text{load}}}{W}$ ← Effective aperture area $A_e$ defined in terms of power density $W$ incident on the antenna

$$A_{e,\max} = \frac{\frac{E^2 l}{8R_r}}{W}$$

$$A_{e,\max} = \frac{\frac{E^2 l}{8R_r}}{W}$$ (maximum effective aperture area) (rewritten)

↓

$$A_{e,\max} = \frac{\frac{E^2 l^2}{8R_r}}{W} \leftarrow W = \frac{1}{2}EH \leftarrow$$ Interpreting with reference to an infinitesimal dipole ←

$$\vec{\mathrm{P}}_{\mathrm{average}} = \frac{1}{2}\mathrm{Re}\,\vec{E} \times \vec{H}^*$$

$$E_\theta = E \text{ and } H_\phi = H$$

↓

$$A_{e,\max} = \frac{\frac{E^2 l^2}{8R_r}}{\frac{1}{2}EH} \leftarrow$$

$$R_r = 80\pi^2 \frac{l^2}{\lambda^2}$$

$$\frac{E}{H} = \eta_0 = 120\pi$$

↓

$$A_{e,\max} = \frac{3\lambda^2}{8\pi}$$ (maximum effective aperture area of an infinitesimal dipole)

↓

$$D_0 = \frac{3}{2} \leftarrow$$ (directivity of an infinitesimal dipole)

$$\frac{A_{e,\max}}{D_0} = \frac{\lambda^2}{4\pi} \leftarrow$$ Ratio of maximum effective aperture area to directivity of an infinitesimal dipole

# Constancy of antenna effective aperture area-to-directive gain ratio

Let us take two arbitrary antennas #1 and #2 separated by a distance $r$ and let $D_{g1}$ and $D_{g2}$ be their directive gains in the direction of antennas #2 and #1 respectively; and further let $P_{t1}$ represent the power transmitted by antenna #1 in all directions.

$$D_{g1} = \frac{W_1}{\frac{P_{t1}}{4\pi r^2}} = \frac{W_1 (4\pi r^2)}{P_{t1}}$$

$$W_0 = \frac{P}{4\pi r^2} \quad \text{(recalled)}$$

$$D_g = \frac{W}{W_0} \quad \text{(recalled)}$$

$$D_g = \frac{W(4\pi r^2)}{P}$$

Directive gain $D_g$ of an antenna in terms of power density $W$ radiated in a given direction and power $P$ radiated in all directions

$$W_1 = \frac{P_{t1} D_{g1}}{4\pi r^2}$$

$$A_e = \frac{P_{\text{load}}}{W} \quad \text{(recalled)}$$

Interpreting $P_{\text{load}}$ as power recived $P_{r2}$ by antenna #2

$$A_{e2} = \frac{P_{r2}}{W_1}$$

Eliminating $W_1$

$$\frac{P_{r2}}{P_{t1}} = \frac{D_{g1} A_{e2}}{4\pi r^2}$$

$$\frac{P_{r2}}{P_{t1}} = \frac{D_{g1} A_{e2}}{4\pi r^2}$$

(recalled) (relation holding good for antenna #1 in transmitting mode and antenna #2 in receiving mode)

$$\frac{P_{r1}}{P_{t2}} = \frac{D_{g2} A_{e1}}{4\pi r^2}$$

← Corresponding relation if we now take antenna #2 in transmitting mode and antenna #1 in receiving mode

$$\frac{P_{r2}}{P_{t1}} = \frac{P_{r1}}{P_{t2}}$$

← Interpreting reciprocity theorem of circuit theory

↓

$$\frac{D_{g1} A_{e2}}{4\pi r^2} = \frac{D_{g2} A_{e1}}{4\pi r^2}$$

↓

$$\frac{A_{e1}}{D_{g1}} = \frac{A_{e2}}{D_{g2}}$$

$$\frac{A_{e1}}{D_{g1}} = \frac{A_{e2}}{D_{g2}}$$ (rewritten)

↓

Since antennas #1 and #2 are arbitrarily chosen, we conclude from the above relation that the ratio of the effective aperture area to directive gain is a constant quantity irrespective of the antenna system. The relation obviously holds good also when we take the maximum value of the effective aperture area ($A_e = A_{e,\,max}$) and the maximum value of the directive gain, the latter being the directivity ($D_{01} = D_{02}$). Therefore, we also get the ratio of the effective aperture area to directive gain as a constant quantity irrespective of the antenna system, enabling us to write:

↓

$$\frac{A_{e1,\max}}{D_{01}} = \frac{A_{e2,\max}}{D_{02}}$$

$$\frac{A_{e,\max}}{D_0} = \text{constant}$$

(irrespective of antenna)

$$\frac{A_{e,\max}}{D_0} = \frac{\lambda^2}{4\pi}$$

(found already for a particular antenna, namely, the infinitesimal dipole)

→

$$\frac{A_{e,\max}}{D_0} = \frac{\lambda^2}{4\pi}$$

(valid for any type of antenna)

# Antenna far-field and Near-field zones

Consider an antenna with its significant linear dimension as *D* and an observation point P at a distance r from its mid-point O. We can find the different ranges of an antenna depending upon the relative value of *r* with respect to the antenna dimension *D* and wavelength $\lambda$.

$$\mathrm{AP} = [(\mathrm{OA})^2 + (\mathrm{OP})^2 - 2(\mathrm{OA})(\mathrm{OP})\cos\angle\mathrm{AOP}]^{1/2}$$

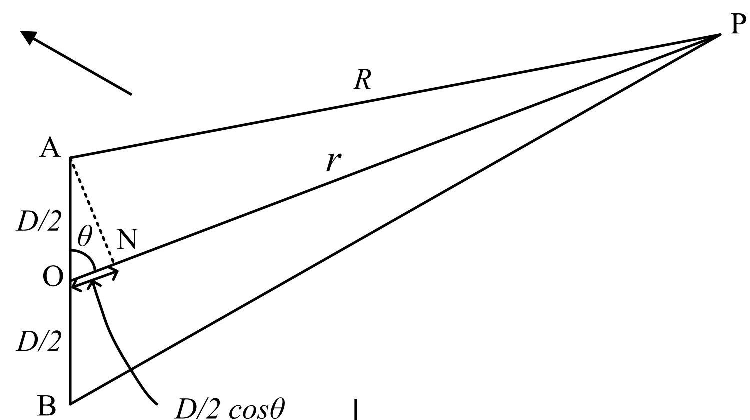
$$\mathrm{OP} = r, \mathrm{AB} = D, \mathrm{OA} = \mathrm{OB} = D/2, \mathrm{AP} = R, \angle\mathrm{AOP} = \theta$$

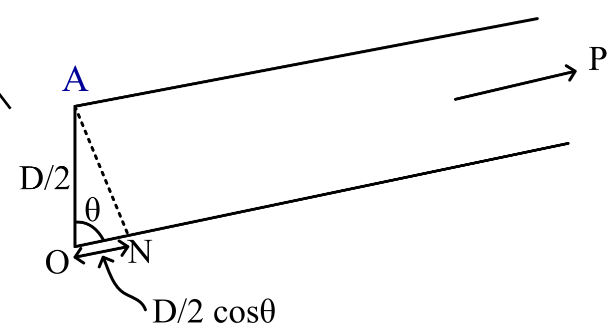
$$R = \left(r^2 + \frac{D^2}{4} - 2r\frac{D}{2}\cos\theta\right)^{1/2}$$

At large distances AP and OP tend to become parallel

$$r - R \cong \mathrm{ON} = \frac{D}{2}\cos\theta$$

$$R \cong r - \frac{D}{2}\cos\theta$$



$$R = (r^2 + \frac{D^2}{4} - 2r\frac{D}{2}\cos\theta)^{1/2}$$ (recalled)

↓

$$R = r\left[1 + \frac{D^2}{4r^2} - \frac{D\cos\theta}{r}\right]^{1/2} = r\left[1 - (\frac{D\cos\theta}{r} - \frac{D^2}{4r^2})\right]^{1/2}$$

$$= (r - \frac{D}{2}\cos\theta) + \frac{D^2 r}{8r^2}(1-\cos^2\theta) + \frac{D^3 r}{16r^3}\cos\theta(1-\cos^2\theta) + .......$$

$$= (r - \frac{D}{2}\cos\theta) + \frac{D^2 r}{8r^2}\sin^2\theta + \frac{D^3 r}{16r^3}\cos\theta\sin^2\theta + ........$$

At large distances, we can ignore the second, third and higher order terms, in view of *r* and its higher powers appearing in the denominators of these terms

↓

$$R \cong r - \frac{D}{2}\cos\theta$$

(expression that agrees to what we obtained earlier from geometrical consideration)

$$R = (r - \frac{D}{2}\cos\theta) + \frac{D^2 r}{8r^2}\sin^2\theta + \frac{D^3 r}{16r^3}\cos\theta\sin^2\theta + ........$$ (recalled)

IRE sets a standard to ignore the second and higher order terms with respect to the first term in the above expression so that one is able to use the following expression in the far-field zone:

$$R \cong r - \frac{D}{2}\cos\theta$$ (far-field expression)

Hence, as per IRE standard, one may assign a phase difference $\Delta\phi$ equal to $\pi/8$ corresponding to the path difference equal to the maximum value of the second term namely

$$\frac{D^2 r}{8r^2}\sin^2\theta$$

↓

$$\text{maximum value of path difference} = \frac{D^2 r}{8r^2}\sin^2\theta = \frac{D^2 r}{8r^2}\sin^2\pi/2 = \frac{D^2 r}{8r^2}$$

↓

$$\text{Phase difference} = \frac{2\pi}{\lambda}(\text{Path difference maximum}) = \frac{2\pi}{\lambda}\frac{D^2 r}{8r^2} = \frac{\pi}{8}$$ (set by IRE standard)

↓

$$r = \frac{2D^2}{\lambda}$$

↓

$$2D^2/\lambda < r < \infty$$ (far-field zone)

$$R = (r - \frac{D}{2}\cos\theta) + \frac{D^2 r}{8r^2}\sin^2\theta + \frac{D^3 r}{16r^3}\cos\theta\sin^2\theta + ........$$ (recalled again)

We next find the condition for distances that enables us to ignore the third and higher order terms in view of *r* and its higher powers appearing in the denominators of these terms, however retaining the second and first terms. Again, according to IRE standard, we set the value π/8 for the phase difference corresponding to the maximum value of the third term, namely

$$\frac{D^3 r}{16r^3}\cos\theta\sin^2\theta$$ (third term) → Condition for the maximum value of the third term

$$\frac{d}{d\theta}(\cos\theta\sin^2\theta) = 0$$

↓

$$\cos\theta(2\sin\theta\cos\theta) + \sin^2\theta(-\sin\theta) = 0$$

↓

$$\tan\theta = \sqrt{2};\ \sin\theta = \frac{\sqrt{2}}{\sqrt{2+1}} = \frac{\sqrt{2}}{\sqrt{3}};\ \cos\theta = \frac{1}{\sqrt{3}}$$

↓

$$\cos\theta\sin^2\theta = \frac{2}{3\sqrt{3}}$$

$$\frac{D^3 r}{16r^3}\frac{2}{3\sqrt{3}}$$

(maximum value of the third term)

$$R = (r - \frac{D}{2}\cos\theta) + \frac{D^2 r}{8r^2}\sin^2\theta + \frac{D^3 r}{16r^3}\cos\theta\sin^2\theta + ........$$

↓

$$\frac{D^3 r}{16r^3}\frac{2}{3\sqrt{3}}$$ (maximum value of the third term)

← Recalled

↓

$$\text{maximum value of path difference} = \frac{D^3 r}{16r^3}\frac{2}{3\sqrt{3}}$$

↓

$$\text{Phase difference} = \frac{2\pi}{\lambda}(\text{Path difference maximum}) = \frac{2\pi}{\lambda}\frac{D^3 r}{16r^3}\frac{2}{3\sqrt{3}} = \frac{\pi}{8}$$ (set by IRE standard)

↓

$$r = \sqrt{\frac{2}{3\sqrt{3}}\frac{D^3}{\lambda}} = 0.62\sqrt{\frac{D^3}{\lambda}}$$

↓

$$0 < r < 0.62\sqrt{D^3/\lambda}$$ (near-field zone)

Thus, with reference to antenna we can identify the following zones:

$$2D^2/\lambda < r < \infty \text{ (far-field zone)}; 0 < r < 0.62\sqrt{D^3/\lambda} \text{ (near-field zone)}$$

$$0.62\sqrt{D^3/\lambda} < r < 2D^2/\lambda \text{ (intermediate zone)}$$

We may note the following points of interest with respect to the above three zones:

Reactive near-field zone: $0 < r < 0.62\sqrt{D^3/\lambda}$

Radiative near-field zone (Fresnel region): $0.62\sqrt{D^3/\lambda} < r < 2D^2/\lambda$

Radiative far-field zone (Fraunhofer region): $2D^2/\lambda < r < \infty$

In reactive near-field zone, the field amplitude factor and the phase contributions from the secondary source elements from the antenna both vary with the change in the position of the receiving point. Also, in this region, the reactive power dominates significantly over the radiative power ( $\vec{\mathrm{P}}_{\mathrm{average}} = 0$ as has been shown earlier with reference to an infinitesimal dipole). In other words, the near-field quantities do not correspond to the propagation of power. The power associated with these quantities is of non-radiative type, which can also be referred to as the reactive power.

In radiative near-field zone (Fresnel region), the phase contributions from the secondary source elements from the antenna vary though the amplitude factor $1/r$ remains constant as the position of the receiving point is varied. Also, in this region, the radiative power is greater than the reactive power.

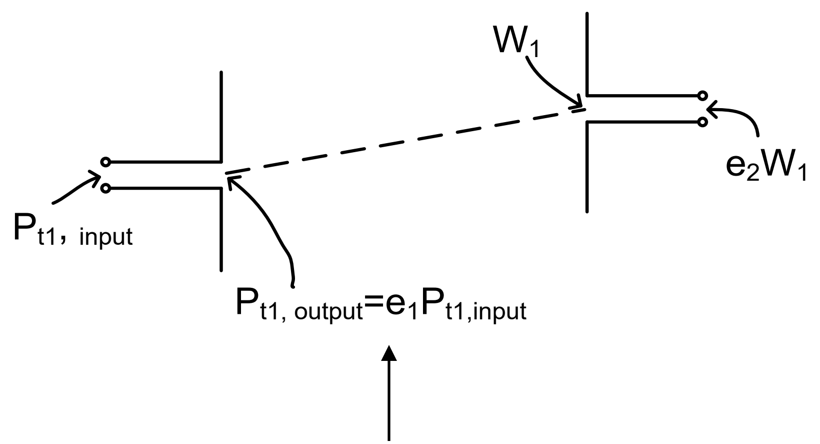
In radiative far-field zone (Fraunhofer region), the amplitude factor $1/r$ as well as the phase contributions from the secondary source elements from the antenna remains constant as the position of the receiving point is varied. Consequently, the field pattern becomes independent of the position of the receiving point. Also, in this region, the radiative power dominates significantly over the reactive power.

# Friis transmission equation

Let us now proceed to relate the power received and delivered to a load by a receiving antenna to the power transmitted by a transmitting antenna at a distance in terms of their respective power gains and wavelength corresponding to the operating frequency.

We will derive the relation under the condition that the receiving antenna is placed in the direction of the maximum power density radiated by the transmitting antenna and that the maximum power is delivered to the load.

In the transmitting mode the antenna is connected to a source of power by a transmission line. Similarly, in the receiving mode, the .antenna is connected to a load by a transmission line. Due to losses in the transmission line, the power at the output end of the transmission line of the antenna is less than the power at its input end by a factor called the efficiency of the antenna.

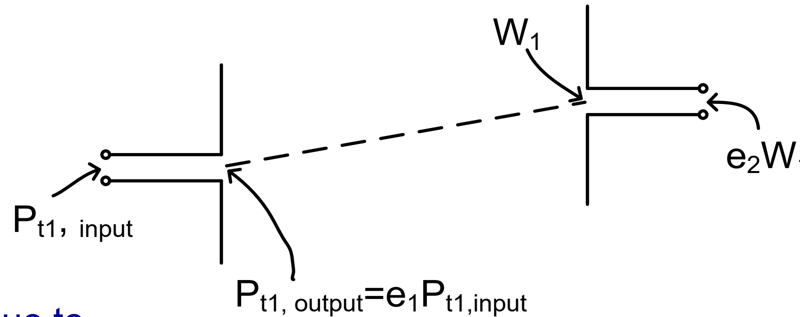



Transmitting and receiving antennas #1and #2 showing the powers at the input and output of the transmission lines connected to the antennas #1and #2 respectively

$$P_{t1,\text{output}} = \mathrm{e}_1 P_{t1,\text{input}}$$

$e_1$ is defined as the efficiency of the transmitting antenna, the subscript 1 referring to the transmitting antenna taken as antenna #1 and the subscript *t* standing for transmitting mode

$$P_{t1,\text{output}} = e_1 P_{t1,\text{input}}$$ (rewritten)




$$W_1 = \frac{P_{t1} D_{g1}}{4\pi r^2}$$ (recalled)

$$D_{g1} = \frac{1}{e_1} \frac{W_1}{\dfrac{P_{t1,\text{input}}}{4\pi r^2}}$$ (recalled)

Power density due to antenna #1 present at the input of the transmission line connected to antenna #2

$$D_{g1} e_1 = \frac{W_1}{\dfrac{P_{t1,\text{input}}}{4\pi r^2}}$$

With the help of the relation

$$D_{g1} = \frac{W_1}{\dfrac{P_{t1}}{4\pi r^2}}$$ (recalled) $(\text{with } P_{t1} \text{ taken as } e_1 P_{t1,\text{input}})$

$$G_1 = \frac{W_1}{\text{average power density}}$$

$$= \frac{W_1}{\dfrac{P_{t1,\text{input}}}{4\pi r^2}} \text{ (defined as)}$$

On comparing

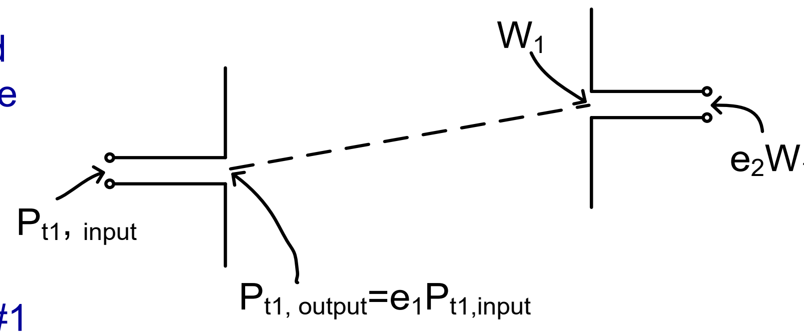
$$G_1 = D_{g1} e_1$$

With reference to antenna #2,
effective aperture area may be put as

$$A_{e2} = \frac{P_{r2,\text{output}}}{\mathrm{e}_2 W_1}$$

Power received at the output of the transmission line connected to antenna #2 (transferred to the load)

Power density due to antenna #1 appearing at the output of the transmission line connected to antenna #2




$$D_{g1}\mathrm{e}_1 = \frac{W_1}{\dfrac{P_{t1,\text{input}}}{4\pi r^2}}$$ (recalled)

Eliminating $W_1$

$$\frac{P_{r2,\text{output}}}{P_{t1,\text{input}}} = \frac{A_{e2} D_{g1}\mathrm{e}_1\mathrm{e}_2}{4\pi r^2}$$

$$\frac{P_{r2,\text{output}}}{P_{t1,\text{input}}} = \frac{A_{e2} D_{g1} e_1 e_2}{4\pi r^2}$$ (rewritten)

$$\frac{A_{e2}}{D_{g2}} = \frac{\lambda^2}{4\pi} \leftarrow \frac{A_{e,\max}}{D_0} = \frac{\lambda^2}{4\pi}$$

(valid for any type of antenna)

Under the assumed condition of maximum power transferred to the load connected to antenna #2

$$\frac{P_{r2,\text{output}}}{P_{t1,\text{input}}} = \frac{D_{g1} e_1 D_{g2} e_2}{4\pi r^2} \frac{\lambda^2}{4\pi}$$

$$G_1 = D_{g1} e_1 \text{ and similarly } G_2 = D_{g2} e_2$$

$$\frac{P_{r2,\text{output}}}{P_{t1,\text{input}}} = G_1 G_2 \left(\frac{\lambda}{4\pi r}\right)^2$$

(Friis transmission equation)

**Let us take up an example to illustrate Friis transmission equation. Take two identical antennas having the same gain —one transmitting power while the other receiving it— both having significant dimension given as 20 cm. The antennas are separated by a distance equal to 1.5 times the minimum distance prescribed by Fraunhofer radiative far-field zone, arranged in a measurement setup using 10 GHz operating frequency. Calculate the antenna gain if the received power measured is 20 dB below the transmitted power.**

Let us begin with finding the distance *r* between the antennas according to its limit given in terms of Fraunhofer radiative far-field zone: $2D^2/\lambda < r < \infty$.

$$D = 20\,\text{cm (given)}$$

$$r = 1.5 \times \frac{2D^2}{\lambda} = 1.5 \times \frac{2(20)^2}{3} = (20)^2 = 400\ \text{cm}$$

$$f = 10\ \text{GHz} = 10 \times 10^9\ \text{Hz (given)}$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^{10}}{10 \times 10^9} = 3\,\text{cm}$$

$$10 \log_{10} \frac{P_t}{P_r} = 20\,\text{dB (given)}$$

$$\log_{10} \frac{P_t}{P_r} = 2$$

$$10^2 = \frac{P_t}{P_r}$$

$$\frac{P_r}{P_t} = \frac{1}{10^2}$$

$$\frac{P_{r2,\text{output}}}{P_{t1,\text{input}}} = \frac{1}{10^2}$$

$$\frac{P_{r2,\text{output}}}{P_{t1,\text{input}}} = G_1 G_2 \left(\frac{\lambda}{4\pi r}\right)^2$$

$$\frac{1}{10^2} = G_1 G_2 \left(\frac{\lambda}{4\pi r}\right)^2$$

$G_1 = G_2 = G$ say (identical antennas)

$$\frac{1}{10^2} = G^2 \left(\frac{\lambda}{4\pi r}\right)^2$$

$$\frac{1}{10} = G\left(\frac{\lambda}{4\pi r}\right)$$

$\lambda = 3\,\text{cm}$, $r = 400\,\text{cm}$ (recalled)

$$G = \frac{4\pi \times 40}{3} = 167.47$$

$$G\,(\text{dB}) = 10 \log_{10} 167.47$$

$$= 10 \times 2.239 = 22.39\,\text{dB}$$

**Let us now appreciate a method, well known as three-antenna method, based on Friis transmission equation, to find the gain of an antenna, if you can measure the power transmitted by an antenna and the power received by another antenna for a given operating frequency.**

Let us have three antennas labeled as #1, #2 and #3 respectively. Since we can measure power received $P_{r2}$ by antenna #2 due to power transmitted $P_{t1}$ by antenna #1 at a known distance $r$, we can use the following relation given by Friis transmission equation in terms of the antenna gains $G_1$ and $G_2$ and the wavelength $\lambda$ corresponding to the operating frequency:

$$\frac{P_{r2}}{P_{t1}} = G_1 G_2 \left(\frac{\lambda}{4\pi r}\right)^2$$ (given by Friis transmission equation)

↓

$$G_1 G_2 = \frac{P_{r2}}{P_{t1}} \left(\frac{4\pi r}{\lambda}\right)^2 \longleftarrow \quad c = f\lambda$$

↓

$$G_1 G_2 = \frac{P_{r2}}{P_{t1}} \left(\frac{4\pi r f}{c}\right)^2$$

↓

Physical quantities in the right hand side can be measured and hence we can determine the gain product $G_1G_2$. Following the same approach, we can then determine $G_1G_2$ and $G_2G_3$ as well.

→

$$\left.\begin{aligned} G_1 &= \sqrt{\frac{(G_1G_2)(G_1G_3)}{G_2G_3}} \\ G_2 &= \sqrt{\frac{(G_2G_3)(G_1G_2)}{G_1G_3}} \\ G_3 &= \sqrt{\frac{(G_2G_3)(G_1G_3)}{G_1G_2}} \end{aligned}\right\}$$

↓

Hence the gains of all the three antennas can be determined.

# Finite-length dipole

$$R_r = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2$$ (radiation resistance of an infinitesimal dipole) (recalled)

↓

Radiation resistance of an infinitesimal dipole is very small since its length $dl << \lambda$.

↓

$$P = \frac{1}{2} I^2 R_r$$ (power radiated out at large distances from an infinitesimal dipole in all directions) (recalled)

↓

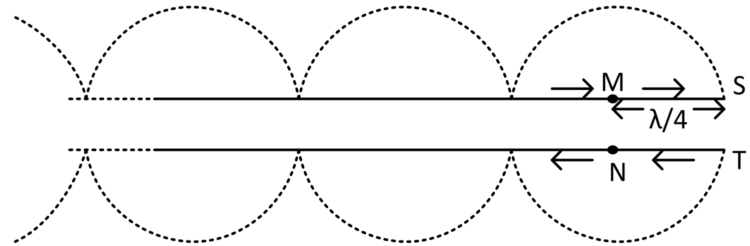
Power $P$ radiated out at large distances from an infinitesimal dipole in all directions becomes very small unless we increase the current $I$ to a very large value.

↓

Therefore, let us look forward to more practical antennas such as the finite-length dipole for a higher value of the radiation resistance and consequently a lesser amount of current through it required for a larger radiated power.

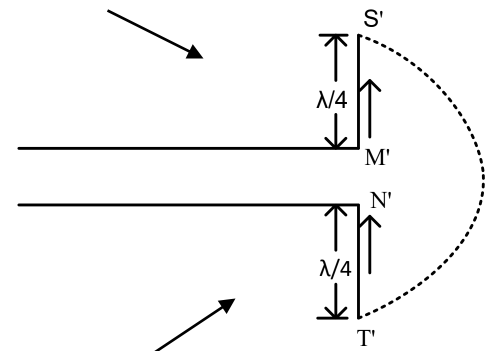
Open-ended two-wire transmission line showing standing-wave current distribution on it by dotted curve

M and N are points on the two wires equidistant from their respective ends S and T. Typically, MS = NT = $\lambda/4$ for a centre-fed, half-wave ($\lambda/2$) dipole to be made out of the wires of the line.




$M^{/}$ and $N^{/}$ are the points of bending equidistant from their respective ends $S^{/}$ and $T^{/}$.

Open-ended two-wire transmission line with wires bent at right angles at the open ends to form a centre-fed, finite-length dipole showing standing-wave current distribution on it by dotted curve




Typically, $M^{/}S^{/} = N^{/}T^{/} = \lambda/4$ for a centre-fed, half-wave ($\lambda/2$) dipole to be made out of the wires of the line.

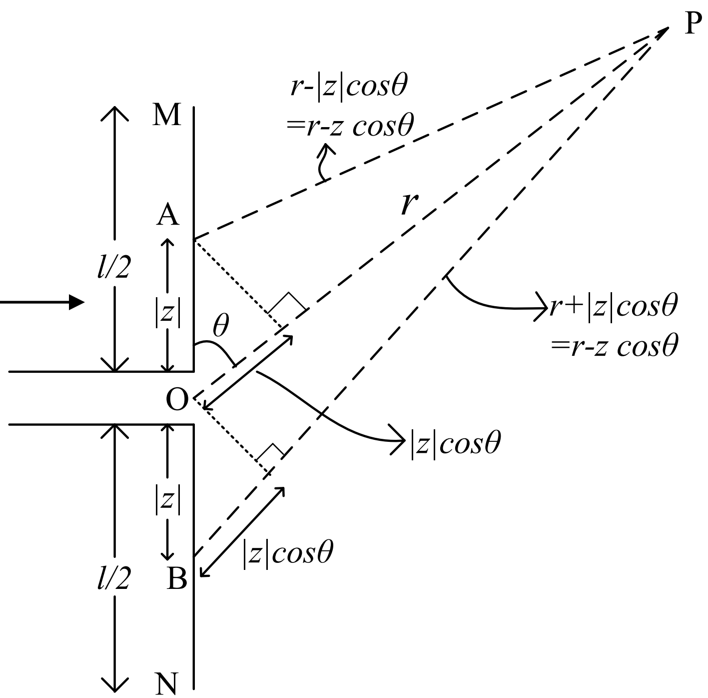
The current is nil at the open ends of the line or dipole and the current distribution typically on a centre-fed, half-wave ($\lambda/2$) dipole of length $l = \lambda/2$ may be represented as

$$I = I_0 \cos\beta z = I_0 \cos\frac{2\pi}{\lambda} z$$

$$[-l/2\,(=-\lambda/4) < 0 < l/2\,(=\lambda/4)]$$

(z is measured from the middle of the dipole: z = 0)

Finite-length dipole of length $l$ = MN aligned vertically along z showing two infinitesimal current elements at the points A and B and a distant point P where the field of the dipole is sought




$$MN = l$$

$$OM = ON = l/2$$

Distance of the point A from the middle O of the dipole

$$= z = |z|$$

Distance of the point B from the middle O of the dipole

$$= z = -|z|$$

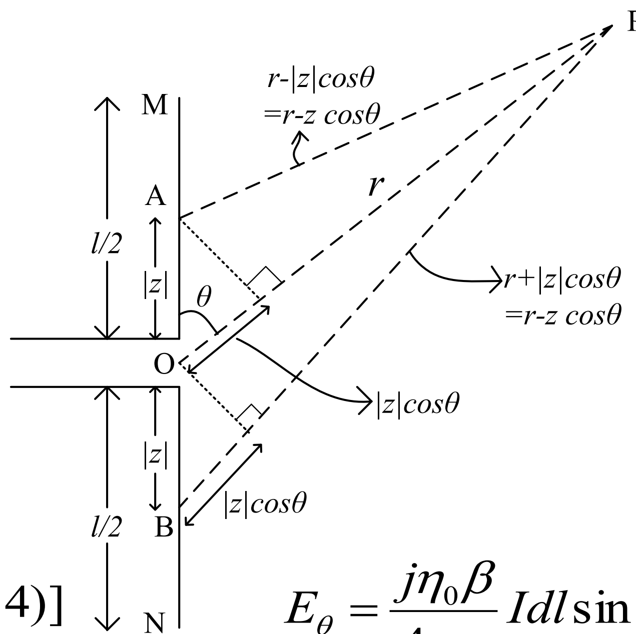
$$OP = r$$

$$AP = r - |z|\cos\theta = r - z\cos\theta$$

$z = |z|$ (interpreted as positive for the positive half OM of the dipole)

$$BP = r + |z|\cos\theta = r - z\cos\theta$$

$z = -|z|$ (interpreted as negative for the negative half ON of the dipole)




$$I = I_0 \cos\beta z = I_0 \cos\frac{2\pi}{\lambda} z$$
$$[-l/2(=-\lambda/4) < 0 < l/2(=\lambda/4)]$$

(dipole current distribution)

$$E_\theta = \frac{j\eta_0\beta}{4\pi r} I dl \sin\theta \exp(-j\beta r)$$

(infinitesimal dipole far-field quantity recalled)

Integrating the contributions from infinitesimal dipoles distributed over the entire length $l = \lambda/2$ of a half-wave dipole comprising two halves each of length $l = \lambda/4$

$$E_\theta = \int dE_\theta = \int_0^{\lambda/4} \frac{j\eta_0\beta}{4\pi(r - z\cos\theta)} I_0 \cos\beta z \sin\theta \exp(-j\beta(r - z\cos\theta))dz$$
$$+ \int_{-\lambda/4}^{0} \frac{j\eta_0\beta}{4\pi(r - z\cos\theta)} I_0 \cos\beta z \sin\theta \exp(-j\beta(r - z\cos\theta))dz$$

(azimuthal electric field component at P due to a half-wave dipole)

$$E_\theta = \int dE_\theta = \int_0^{\lambda/4} \frac{j\eta_0\beta}{4\pi(r - z\cos\theta)} I_0 \cos\beta z \sin\theta \exp(-j\beta(r - z\cos\theta))dz$$

$$+ \int_{-\lambda/4}^{0} \frac{j\eta_0\beta}{4\pi(r - z\cos\theta)} I_0 \cos\beta z \sin\theta \exp(-j\beta(r - z\cos\theta))dz$$

(azimuthal electric field component at P due to a half-wave dipole) (recalled)

Approximated by putting in the denominators: $r - z\cos\theta \cong r$

$$E_\theta = \frac{j\eta_0 \sin\theta\beta I_0}{4\pi r}\left(\int_0^{\lambda/4} \cos\beta z \exp(-j\beta(r - z\cos\theta))dz + \int_{-\lambda/4}^{0} \cos\beta z \exp(-j\beta(r - z\cos\theta))dz)\right)$$

After rearrangement of terms

$$E_\theta = \frac{j\eta_0 \sin\theta\beta I_0}{4\pi r} \exp(-j\beta r)$$

$$\times\left(\int_0^{\lambda/4} \cos\beta z \exp(j\beta z\cos\theta))dz + \int_{-\lambda/4}^{0} \cos\beta z \exp(j\beta z\cos\theta))dz)\right)$$

$$E_\theta = \frac{j\eta_0 \sin\theta\beta I_0}{4\pi r}\exp(-j\beta r)$$

$$\times\left(\int_0^{\lambda/4}\cos\beta z \exp(j\beta z\cos\theta))dz + \int_{-\lambda/4}^{0}\cos\beta z\exp(j\beta\, z\cos\theta))dz)\right)$$

(rewritten)

In view of $\int \cos\beta z \exp(j\beta z\cos\theta))dz = \frac{\exp(j\beta\cos\theta)z}{\beta\sin^2\theta}(j\cos\theta\cos\beta z + \sin\beta z)$

$$E_\theta = \frac{j\eta_0 \sin\theta\beta I_0}{4\pi r}\exp(-j\beta r)\left[\left(\frac{\exp(j\beta\cos\theta)z}{\beta\sin^2\theta}(j\cos\theta\cos\beta z + \sin\beta z)\right)_0^{\lambda/4}\right.$$

$$\left. + \left(\frac{\exp(j\beta\cos\theta)z}{\beta\sin^2\theta}(j\cos\theta\cos\beta z + \sin\beta z)\right)_{-\lambda/4}^{0}\right].$$

Rewritten

$$E_\theta = \frac{j\eta_0 \sin\theta \beta I_0}{4\pi r}\exp(-j\beta r)\left[\left(\frac{\exp(j\beta\cos\theta)z}{\beta\sin^2\theta}(j\cos\theta\cos\beta z+\sin\beta z)\right)_0^{\lambda/4} + \left(\frac{\exp(j\beta\cos\theta)z}{\beta\sin^2\theta}(j\cos\theta\cos\beta z+\sin\beta z)\right)_{-\lambda/4}^{0}\right].$$

In view of the relation $\beta = 2\pi/\lambda$

$$E_\theta = \frac{j\eta_0 \sin\theta \beta I_0}{4\pi r}\exp(-j\beta r)\left(\frac{\exp(j\frac{\pi}{2}\cos\theta - j\cos\theta}{\beta\sin^2\theta} + \frac{j\cos\theta + \exp(-j\frac{\pi}{2}\cos\theta}{\beta\sin^2\theta}\right)$$

Simplifies to

$$E_\theta = \frac{j\eta_0 \sin\theta \beta I_0}{4\pi r}\exp(-j\beta r)\left(\frac{\exp j\frac{\pi}{2}\cos\theta + \exp - j\frac{\pi}{2}\cos\theta}{\beta\sin^2\theta}\right)$$

$$E_\theta = \frac{j\eta_0 \sin\theta \beta I_0}{4\pi r}\exp(-j\beta r)\left(\frac{\exp j\frac{\pi}{2}\cos\theta + \exp - j\frac{\pi}{2}\cos\theta}{\beta \sin^2\theta}\right)$$

$$\longleftarrow \quad \frac{\exp j\psi + \exp - j\psi}{2} = \cos\psi$$

$$E_\theta = \frac{j\eta_0 I_0}{4\pi r}\exp(-j\beta r)\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

(half-wave dipole)

$$\longleftarrow \quad \frac{E_\theta}{H_\phi} = \eta_0$$

(relation derived earlier for an infinitesimal dipole continuing to be valid for a half-wave dipole)

$$H_\phi = \frac{jI_0}{4\pi r}\exp(-j\beta r)\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$H_\phi^* = \frac{-jI_0}{4\pi r}\exp(j\beta r)\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

(half-wave dipole)

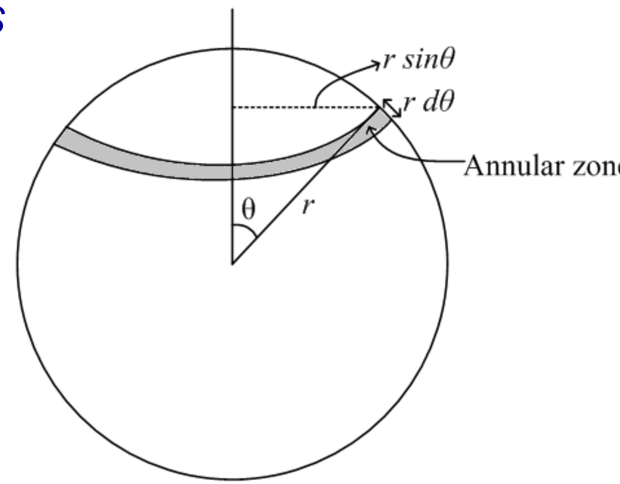
$$E_\theta = \frac{j\eta_0 I_0}{4\pi r}\exp(-j\beta r)\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$H_\phi^* = \frac{-jI_0}{4\pi r}\exp(j\beta r)\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

(half-wave dipole)

(rewritten)

*dP* being the element of power radiating through an annular disc of element of area *dS* = $2\pi r\sin\theta r d\theta$ at an angle $\theta$ on the surface of a sphere of radius *r*




$$P = \int dP = \int \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)\cdot dS\vec{a}_r$$

$$= \int \frac{1}{2}\mathrm{Re}\, E_\theta H_\phi^* \vec{a}_r \cdot \vec{a}_r 2\pi r\sin\theta r d\theta$$

$$= \int_0^\pi \eta_0\left(\frac{I_0}{4\pi r}\right)^2\left(\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2 \pi r\sin\theta\, r d\theta$$ (half-wave dipole)

(power radiating through a sphere of radius *r* thus found using the same approach as that followed earlier while finding such power for an infinitesimal dipole)

$$P = \int_0^{\pi} \eta_0 \left( \frac{I_0}{4\pi r} \right)^2 \left( \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right)^2 \pi r \sin\theta \, r d\theta$$ (rewritten)

$$\eta_0 = 120\pi$$

$$P = 30 I_0^{\,2} \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta$$

$$P = \frac{1}{2} I_0^{\,2} R_r$$ (power related to radiation resistance recalled)

$$R_r = 60 \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta$$

$$\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta = 1.218$$

(evaluated numerically)

$$R_r = 60 \times 1.218 = 73.08 \ \text{ohm}$$ (half-wave dipole)

(radiation resistance of a half-wave dipole)

$$P = 30 I_0{}^2 \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta$$ (recalled)

← Dividing by $4\pi r^2$

$$W_0 = \frac{30 I_0{}^2 \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta}{4\pi r^2}$$

(Average power per unit area radiated out of a half-wave dipole or power density of an isotropic radiator equivalent to half-wave dipole)

Power density of the half-wave dipole in the direction

$$W = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^{*}) = \frac{1}{2}\mathrm{Re}(E_\theta H_\phi^{*}) = \frac{1}{2}\eta_0 I_0^{2}\left(\frac{1}{4\pi r}\right)^{2}\left(\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^{2}$$

$$E_\theta = \frac{j\eta_0 I_0}{4\pi r}\exp(-j\beta r)\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$H_\phi^{*} = \frac{-jI_0}{4\pi r}\exp(j\beta r)\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

(recalled)

$$\eta_0 = 120\pi$$

$$W = 15 I_0^{2}\frac{1}{4\pi r^{2}}\left(\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^{2}$$

(half-wave dipole)

$$W = 15 I_0^{\,2} \frac{1}{4\pi r^2} \left( \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right)^2$$

(rewritten)

$$W_0 = \frac{30 I_0^{\,2} \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta}{4\pi r^2}$$

$$D_g = \frac{W}{W_0} = \frac{2\left( \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right)^2}{\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta}$$

$$\int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta = 1.218$$

$$D_g = \frac{2\left( \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right)^2}{1.218}$$

(directive gain) (half-wave dipole)

$$D_g = \frac{2\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2}{1.218}$$ (directive gain) (rewritten)

↓ ← Maximised by taking $\theta = \frac{\pi}{2}$

$$D_0 = \frac{2\left(\frac{\cos\left(\frac{\pi}{2}\cos\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}}\right)^2}{1.218} = \frac{2\left(\frac{\cos(\frac{\pi}{2}\times 0)}{\sin\frac{\pi}{2}}\right)^2}{1.218} = \frac{2}{1.218} = 1.64$$ (directivity) (half-wave dipole)

$$E_\theta = \frac{j\eta_0 I_0}{4\pi r}\exp(-j\beta r)\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$ (half-wave dipole) (recalled)

↓

$$(E_\theta)_{\text{amplitude}} = \frac{\eta_0 I_0}{4\pi r}\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$(E_\theta)_{\text{amplitude}}\Big|_{\max} = \frac{\eta_0 I_0}{4\pi r}\frac{2\cos\left(\frac{\pi}{2}\cos\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}} = \frac{2\eta_0 I_0}{4\pi r}$$

↓

$$\frac{(E_\theta)_{\text{amplitude}}}{(E_\theta)_{\text{amplitude}}\Big|_{\max}} = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} = \frac{1}{\sqrt{2}}$$ ←

Value so assigned makes the square of the normalised quantity equal to ½ corresponding to half power bandwidth HPBW

↑

The angle between the two solutions can be found numerically as $78^0$.

↓

HPBW = $78^0$

## Comparison between infinitesimal and half-wave dipoles

| Antenna type | Radiation resistance | Directivity | HPBW |
|---|---|---|---|
| Infinitesimal dipole | Insignificant | 1.5 | $90^0$ |
| Half-wave dipole | 73.08 ohm | 1.64 | $78^0$ |

The radiation resistance of a finite-length antenna such as the half-wave dipole is higher than that of an infinitesimal dipole. This calls for lesser required current for a finite-length antenna than for an infinitesimal dipole.

It often becomes necessary to beam power from an antenna in one specified direction.

How much an antenna is capable of doing so can be estimated both by the antenna HPBW and the directivity, the latter rather more quantitatively.

Both the HPBW and the directivity of a finite-length antenna are greater than those of an infinitesimal dipole.

One can increase the directivity of an antenna by designing its geometry/shape and size. However, such an approach becomes somewhat difficult to implement and often leads to an inconvenient antenna geometry/shape and size for a practical antenna design.

The alternative approach is to use linear (one-dimensional), planar (two-dimensional) or volume (three-dimensional) array of identical antenna elements. Moreover, such an array antenna provides means to steer the beam of the antenna electronically rather than mechanically, say, by physically rotating a bulky antenna.

**For the sake of simplicity, let us consider here for analysis a uniform linear array of identical antenna elements.**

# Linear array of antenna elements

Successive (progressive) phase difference $\psi$ between the electric fields due to uniform array of identical antenna elements at a large distance from the array

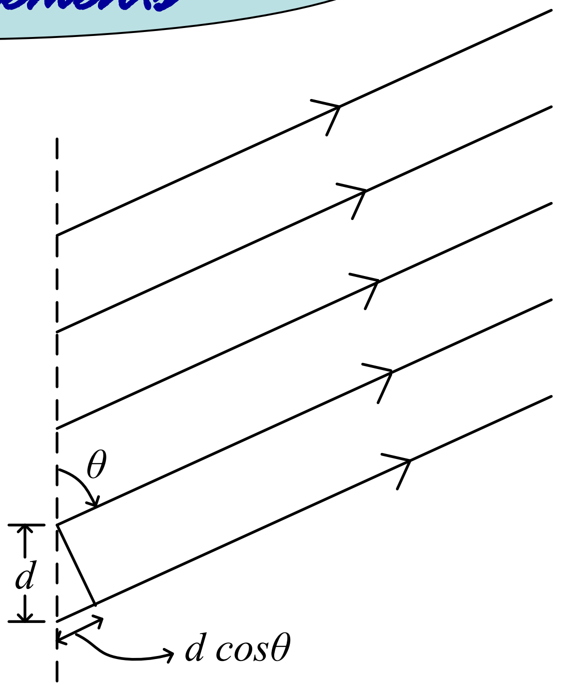
$$\psi = \beta d \cos\theta + \xi$$

$\beta$ = wave phase propagation constant

$d$ = distance between successive array elements

$\theta$ = angular direction of the observation point from the line of array

$\xi$ = excitation phase difference between successive array elements, being the excitation phase lead of an element with respect to the next lower order element




An array of elements showing the direction of a distant observation point and correspondingly parallel rays of radiated waves from the elements in the same direction

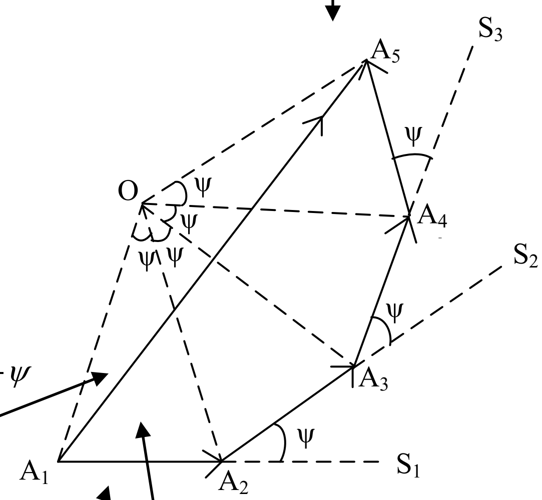
With due consideration to the above progressive phase difference $\psi$, let us next draw the vector diagram for the electric field due to each of the array elements at a large distance from the array and subsequently take the vector sum of the contributions from all the elements of the array.

Vector diagram for electric fields due to array elements #1, #2, #3 and #4 at a large distance from the array considering typically four elements, showing also the vector sum $\overrightarrow{A_1A_5}$

$$\overrightarrow{A_1A_5} = \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \overrightarrow{A_3A_4} + \overrightarrow{A_4A_5}$$

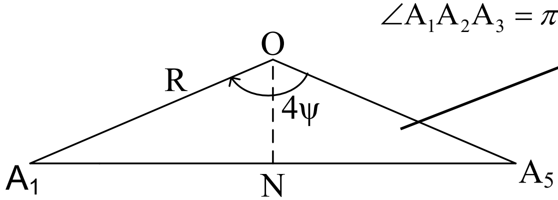
where $\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \overrightarrow{A_3A_4}$ and $\overrightarrow{A_4A_5}$

represent electric fields at the observation point due to antenna elements #1, #2, #3 and #4 respectively




$$\angle S_1A_2A_3 = \angle S_2A_3A_4 = \angle S_3A_4A_5 = \psi$$

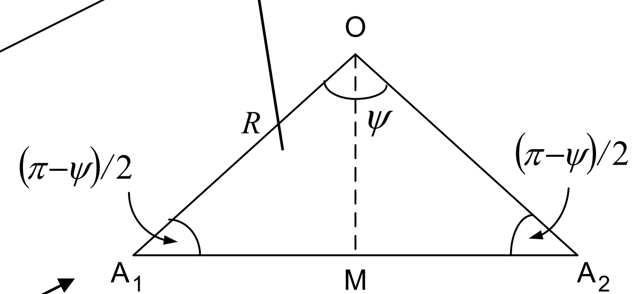
$\angle A_1A_2A_3 = \pi - \psi$




$$\angle A_1A_2A_3 = \pi - \psi$$

$$\angle A_1A_2O = \frac{\pi - \psi}{2}$$

$$\angle A_2A_1O = \frac{\pi - \psi}{2}$$




$$\angle A_1A_2O + \angle A_2A_1O + \angle A_1OA_2 = \pi$$

$$\angle A_1OA_2 = \pi - \angle A_1A_2O - \angle A_2A_1O$$

$$\angle A_1A_2O = \frac{\pi-\psi}{2} \qquad \angle A_2A_1O = \frac{\pi-\psi}{2}$$

(recalled)

$$\downarrow$$

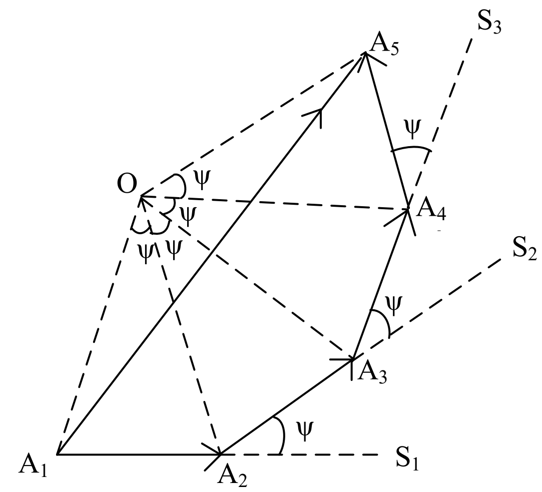
$$\angle A_1OA_2 = \pi - \angle A_1A_2O - \angle A_2A_1O$$ (recalled)

$$\downarrow$$

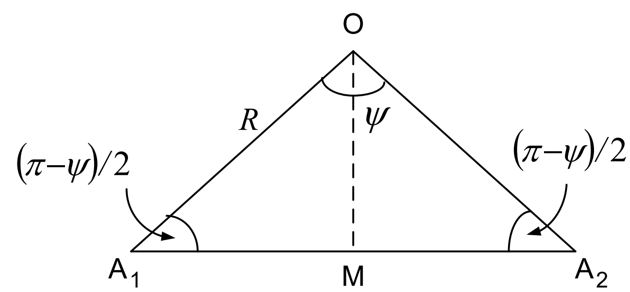
$$\angle A_1OA_2 = \pi - \frac{\pi-\psi}{2} - \frac{\pi-\psi}{2} = \psi$$

$$\downarrow$$

$$\angle A_2OA_3 = \angle A_3OA_4 = \angle A_4OA_5 = \psi$$




(recalled)




$$\angle A_1OA_5 = \angle A_1OA_2 + \angle A_2OA_3 + \angle A_3OA_4 + \angle A_4OA_5$$ (recalled)

$$= \psi + \psi + \psi + \psi = 4\psi.$$

$$A_1M = OA_1 \sin\angle A_1OM = OA_1 \sin\frac{\psi}{2} = R\sin\frac{\psi}{2} \longleftarrow \qquad OA_1 = R$$

$$A_1M = R\sin\frac{\psi}{2}$$ (recalled)

$$A_1A_2 = E_0 \text{ say}$$

$$E_0 = A_1A_2 = 2A_1M = 2R\sin\frac{\psi}{2}$$

(electric field magnitude due to a single array element)

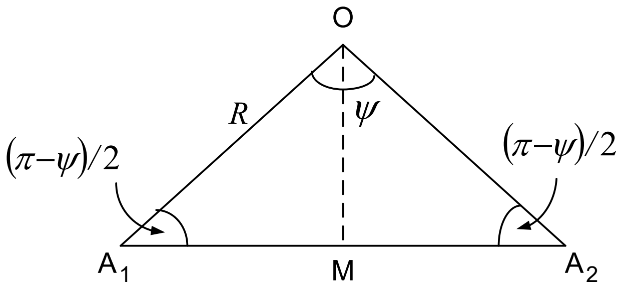
$$R = \frac{E_0}{2\sin\frac{\psi}{2}}$$

$$A_1N = OA_1 \sin\angle A_1ON = R\sin\frac{4\psi}{2}$$

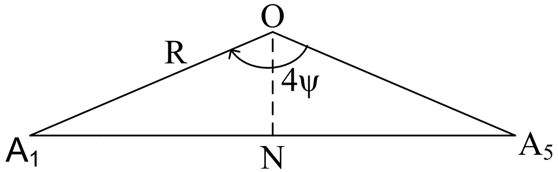
$$E_R = A_1A_5 = 2A_1N = 2R\sin\frac{4\psi}{2}$$

$$E_R = \frac{E_0}{\sin\frac{\psi}{2}}\sin\frac{4\psi}{2}$$

(resultant electric field typically due to 4 array elements)




(recalled)

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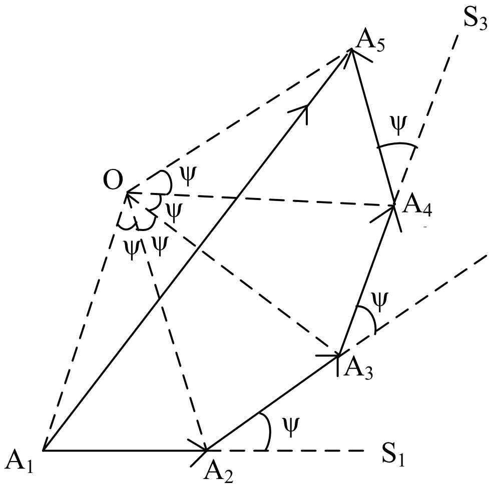



(recalled)

$$E_R = \frac{E_0}{\sin\frac{\psi}{2}} \sin\frac{4\psi}{2}$$ (rewritten)

(resultant electric field typically due to 4 array elements)

↓ ← Generalisation taking *N* elements

$$E_R = \frac{E_0}{\sin\frac{\psi}{2}} \sin\frac{N\psi}{2}$$ ← $$\mathrm{AF} = \frac{\sin\frac{N\psi}{2}}{\sin\frac{\psi}{2}}$$ (array factor defined as)

↓

$$E_R = (E_0)(\mathrm{AF})$$

↓

Product of the array factor with the electric field amplitude due an individual array element gives the electric field amplitude due to the array of all the elements put together.

$$(\mathrm{AF})_{\max} = N$$ (maximum array factor)

$$(\mathrm{AF})_{\mathrm{n}} = \frac{\mathrm{AF}}{(\mathrm{AF})_{\max}}$$ (normalised array factor defined as)

↓

$$(\mathrm{AF})_{\mathrm{n}} = \frac{\mathrm{AF}}{(\mathrm{AF})_{\max}} = \frac{1}{N}\,\frac{\sin\frac{N\psi}{2}}{\sin\frac{\psi}{2}}$$

$$(\mathrm{AF})_n = \frac{\mathrm{AF}}{(\mathrm{AF})_{max}} = \frac{1}{N}\frac{\sin\frac{N\psi}{2}}{\sin\frac{\psi}{2}}$$ (recalled)

$$E_R = (E_0)(\mathrm{AF})$$ (recalled)

For smaller values of the successive phase difference $\psi$ between the electric fields due to the elements, we may take $\sin(\psi/2) \approx \psi/2$

$$\frac{E_R}{E_0} = \mathrm{AF}$$

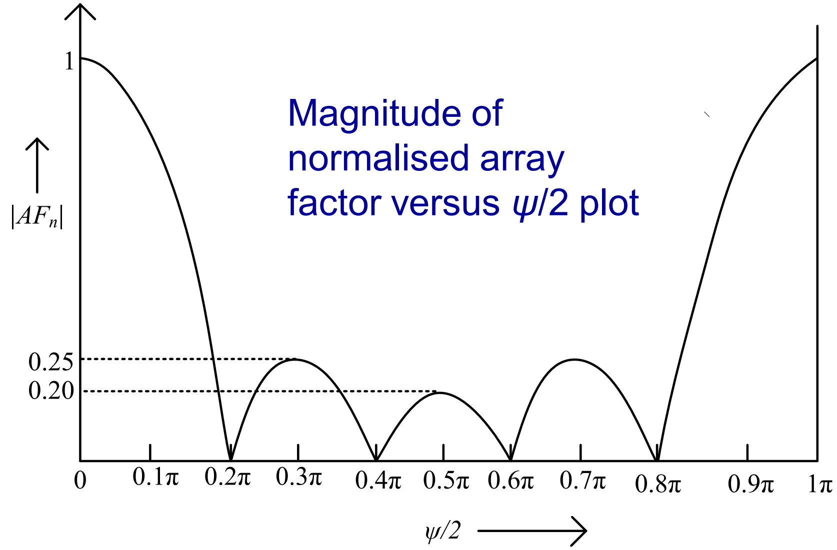
$$(\mathrm{AF})_n \cong \frac{\sin\frac{N\psi}{2}}{\frac{N\psi}{2}}$$

$$\frac{\frac{E_R}{E_0}}{(\frac{E_R}{E_0})_{max}} = \frac{\mathrm{AF}}{(\mathrm{AF})_{max}} = (\mathrm{AF})_n$$

$$\frac{\frac{E_R}{E_0}}{(\frac{E_R}{E_0})_{max}} = \frac{\mathrm{AF}}{(\mathrm{AF})_{max}} = (\mathrm{AF})_n = \frac{\sin\frac{N\psi}{2}}{\frac{N\psi}{2}}$$

$$\frac{\frac{E_R}{E_0}}{\left(\frac{E_R}{E_0}\right)_{\max}} = \frac{\mathrm{AF}}{(\mathrm{AF})_{\max}} = (\mathrm{AF})_{\mathrm{n}} = \frac{\sin\frac{N\psi}{2}}{\frac{N\psi}{2}}$$

(rewritten)



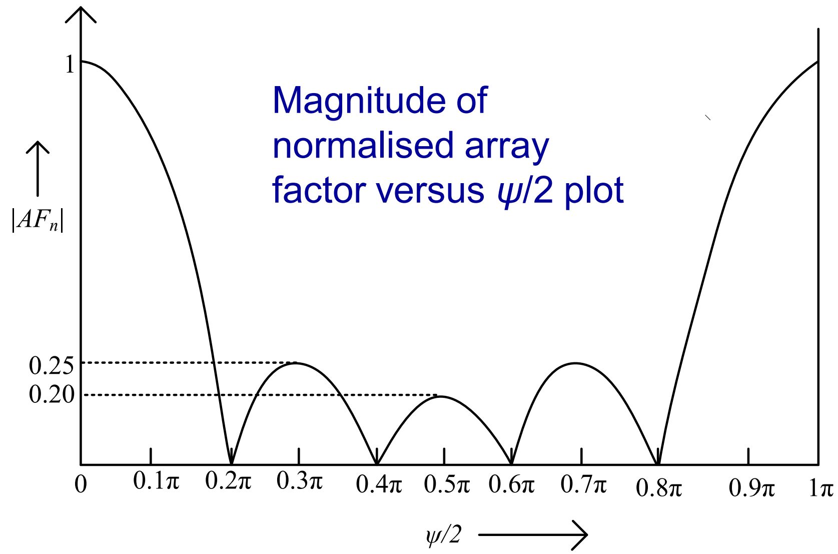

Magnitude of normalised array factor versus $\psi/2$ plot

The field pattern thus follows the normalised array factor pattern which follows the well known sin$x/x$ function (here $N\psi/2$ interpreted as $x$).

The minima of the function takes place at $N\psi/2 = \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots\ldots$ where the function becomes a null.

The principal maximum of the function occurs at $N\psi/2 = 0$ which also corresponds to $\psi = 0$.

The first negative maximum occurs at $N\psi/2 = 1.43\pi$ and the first positive maximum at $N\psi/2 = 2.46\pi$.




However, depending on the value of the number of elements, some of the minima of the normalised array factor predicted above at $N\psi/2 = \pi, 2\pi, 3\pi, 4\pi, 5\pi$, ……would be 'missing' being replaced by the principal maxima.

For example, if we take the number of elements of the array as $N = 5$, then the $N^{th}$ minimum, here, the fifth minimum predicted above as $N\psi/2 = 5\pi$ or $\psi/2 = 5\pi/N = 5\pi/5 = \pi$ will be missing only to be replaced by a principal maximum.

We can appreciate this by noting that, at $\psi/2 = \pi$, the value of the normalised array factor takes on the maximum value of unity [(1)($N$)/$N$ = 1], which corresponds to the principal maximum that occurs at $\psi/2 = 0$.

# Broadside array

Let us design the linear array to make it a broadside array such that we obtain a principal maximum in a direction normal to the line of the array, that is, at $\theta = \pi/2$.

$$\psi = \beta d \cos\theta + \xi$$

$$\theta = \frac{\pi}{2}$$

$$\psi = 0$$

(corresponding to principal maximum)
(broad-side array)

Broad-side array condition

$$0 = \beta d \cos\frac{\pi}{2} + \xi = (\beta d)(0) + \xi = \xi$$

Excitation phase difference $\xi$ between successive elements has to be nil for a broad-side array.

$$\psi = \beta d \cos\theta + \xi$$

$$\psi = \beta d \cos\theta \quad \text{(broad-side array)}$$

Phase difference between the electric fields due to successive array elements at a large distance from the broadside array

$\psi = \beta d \cos\theta$ (rewritten) $\longleftarrow$ $\psi = 2\pi / N$ $\longleftarrow$ $N\psi / 2 = \pi$ (first null)

(broad-side array) (recalled)

$\downarrow$

$$\frac{2\pi}{N} = \beta d \cos\theta$$ (first null of broadside array)

$\downarrow \longleftarrow \beta = \frac{2\pi}{\lambda}$

$$\frac{2\pi}{N} = \frac{2\pi}{\lambda} d \cos\theta \longrightarrow \cos\theta = \frac{\lambda}{Nd} \longrightarrow \theta = \cos^{-1}\frac{\lambda}{Nd}$$ (first null) (broadside array)

Since the nulls in the field pattern take place symmetrically on both sides of the principal maximum, we can find the first null bandwidth FNBW as the separation between the first nulls on both sides of the principal maxima as twice the angular separation between one of these first nulls and the principal maximum, the latter taking place at $\theta = \pi$, as follows:

$$\text{FNBW} = 2 \times \left(\frac{\pi}{2} - \cos^{-1}\frac{\lambda}{Nd}\right) = \pi - 2\cos^{-1}\frac{\lambda}{Nd}$$ (broadside array)

(expression for first null bandwidth FNBW of a broadside array)

We can find the half power bandwidth HPBW of the broadside array putting

$$(\mathrm{AF})_n = \frac{1}{\sqrt{2}}$$

← Value so assigned makes the square of the normalised quantity equal to ½ corresponding to half power bandwidth HPBW

$$(\mathrm{AF})_n = \frac{\sin \frac{N\psi}{2}}{\frac{N\psi}{2}}$$

$$\frac{\sin \frac{N\psi}{2}}{\frac{N\psi}{2}} = \frac{1}{\sqrt{2}} \longrightarrow \frac{N\psi}{2} = \frac{N(\beta d \cos\theta)}{2} = 1.39$$

$\psi = \beta d \cos\theta$ (broad-side array)

$$\beta = \frac{2\pi}{\lambda}$$

$$\frac{N\pi d \cos\theta}{\lambda} = 1.39$$ (broad-side array)

$$\theta = \cos^{-1} \frac{1.39\lambda}{N\pi d}$$

(half-power angular location)

← Using the same approach as followed to derive FNBW of the broadside array

$$\mathrm{HPBW} = 2 \times \left( \frac{\pi}{2} - \cos^{-1} \frac{1.39\lambda}{N\pi d} \right) = \pi - 2\cos^{-1} \frac{1.39\lambda}{N\pi d}$$ (broad-side array)

(expression for half power bandwidth HPBW of a broadside array)

# End-fire array

Let us design the linear array to make it the so-called end-fire array such that we obtain a principal maximum corresponding to $\psi$ = 0 along the line or axis of the array, that is, at either $\theta$ = 0 or $\theta = \pi$.

$$\psi = \beta d \cos\theta + \xi$$

$\theta = 0$ or $\pi$

$\psi = 0$ (corresponding to principal maximum)

← End-fire array condition

$$\xi = -\beta d \text{ or } \beta d$$

$$\psi = \beta d \cos\theta - \beta d = \beta d(\cos\theta - 1)$$

or

$$\psi = \beta d \cos\theta + \beta d = \beta d(\cos\theta + 1)$$

(end-fire array)

$$\psi = \beta d \cos\theta - \beta d = \beta d(\cos\theta - 1)$$

Corresponds to principal maximum at $\theta = 0$ at the axis of the array of the end-fire array

(recalled)

$$N\psi / 2 = \pi \quad (\text{first null})$$

$$N\psi = N(\beta d \cos\theta - \beta d) = N\beta d(\cos\theta - 1) = 2\pi$$

$$\beta = \frac{2\pi}{\lambda}$$

$$\cos\theta = 1 + \frac{\lambda}{Nd}$$

The solution is however inadmissible since $\cos\theta$ cannot be greater than 1.

Therefore, let us go for an alternative solution leading to admissible $\cos\theta$ less than 1.

$$\psi = \beta d\cos\theta - \beta d = \beta d(\cos\theta - 1)$$

Corresponds to principal maximum at $\theta = 0$ at the axis of the array of the end-fire array

(recalled)

$$N\psi/2 = -\pi \text{ instead of } N\psi/2 = \pi \text{ (first null)}$$

$$N\psi = N(\beta d\cos\theta - \beta d) = N\beta d(\cos\theta - 1) = -2\pi$$

$$\beta = \frac{2\pi}{\lambda}$$

$$\cos\theta = 1 - \frac{\lambda}{Nd}$$

(first null) (end-fire array)

The solution is now admissible since $\cos\theta$ comes out to be less than 1.

Since the first nulls would occur on both sides of the principal maximum at $\theta = 0$ of the end-fire array, the first null bandwidth FNBW of the end-fire array is obtained as twice the value of

$$\text{FNBW}\,(= 2\theta) = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right)$$

(expression for first-null bandwidth FNBW of an end-fire array)

We can next find HPBW of the end-fire array following the same procedure as used in for the broadside array. However, for a valid solution now, instead of $N\psi/2 = 1.39$, we have to take $N\psi/2 = -1.39$. that is, $\psi = -2\times 1.39/N$ as follows:

$$N\psi/2 = -1.39 \longrightarrow \psi = -\frac{2\times 1.39}{N}$$

$$\downarrow$$

$$\psi = \beta d\cos\theta - \beta d = \beta d(\cos\theta - 1)$$ (recalled)

(corresponding to principal maximum at $\theta = 0$ at the axis of the array of the end-fire array)

$$\downarrow$$

$$\beta d(\cos\theta - 1) = -\frac{2\times 1.39}{N}$$ (end-fire array)

$$\longleftarrow$$

$$\cos\theta = 1 - \frac{1.39\lambda}{N\pi d}$$

$$\downarrow$$

$$\theta = \cos^{-1}\left(1 - \frac{1.39\lambda}{N\pi d}\right)$$ (half-power angular location)

$$\downarrow$$

$$\text{HPBW} = 2\theta = 2\cos^{-1}\left(1 - \frac{1.39\lambda}{N\pi d}\right)$$ (half-power points being located on both sides of the principal maximum at $\theta = 0$ of the end-fire array)

(end-fire array)

## Phased array scanning for electronic steering of the antenna beam

We have seen how the excitation phase difference between successive elements $\xi$ needs to be different for the broadside ( $\xi = 0$) and end-fire ($\xi = \mp\beta d$ ) arrays that makes us obtain a principal maximum in a direction normal to and along the axis of the array respectively. Therefore, by controlling the progressive phase excitation $\xi$ between the elements of the array, we can steer the beam of power of an antenna electronically rather than mechanically (the latter by physically rotating the antenna). This is the principle of the phased-array scanning for electronic steering of the beam of antenna power (which can be implemented by controlling the value of $\xi$ by adjusting the bias voltage of a hybrid-coupled varactor or the current of the coil wrapping a ferrite phase shifter).

## *Yagi-Uda array*

Yagi-Uda also known as fish-bone antenna is a highly directive antenna

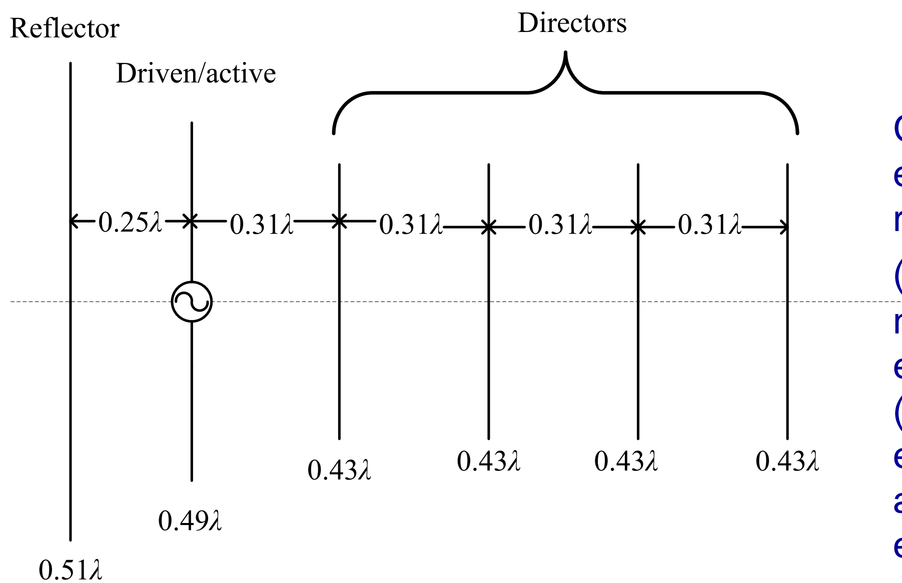
↓

Transmits or receives a highly directive beam of power in the direction of the axis of the array (as in an end-fire array) of parallel dipole elements

Yagi-Uda (fish-bone) antenna

↓

Consists of a driven element (active), a reflector element (passive) and a number of director elements (passive) (thus all the elements not being active unlike in an end-fire array)



**Some basic features**

- The thickness of each array is significantly less than a wavelength.
- Only one of the elements is active or driven and the remaining elements are all non-excited, passive or parasitic (unlike in conventional arrays like the end-fire array).
- One of the passive elements called the reflector is positioned along the array axis in the direction opposite to the direction in which the power is to be directed.
- The passive elements positioned along the array axis in in the direction in which power is to be directed are called the directors.
- Power is received by the driven element in the receiving mode and delivered to the load.
- The length of the driven element is slightly less than a half wavelength.
- The reflector element is slightly longer than the driven element.
- The director elements are shorter than the driven element.

<u>A typical simple example of a six-element Yagi-Uda array</u>: See the accompanying figure (on the preceding slide) for the dimensions with respect to the lengths of the elements and the distance between the elements.

The reported directivity of the this Yagi-Uda array is 7.54 (that is 8.77 dB) as compared to that of a half-wave dipole 1.64 (that is 2.15 dB).

# Summarising Notes

√ Expressions for energy and energy density in electrostatic field has been derived in terms of the electric field and electric displacement (or electric flux density).

√ Expressions for energy and energy density in magnetostatic field analogous to the corresponding expression for energy density in electrostatic field has been appreciated.

√ Expression for energy stored in a capacitor in terms of the capacitance of the capacitor and the charge of the capacitor or, alternatively, voltage across it has been obtained using the expression for energy density in electric field.

√ With reference to a parallel-plate capacitor, the expression for energy density stored in electric field in terms of the electric field and electric displacement or electric flux density has been found to be valid.

√ With reference to a solenoid, the expression for energy density stored in magnetic field in terms of the magnetic field and magnetic flux density has been found to be valid.

√ Poynting vector (power density vector) has been introduced.

√ Poynting theorem has been derived involving instantaneous Poynting vector.

√ Poynting theorem encapsulates the phenomenon of the storage, loss and flow of electromagnetic energy.

√ Poynting theorem has been used to appreciate Joule’s circuit law for the power loss in a wire of circular cross section and of finite resistance carrying a direct current.

√ Poynting theorem has been used to derive the expression for energy density in electric field with reference to the problem of a parallel-plate capacitor of circular cross section.

√ Poynting theorem has been applied to the problem of an inductor in the form of a solenoid of circular cross section and hence an expression for energy density in magnetic field has been derived.

√ Complex Poynting theorem gives the concept of time averaged electromagnetic power flow.

√ Complex form of the Poynting theorem has been derived that can be used to study time-averaged power flow through a bounded or an unbounded medium and associated power loss due to the presence of a lossy conducting medium.

√ Complex Poynting vector has been identified as half the cross product of the electric field vector and the complex conjugate of magnetic field vector.

√ Time averaged complex Poynting vector has been identified as the real part of the complex Poynting vector.

√ Average power going out of a volume enclosure can be found as the outward flux of the time averaged complex Poynting vector through the enclosure.

√ Concept of reactive power flowing into a volume enclosure and its relevance to average energies stored in electric and magnetic fields in the volume has been developed.

√ Concepts of power flow developed finds extensive applications, for instance, in hollow-pipe waveguides (used for the transmission of power in the microwave frequency range) and antennas.

√ Expression for the power loss per unit area in a conductor in terms of the surface resistance and surface current density of the conductor has been deduced — an expression that has applications for instance in estimating the attenuation of power in a hollow-pipe waveguide.

√ Concepts of power flow have been exemplified in their application to conduction current antennas.

√ With the help of the fundamentals of Hertzian infinitesimal dipole, the antenna concepts have been developed such as

◊ directive gain

◊ power gain

◊ radiation resistance

◊ effective aperture area and

◊ Friis transmission equation.

√ Study on antennas has characterized not only the infinitesimal dipole but also practical antennas such as

◊ finite-length dipole

◊ broad-side antenna array

◊ end-fire antenna array and

◊ Yagi-Uda antenna.

*Readers are encouraged to go through Chapter 8 of the book for more topics and more worked-out examples and review questions.*