Engineering Electromagnetics Essentials

Chapter 7

Electromagnetic boundary conditions

Objective

Development of understanding of general electromagnetic boundary conditions and their interpretation for dielectric and conducting media forming the interfaces as well as for time-independent and time-dependent situations

Topics dealt with

Derivation of general boundary conditions in vector form at the interface between two physical media

Surface charge density and surface current density and their interpretation in conducting and dielectric media

Boundary conditions at dielectric-dielectric and conductor-dielectric interfaces

Reflection of electromagnetic waves from a good conductor

Reflection and refraction of electromagnetic waves at a dielectric-dielectric interface

Brewster's phenomenon

Total internal reflection

Refraction of current at the conductor-conductor interface

Background

Concepts of integral form of Maxwell's equations and relaxation time developed in Chapter 5

General electromagnetic boundary conditions

Field quantities, for both steady (time-independent) and time-varying (time-dependent) situations, will, in general, get modified when the medium is perturbed by the presence of another medium because of the abrupt change in the medium properties at the interface (common boundary) between the media. However, the field quantities would pass through a common set of electromagnetic boundary conditions at the interface or common boundary between the media.

$$\oint_S \vec{D}\cdot\vec{a}_n dS = \int_\tau \rho\, d\tau$$ (recalled)

(Maxwell's equation in integral form)

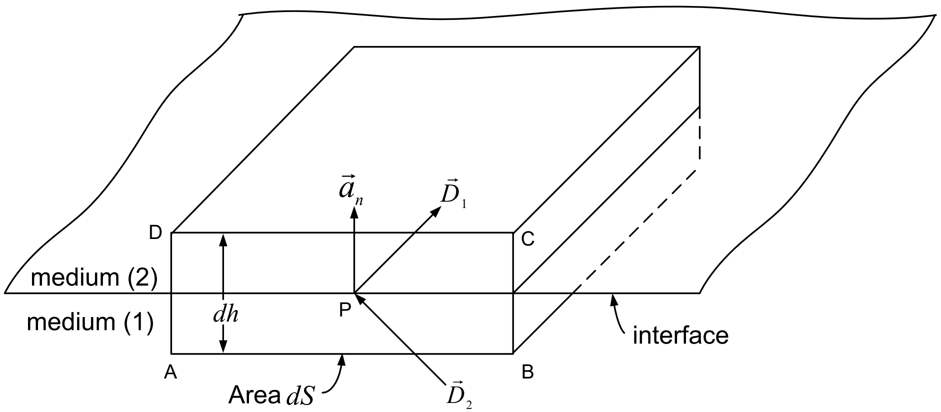
↓

Applied to volume element $d\tau$

↓

$$\vec{D}_2.\vec{a}_n dS + \vec{D}_1.(-\vec{a}_n)dS \cong \rho d\tau$$

(how to be explained later)



$d\tau$ — a parallelepiped volume element $d\tau$ in the form of a pill box of infinitesimal thickness *dh* and of infinitesimal area dS of each of its bottom and top faces, enclosing the point P where the boundary condition is sought

$\vec{a}_n$ = unit vector directed from region 1 to 2

$\vec{D}_1$ = electric displacement in region 1 at the point P on the interface

$\vec{D}_2$ = electric displacement in region 2 at the point P on the interface

dS = element of area on each of top and bottom faces of volume element

dh = infinitesimal thickness of volume element

$d\tau = dS\,dh$ = volume element

$$\vec{D}_2.\vec{a}_n dS + \vec{D}_1.(-\vec{a}_n) dS \cong \rho d\tau$$ (recalled)

It has been so obtained as mentioned before by applying Maxwell's equation to the element of volume enclosure $d\tau$. However, how it has been so obtained has not been explained. Let us deduce it.

$$\oint_S \vec{D} \cdot \vec{a}_n dS = \int_\tau \rho\, d\tau$$ (Maxwell's equation in integral form)

Contribution to the left hand side by the area element *dS* on top face $= \vec{D}_2.\vec{a}_n dS$

Contribution to the left hand side by the area element *dS* on bottom face $= (\vec{D}_2).(-\vec{a}_n) dS = -\vec{D}_2.\vec{a}_n dS$

(outward unit vector at the bottom face being downward being opposite to that at the top face)

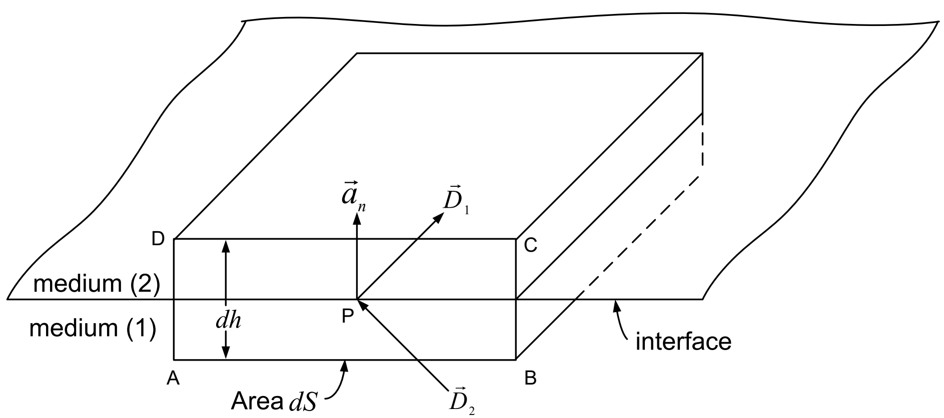
(taking the electric displacements to be constant over the area elements)

(ignoring the contribution of area elements on the side faces of volume element considering such area elements to be insignificant taking negligible infinitesimal thickness *dh* of the volume element and hence prompting us to use the approximate sign of equality)

We can remove the approximation from the sign of equality by taking *dh* tending to zero in the limit:

$$\vec{D}_2.\vec{a}_n dS + \vec{D}_1.(-\vec{a}_n) dS = \underset{dh \to 0}{Lt} \rho\, dS dh$$

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \underset{dh \to 0}{Lt} \rho\, dh$$



$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \underset{dh \to 0}{Lt} \rho \, dh$$

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_s$$

This is one of four general electromagnetic boundary conditions.

Let us define the surface charge density ρ_s as

$$\rho_s = \underset{dh \to 0}{Lt} \rho \, dh$$

(C/m^2)

(since ρ is in C/m^3 and *dh* is in m)

We obtained starting from

$$\oint_S \vec{D} \cdot \vec{a}_n dS = \int_\tau \rho \, d\tau$$ (Maxwell's equation in integral form) $\longrightarrow$ $(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_s$

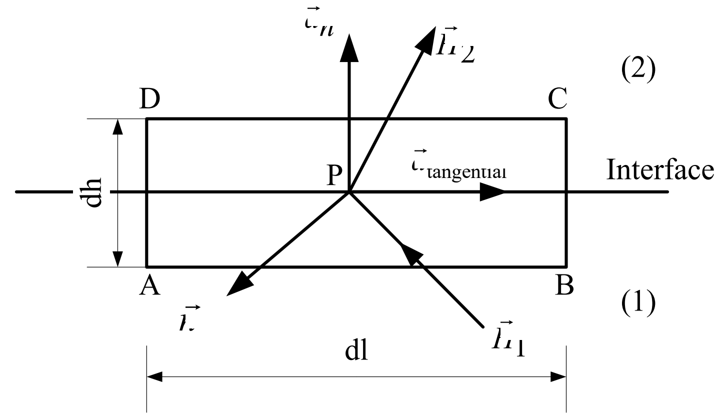
Similarly, following the same procedure as above, however now starting from another Maxwell's equation, we obtain

$$\oint_S \vec{B} \cdot d\vec{S} = \oint_S \vec{B} \cdot \vec{a}_n \, dS = 0$$ (Maxwell's equation in integral form) $\longrightarrow$ $(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$

free magnetic charge being absent or magnetic flux lines being continuous

This is another general electromagnetic boundary condition out of the four.

Let us next consider a rectangle element of infinitesimal length *dl*, infinitesimal thickness *dh* and area element *ds* = (*dh*)(*dl*) enclosing the point P on the interface between the medium 1 and medium 2 (where the boundary condition is sought) such that the bottom and top lengths of the rectangle lay in medium 1 and medium 2 respectively.



$$\vec{a}_n = \text{unit vector directed from medium 1 to medium 2}$$
$$\vec{n} = \text{unit vector normal to the area element } dS$$
$$\vec{a}_{\text{tangential}} = \text{unit vector tangential to the interface}$$

We take $\vec{n}$ such that it takes its direction as the direction of the linear motion that a screw would have if it were rotated following along the sequence of the closed line integral from A to B; B to C; C to D; and then back to A from D in the left hand side of Maxwell's equation:

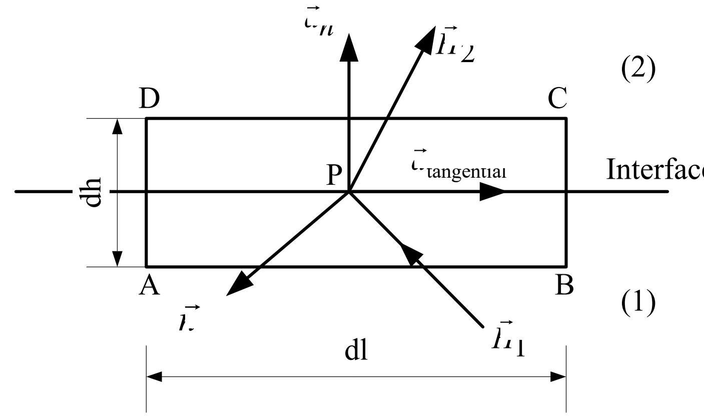
$$\oint_l \vec{H}.d\vec{l} = \int_S \vec{J}_{\text{total}}.\vec{n}\, dS \longleftarrow \oint_l \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{n}\, dS$$

(Maxwell's equation so chosen to be expressed)

$$\vec{J}_{\text{total}} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

(so defined for the sake of convenience)

$$\oint_l \vec{H}.d\vec{l} = \int_S \vec{J}_{\text{total}}.\vec{n}\,dS$$ (recalled)



Applied to the left hand side following the sequence from A to B; B to C; C to D; and then back to A from D

$$\vec{H}_1.\vec{a}_{\text{tangential}}\,dl + \vec{H}_2.(-\vec{a}_{\text{tangential}})\,dl \cong \vec{J}_{\text{total}}.\vec{n}\,dS$$

(ignoring the contribution of length elements on the side faces of area element considering such length elements to be insignificant taking negligible infinitesimal thickness *dh* of the area element and hence prompting us to use the approximate sign of equality)

Subscripts 1 and 2 refer to regions 1 and 2 respectively.

$$(\vec{H}_1 - \vec{H}_2).\vec{a}_{\text{tangential}}\,dl \cong \vec{J}_{\text{total}}.\vec{n}\,dS$$

$$dS = (dh)(dl)$$

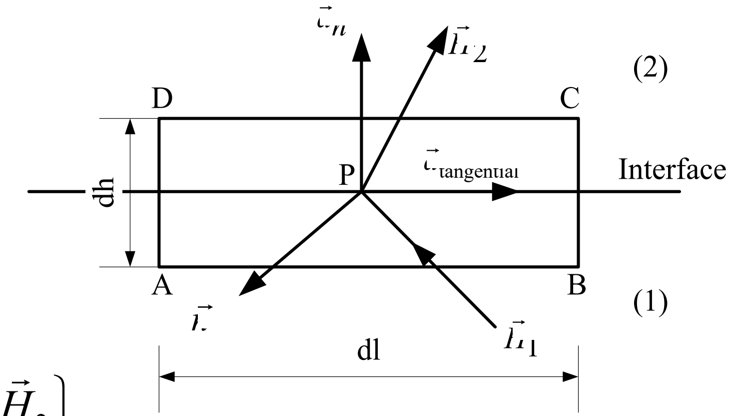
$$(\vec{H}_1 - \vec{H}_2).\vec{a}_{\text{tangential}} \cong \vec{J}_{\text{total}}.\vec{n}\,dh$$

The unit vectors satisfy the relation of cross product:

$$\vec{a}_n \times \vec{n} = \vec{a}_{\text{tangential}}$$

$$(\vec{H}_1 - \vec{H}_2).\vec{a}_n \times \vec{n} \cong \vec{J}_{\text{total}}.\vec{n}\,dh$$

$$(\vec{H}_1-\vec{H}_2).\vec{a}_n\times\vec{n}\cong\vec{J}_{\text{total}}.\vec{n}\,dh$$



$$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\left.\begin{array}{l}\vec{A}=\vec{H}_1-\vec{H}_2\\ \vec{B}=\vec{a}_n\\ \vec{C}=\vec{n}\end{array}\right\}$$

$$\vec{n}.(\vec{H}_1-\vec{H}_2)\times\vec{a}_n\cong\vec{J}_{\text{total}}.\vec{n}\,dh$$

$$\vec{n}.(\vec{H}_1-\vec{H}_2)\times\vec{a}_n\cong\vec{n}.\vec{J}_{\text{total}}\,dh$$

$$\vec{n}.[(\vec{H}_1-\vec{H}_2)\times\vec{a}_n-\vec{J}_{\text{total}}dh]\cong 0 \longleftarrow \vec{G}=(\vec{H}_1-\vec{H}_2)\times\vec{a}_n-\vec{J}_{\text{total}}dh,\text{ say}$$

$$\vec{n}.\vec{G}\cong 0$$

The orientation of the area element ABCD is arbitrary. That makes the value of θ and hence that of $\cos\theta$ arbitrary

$$\vec{n}\cdot\vec{G}=(1)(G)\cos\theta=G\cos\theta\cong 0 \longrightarrow G\cong 0$$

θ is the angle between $\vec{n}$ and $\vec{G}$

$$G \cong 0 \longleftarrow \vec{G} = (\vec{H}_1 - \vec{H}_2)\times\vec{a}_n - \vec{J}_{\text{total}}dh$$

$$(\vec{H}_1 - \vec{H}_2)\times\vec{a}_n - \vec{J}_{\text{total}}dh \cong 0$$

$$(\vec{H}_1 - \vec{H}_2)\times\vec{a}_n \cong \vec{J}_{\text{total}}dh \longleftarrow \vec{J}_{\text{total}} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$(\vec{H}_1 - \vec{H}_2)\times\vec{a}_n \cong \vec{J}dh + \frac{\partial\vec{D}}{\partial t}dh$$

We can remove the approximation from the sign of equality by taking *dh* tending to zero in the limit (that ensures nil contribution to the evaluation in the surface integral of Maxwell's equation):

$$(\vec{H}_1 - \vec{H}_2)\times\vec{a}_n = \underset{dh\to 0}{Lt}\,\vec{J}dh + \underset{dh\to 0}{Lt}\,\frac{\partial\vec{D}}{\partial t}dh$$

$$\vec{J}_S = \underset{dh\to 0}{Lt}\,\vec{J}dh$$

The first term takes a finite value in the limit as $\vec{J}\to\infty$ with $dh\to 0$

$$\underset{dh\to 0}{Lt}\,\frac{\partial\vec{D}}{\partial t}dh = 0$$

The second term becomes nil in the limit $dh\to 0$ as $\vec{D}$ is finite and hence $\partial\vec{D}/\partial t$ is finite

$$(\vec{H}_1 - \vec{H}_2)\times\vec{a}_n = (\vec{J}_S) + (0)$$

$$(\vec{H}_1 - \vec{H}_2)\times\vec{a}_n = \vec{J}_S$$

$$(\vec{H}_1 - \vec{H}_2) \times \vec{a}_n = \vec{J}_S \longleftarrow \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} = (\vec{B}) \times (-\vec{A}) \longleftarrow \left.\begin{array}{l} \vec{A} = \vec{H}_1 - \vec{H}_2 \\ \vec{B} = \vec{a}_n \end{array}\right\}$$

$$\downarrow$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$$

This is yet another general electromagnetic boundary condition out of the four.

Let us recall the following:

$$\oint_l \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{a}_n \, dS$$

(Maxwell's equation in integral form)

Appling to the left hand side following the sequence from A to B; B to C; C to D; and then back to A from D

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = \underset{dh \to 0}{Lt} \left(-\frac{\partial \vec{B}}{\partial t} dh \right)$$

$$\downarrow$$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

Following the same approach, the right hand side becomes nil in the limit since $dh \to 0$ as $\vec{B}$ is finite and hence $\partial \vec{B} / \partial t$ is finite.

Let us recall the following:

$$\oint_l \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{a}_n \, dS$$

(Maxwell's equation in integral form)

Appling to the left hand side following the sequence from A to B; B to C; C to D; and then back to A from D

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \underset{dh \to 0}{Lt} \vec{J} dh$$

$$\downarrow$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_S$$

This is the fourth of four general electromagnetic boundary conditions.

Maxwell's equation in integral form		Electromagnetic boundary condition
$\oint_S \vec{D}\cdot\vec{a}_n dS = \int_\tau \rho\, d\tau$	$\longrightarrow$	$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_s$
$\oint_S \vec{B}\cdot d\vec{S} = \int_S \vec{B}\cdot\vec{a}_n\, dS = 0$	$\longrightarrow$	$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$
$\oint_l \vec{H}\cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot\vec{a}_n\, dS$	$\longrightarrow$	$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_S$
$\oint_l \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\vec{a}_n\, dS$	$\longrightarrow$	$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$

Electromagnetic boundary condition	Meaning	At a point on the interface between two media
$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_s$	$\longrightarrow$	normal component of electric displacement is discontinuous, the amount of discontinuity being equal to the surface charge density
$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$	$\longrightarrow$	normal component of magnetic flux density is continuous
$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_S$	$\longrightarrow$	tangential component of magnetic field is discontinuous, the amount of discontinuity being equal to the surface current density
$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$	$\longrightarrow$	tangential component of electric field is continuous

We can interpret general electromagnetic boundary conditions for dielectric-dielectric interface and conductor-dielectric/free-space interface. For this purpose, it is worth reviewing some of the basic behaviours of conductor and dielectric media with regard to relaxation time, existence of a free charge in the bulk of the media, surface resistance, surface current density, and electric field and magnetic in the media.

Dielectric	**Conductor**
Relaxation time of is very large	Relaxation time is very small
A charge can stay longer inside the bulk of a dielectric without appearing at its surface and a finite volume charge density can be established inside the bulk resulting in a zero surface charge density at the surface of a dielectric for both time-independent and time-dependent situations.	A charge inside the bulk of a good conductor decays very fast to appear with a large volume charge density concentrated over a thin layer on the surface of the conductor, resulting in a finite surface charge density on the conductor surface for both time-independent and time-dependent situations.

Continued

In continuation

Dielectric	**Conductor**
The electric field and the electric displacement can be established inside a dielectric for both time-independent and time-dependent situations.	The bulk of a conductor cannot be electrically charged, resulting in no electric field or electric displacement inside the conductor for both time-independent and time-dependent situations.
A finite magnetic field or magnetic flux density can be established inside a dielectric independently of electric field, for both time-independent and time-dependent situations.	A finite magnetic field or magnetic flux density can be established inside a conductor independent of electric field for time-independent situations. However, for time-dependent situations, the magnetic field or magnetic flux density is nil inside a conductor since it is coupled to the electric field which is nil inside the conductor for such situations.

Continued

In continuation

Dielectric	Conductor
No current flows through a dielectric and therefore the surface current density becomes zero at the dielectric surface for both time-independent and time-dependent situations.	A finite current can be made to flow through the bulk of a conductor for time-independent situations, resulting in zero surface current density. However, for time-dependent situations a large current density can be concentrated over a thin layer on the conductor surface resulting in a finite surface current density.

As mentioned earlier, the above behaviours of a dielectric and a conductor help in the interpretation of general electromagnetic boundary conditions for dielectric-dielectric interface and conductor-dielectric/free-space to be taken up next in our study, which is of practical relevance.

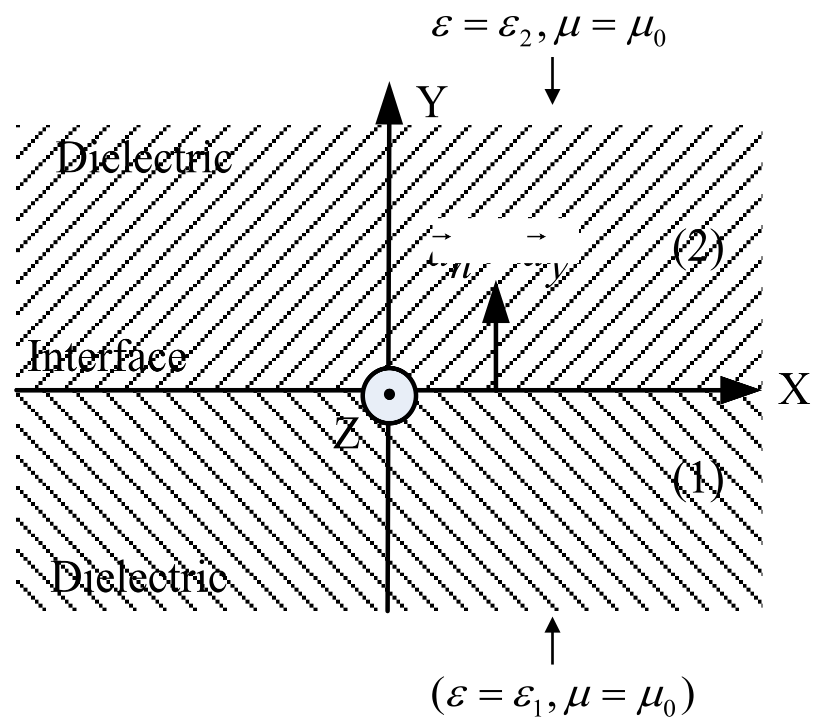
Electromagnetic boundary conditions at dielectric-dielectric interface

$$\rho \neq 0 \;\rightarrow\; \rho_s = \underset{dh \rightarrow 0}{Lt} \rho\, dh = 0$$

$$\vec{E}_{1,2} \neq 0, \vec{D}_{1,2} \neq 0$$

$$\vec{H}_{1,2} \neq 0, \vec{B}_{1,2} \neq 0$$

Subscripts 1 and 2 refer to quantities in region1 and 2 respectively



(mentioned earlier in a Table under the behaviour of a dielectric)

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_S$$

$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

(general electromagnetic boundary conditions)

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = 0$$

$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = 0$$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

(electromagnetic boundary conditions at dielectric-dielectric interface)

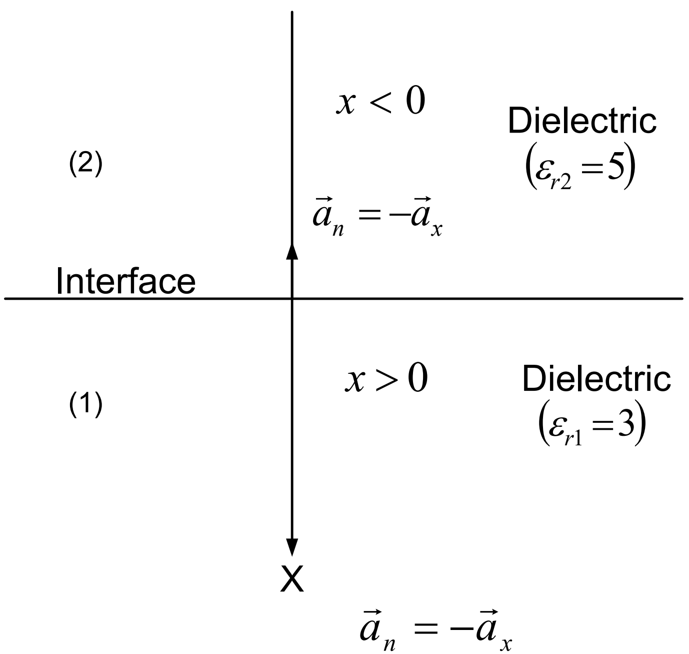
Let us illustrate the application of the boundary conditions at a dielectric-dielectric interface by taking up the problem of finding the electric displacements in region 1 (*x*>0) containing a dielectric of relative permittivity ε_{r1} = 3 and region 2 (*x*<0) containing another dielectric of relative ε_{r2} = 5, the two regions forming an interface at *x* = 0 if the electric field in region 2 is given as: $\vec{E}_2 = 40\vec{a}_x + 60\vec{a}_y - 80\vec{a}_z$ V/m.

$$\vec{E}_2 = 40\vec{a}_x + 60\vec{a}_y - 80\vec{a}_z \text{ V/m.}$$

(given)

$$\vec{D}_2 = \varepsilon_2\vec{E}_2 = \varepsilon_0\varepsilon_{r2}\vec{E}_2 = 5\varepsilon_0\vec{E}_2$$
$$= 5\varepsilon_0 \times (40\vec{a}_x + 60\vec{a}_y - 48\vec{a}_z) \text{ V/m}$$

$$\vec{D}_2 = (200\vec{a}_x + 300\vec{a}_y - 400\vec{a}_z)\varepsilon_0 \text{ C/m}^2$$



Let us recall the boundary condition

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = 0$$ (recalled)

$$[(D_{2x} - D_{1x})\vec{a}_x + (D_{2y} - D_{1y})\vec{a}_y + (D_{2z} - D_{1z})\vec{a}_z].[-\vec{a}_x] = 0$$

$$-(D_{2x} - D_{1x}) = 0$$

$$\vec{D}_2 = (200\vec{a}_x + 300\vec{a}_y - 400\vec{a}_z)\varepsilon_0 \text{ C/m}^2$$ (recalled)

$$D_{1x} = D_{2x} = 200\varepsilon_0 \text{ C/m}^2$$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$ (boundary condition recalled)

$$\downarrow$$

$$-\vec{a}_x \times [(E_{2x} - E_{1x})\vec{a}_x + (E_{2y} - E_{1y})\vec{a}_y + (E_{2z} - E_{1z})\vec{a}_z)] = 0$$

$$\downarrow$$

$$-(E_{2y} - E_{1y})\vec{a}_z + (E_{2z} - E_{1z})\vec{a}_y = 0$$

$$\downarrow$$

$$E_{1y} = E_{2y} \text{ and } E_{1z} = E_{2z}$$

$$\vec{E}_2 = 40\vec{a}_x + 60\vec{a}_y - 80\vec{a}_z \text{ V/m.}$$ (given)

$$\downarrow$$

$$E_{2y} = 60\,\text{V/m and } E_{2z} = -80\,\text{V/m}$$

$$\downarrow$$

$$E_{1y} = 60\,\text{V/m and } E_{1z} = -80\,\text{V/m}$$

$$\downarrow$$

$$D_{1y} = \varepsilon_1 E_{1y} = \varepsilon_0 \varepsilon_{r1} E_{1y} = (\varepsilon_0)(3)E_{1y} = (\varepsilon_0)(3)(60) = 180\varepsilon_0 \text{ C/m}^2$$

$$D_{1z} = \varepsilon_1 E_{1z} = \varepsilon_0 \varepsilon_{r1} E_{1z} = (\varepsilon_0)(3)E_{1z} = (\varepsilon_0)(3)(-80) = -240\varepsilon_0 \text{ C/m}^2 \longleftarrow E_{1z} = -80 \text{ V/m}$$ (recalled)

$$\downarrow$$

$$\vec{D}_1 = D_{1x}\vec{a}_x + D_{1y}\vec{a}_y - D_{1z}\vec{a}_z)\varepsilon_0 \longleftarrow$$

$$D_{1x} = D_{2x} = 200\varepsilon_0 \text{ C/m}^2$$ (recalled)

$$D_{1y} = 180\varepsilon_0 \text{ C/m}^2$$ (recalled)

$$\downarrow$$

$$\vec{D}_1 = (200\vec{a}_x + 180\vec{a}_y - 240\vec{a}_z)\varepsilon_0 \text{ C/m}^2$$

$$\vec{D}_2 = (200\vec{a}_x + 300\vec{a}_y - 400\vec{a}_z)\varepsilon_0 \text{ C/m}^2$$ (recalled)

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Take up another similar problem as an exercise to illustrate the application of boundary conditions at a dielectric-dielectric interface in which to find the electric field in region 2 (*y*>0) containing a dielectric of relative permittivity ε_{r2} separated at the interface (*y* = 0) from region 1 (*y*<0) containing another dielectric of relative permittivity ε_{r1} if the electric field in region 1 is given as: $\vec{E}_1 = l\vec{a}_x + m\vec{a}_y - n\vec{a}_z.$

The approach to getting the solution to the problem has already been elaborated in the preceding illustration. Some of the steps are provided as a hint as follows.

$$\vec{D}_1 = \varepsilon_1 \vec{E}_1 = \varepsilon_0 \varepsilon_{r1}(l\vec{a}_x + m\vec{a}_y + n\vec{a}_z) \longleftarrow \vec{E}_1 = l\vec{a}_x + m\vec{a}_y - n\vec{a}_z$$

(electric field in region 1) (given)

$$\downarrow$$

$$\left.\begin{aligned} D_{1x} &= \varepsilon_0 \varepsilon_{r1} l \\ D_{1y} &= \varepsilon_0 \varepsilon_{r1} m \\ D_{1z} &= \varepsilon_0 \varepsilon_{r1} n \end{aligned}\right\}$$

$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = 0$ (boundary condition) $\longleftarrow \vec{a}_n = \vec{a}_y$

$$\downarrow$$

$$[D_{2x}\vec{a}_x + D_{2y}\vec{a}_y + D_{2z}\vec{a}_z - (D_{1x}\vec{a}_x + D_{1y}\vec{a}_y + D_{1z}\vec{a}_z)]\cdot\vec{a}_y = 0$$

$$\left.\begin{aligned} \vec{a}_x \cdot \vec{a}_y &= 0 \\ \vec{a}_y \cdot \vec{a}_y &= 1 \\ \vec{a}_z \cdot \vec{a}_y &= 0 \end{aligned}\right\}$$

$$\downarrow$$

$$D_{2y} - D_{1y} = 0$$

$\vec{E}_1 = l\vec{a}_x + m\vec{a}_y - n\vec{a}_z$ (given)

$$\downarrow$$

$$E_{1y} = m \longrightarrow D_{2y} = D_{1y} = \varepsilon_1 E_{1y} = \varepsilon_0 \varepsilon_{r1} E_{1y} = \varepsilon_0 \varepsilon_{r1} m$$

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$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0 \longleftarrow \left.\begin{array}{l} \vec{a}_n = \vec{a}_y \\ \vec{E}_1 = l\vec{a}_x + m\vec{a}_y - n\vec{a}_z \end{array}\right\}$$

$$\downarrow$$

$$\vec{a}_y \times [(E_{2x}\vec{a}_x + E_{2y}\vec{a}_y + E_{2z}\vec{a}_z) - (l\vec{a}_x + m\vec{a}_y + n\vec{a}_z)] = 0 \longleftarrow \left.\begin{array}{l} \vec{a}_y \times \vec{a}_x = -\vec{a}_z \\ \vec{a}_y \times \vec{a}_y = 0 \\ \vec{a}_y \times \vec{a}_z = \vec{a}_x \end{array}\right\}$$

$$\downarrow$$

$$-(E_{2x} - l)\vec{a}_z + (E_{2z} - n)\vec{a}_x = 0$$

$$\downarrow$$

$$\left.\begin{array}{l} E_{2x} - l = 0 \\ E_{2z} - n = 0 \end{array}\right\}$$

$D_{2y} = \varepsilon_0 \varepsilon_{r1} m$ (recalled)

$$\downarrow$$

$$\left.\begin{array}{l} E_{2x} = l \\ E_{2z} = n \end{array}\right\} \qquad E_{2y} = \frac{D_{2y}}{\varepsilon_2} = \frac{D_{2y}}{\varepsilon_0 \varepsilon_{r2}} = \frac{\varepsilon_{r1} m}{\varepsilon_{r2}}$$

$$\vec{E}_1 = l\vec{a}_x + m\vec{a}_y - n\vec{a}_z$$

$$\vec{E}_2 = E_{2x}\vec{a}_x + E_{2y}\vec{a}_y + E_{2z}\vec{a}_z = l\,\vec{a}_x + m\frac{\varepsilon_{r1}}{\varepsilon_{r1}}\vec{a}_y + n\,\vec{a}_z$$

In the next illustration, take a medium divided in region 1 of relative permeability 60 (iron) and region 2 (free space) across the interface (*z* = 0) and hence find the magnetic flux density in region 2 if the magnetic flux in region 1 is $6\vec{a}_x + 12\vec{a}_y$ T.

$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$ (boundary condition)

Subscripts 1 and 2 refer to quantities in region1 and 2 respectively

$\vec{a}_n = \vec{a}_z$

$$\left.\begin{aligned} \vec{B}_1 &= 6\vec{a}_x + 12\vec{a}_y \text{ T (given)} \\ \vec{B}_2 &= B_{2x}\vec{a}_x + B_{2y}\vec{a}_y + B_{2z}\vec{a}_z \end{aligned}\right\}$$

$$\left.\begin{aligned} &\text{Region 1}: y \le 0 \\ &\text{Region 2}: y \ge 0 \end{aligned}\right\}$$

$$[(B_{2x}\vec{a}_x + B_{2y}\vec{a}_y + B_{2z}\vec{a}_z) - (6\vec{a}_x + 12\vec{a}_y)]\vec{a}_z = 0$$

$$B_{2z} = 0$$

$\vec{a}_n = \vec{a}_z$

$$\left.\begin{aligned} \vec{B}_1 &= \mu_1\vec{H}_1 \\ \vec{B}_2 &= \mu_2\vec{H}_2 \end{aligned}\right\}$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = 0$$

(boundary condition)

$$\vec{a}_z \times \left(\frac{B_{2x}\vec{a}_x + B_{2y}\vec{a}_y + B_{2z}\vec{a}_z}{\mu_2} - \frac{6\vec{a}_x + 12\vec{a}_y}{\mu_1} \right) = 0$$

$$\left(\frac{B_{2x}}{\mu_2} - \frac{6}{\mu_1} \right)\vec{a}_y - \left(\frac{B_{2y}}{\mu_2} - \frac{12}{\mu_1} \right)\vec{a}_x = 0$$

$$\left(\frac{B_{2x}}{\mu_2}-\frac{6}{\mu_1}\right)\vec{a}_y-\left(\frac{B_{2y}}{\mu_2}-\frac{12}{\mu_1}\right)\vec{a}_x=0$$

$$\downarrow$$

$$\left.\begin{aligned} B_{2x}&=6\frac{\mu_2}{\mu_1}\\ B_{2y}&=12\frac{\mu_2}{\mu_1}\\ B_{2z}&=0\ \text{(recalled)}\end{aligned}\right\}\longrightarrow \vec{B}_2=B_{2x}\vec{a}_x+B_{2y}\vec{a}_y+B_{2z}\vec{a}_z$$

$$\downarrow$$

$$\left.\begin{aligned}\mu_{r1}&=60\ \text{(iron)}\\ \mu_{r2}&=1\ \text{(freespace)}\end{aligned}\right\}$$ (given)

$$\downarrow$$

$$\left.\begin{aligned}\mu_1&=\mu_0\mu_{r1}=60\mu_0\\ \mu_2&=\mu_0\mu_{r2}=\mu_0\end{aligned}\right\}$$

$$\vec{B}_2=6\frac{\mu_2}{\mu_1}\vec{a}_x+12\frac{\mu_2}{\mu_1}\vec{a}_y$$

$$\downarrow$$

$$\vec{B}_2=0.1\vec{a}_x+1.2\vec{a}_y\ \text{T}$$

Electromagnetic boundary conditions at conductor-dielectric interface

$$\vec{J}_S = \underset{dh \to 0}{Lt} \vec{J} dh$$

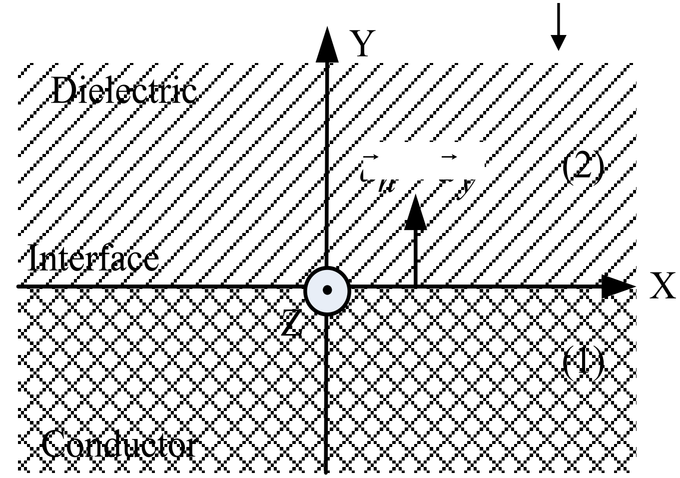
$$\vec{E}_1 = 0, \vec{D}_1 = 0$$

$$\vec{H}_1 \neq 0,\ \vec{B}_1 \neq 0 \ \text{(for time - independent situations)}$$

$$\vec{H}_1 = \vec{B}_1 = 0 \ \text{(for time - dependent situations)}$$

(mentioned earlier in a Table under the behaviour of a conductor)

$\varepsilon = \varepsilon_2, \mu = \mu_0$



$\varepsilon = \varepsilon_0, \mu = \mu_0$

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_S$$

$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

(general electromagnetic boundary conditions)

$$\vec{D}_2.\vec{a}_n = \rho_S$$

$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = 0$$

$$\vec{a}_n \times \vec{E}_2 = 0$$

(electromagnetic boundary conditions for time-independent situations)

$$\vec{D}_2.\vec{a}_n = \rho_S$$

$$\vec{B}_2.\vec{a}_n = 0$$

$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$$

$$\vec{a}_n \times \vec{E}_2 = 0$$

(electromagnetic boundary conditions for time-dependent situations)

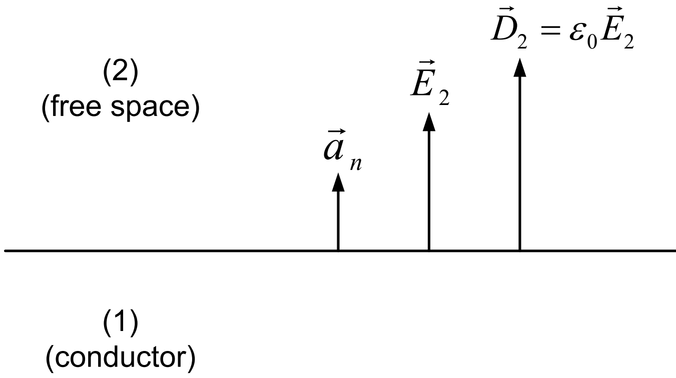
In order to illustrate electromagnetic boundary condition at conductor-dielectric interface, let us take up the problem of finding the surface charge density developed on the surface of a good conductor placed in free space where the electric field is given as: $4\vec{a}_x - 3\vec{a}_z$ V/m.

$$\vec{E}_2 = E_2 \vec{a}_n \longleftarrow \vec{a}_n \times \vec{E}_2 = 0$$ (boundary condition) (recalled)

$$\vec{D}_2 = \varepsilon_0 E_2 \vec{a}_n$$

(remembering the nomenclature that the subscript 2 refers to region 2, here free-space region)

$\vec{D}_2 \cdot \vec{a}_n = \rho_s$ (electromagnetic boundary condition recalled)



$$\varepsilon_0 E_2 \vec{a}_n \cdot \vec{a}_n = \rho_s$$

$\vec{E}_2 = 4\vec{a}_x - 3\vec{a}_z$ V/m (given)

$$\rho_s = \varepsilon_0 E_2 \longleftarrow E_2 = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5 \text{ V/m}$$

$$\vec{E}_2 = \vec{E} = 4\vec{a}_x - 3\vec{a}_z \text{ V/m}$$

$$\rho_s = 5\varepsilon_0 \text{ C/m}^2$$

In another illustration let us find the surface charge density developed on the surface of a conducting plate of a parallel-plate capacitor in terms of the potential difference between the plates and hence derive an expression for the capacitance of the capacitor in terms of the dimensions and permittivity of the dielectric filling the region between the plates of the capacitor.

The problem enjoys rectangular symmetry, and for large plates perpendicular to z, we may treat the problem as one-dimensional considering the potential to vary only along z:

$(\partial/\partial x = \partial/\partial y = 0; \partial/\partial z \neq 0)$.

Let the plates be located at $z = 0$ and $z = d$ respectively and let the potential of the plate at $z = 0$ be raised to potential $V = V_0$ with respect to the potential $V = 0$ of the other plate at $z = d$.

$\vec{E} = \frac{V_0}{d}\vec{a}_z$ (electric field in the region between the plates)

(d = distance between plates)

$\vec{D} = \varepsilon \frac{V_0}{d}\vec{a}_z$ (electric displacement in the region between the plates)

$$\vec{D}\cdot\vec{a}_n = \vec{D}\cdot\vec{a}_z = \rho_s$$

$\vec{D}_2 \cdot \vec{a}_n = \rho_s$ (electromagnetic boundary condition recalled)

$\rho_s = \vec{D}\cdot\vec{a}_z = \varepsilon\frac{V_0}{d}\vec{a}_z\cdot\vec{a}_z = \frac{\varepsilon V_0}{d}$ (charge density on the plate)

$$\text{Charge on the plate} = \rho_s A = \frac{\varepsilon V_0}{d} A \longleftarrow \rho_s = \frac{\varepsilon V_0}{d}$$ (charge density on the plate) (recalled)

$$\text{Area of the plate} = A$$

$$C = \frac{\text{Charge on the plate}}{V_0} = \frac{\rho_s A}{V_0} = \frac{\frac{\varepsilon V_0}{d} A}{V_0} = \frac{\varepsilon A}{d}$$

(expression for the capacitance of a parallel-plate capacitor)

We can appreciate from the above expression for capacitance that the practical unit of permittivity ε is F/m since the units of capacitance *C*, area *A* and distance *d* between plates are F, m^2 and m respectively.

In an illustration similar to the above on finding the capacitance of a parallel-plate capacitor, let us now find the surface charge density developed on the conducting surface of a long cylindrical capacitor comprising two coaxial long conducting cylinders in terms of the potential difference between the conducting cylinders and hence derive an expression for the capacitance per unit length of the capacitor in terms of the inner and outer radii of the capacitor and the permittivity of the dielectric filling the region between the conducting cylinders of the capacitor. The problem is similar to finding the capacitance per unit length of a coaxial cable.

The problem enjoys cylindrical symmetry, and for long coaxial conductors we may treat the problem as one-dimensional, considering the potential to vary only along *r*:

$$\partial / \partial\theta = \partial / \partial z = 0; \partial / \partial r \neq 0$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) = 0$$

(Laplace's equation holding good in the region between the inner and outer conductors)

In view of $\partial/\partial\theta = \partial/\partial z = 0; \partial/\partial r \neq 0$

$$\frac{\partial}{\partial r}\partial\left(r\frac{\partial V}{\partial r}\right)=0 \longrightarrow \frac{d}{dr}\left(r\frac{dV}{dr}\right)=0 \xrightarrow{\text{Integrating}} r\frac{dV}{dr} = C_1 \longrightarrow \frac{dV}{dr} = \frac{C_1}{r}$$

Integrating

C_1 and C_2 are integration constants

$$V = C_1 \ln r + C_2$$

$V = V_0$ at $r = a$

(potential at inner conductor)

$$V_0 = C_1 \ln a + C_2$$

$V = 0$ at $r = b$

$$0 = C_1 \ln b + C_2$$

On solving

$$\left.\begin{aligned} C_1 &= \frac{V_0}{\ln\frac{a}{b}} \\ C_2 &= -\frac{V_0}{\ln\frac{a}{b}}\ln b \end{aligned}\right\}$$

$$V = \frac{\ln\frac{r}{b}}{\ln\frac{a}{b}}V_0$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\vec{a}_r = -\frac{dV}{dr}\vec{a}_r \longleftarrow \quad V = \frac{\ln\frac{r}{b}}{\ln\frac{a}{b}}V_0$$ (recalled)

$$\downarrow$$

$$\vec{E} = \frac{V_0}{\ln\frac{b}{a}}\frac{1}{r}\vec{a}_r \longrightarrow \vec{D} = \varepsilon\vec{E} = \varepsilon\frac{V_0}{\ln\frac{b}{a}}\frac{1}{r}\vec{a}_r$$

$$\downarrow$$

$$\vec{D} = \varepsilon\frac{V_0}{\ln\frac{b}{a}}\frac{1}{a}\vec{a}_r$$ (electric displacement at the surface of the inner conductor *r* = *a*)

$$\downarrow$$

$$\rho_s = \vec{D}\cdot\vec{a}_n = \vec{D}\cdot\vec{a}_r = \varepsilon\frac{V_0}{\ln\frac{b}{a}}\frac{1}{a}\vec{a}_r\cdot\vec{a}_r = \varepsilon\frac{V_0}{\ln\frac{b}{a}}\frac{1}{a}$$

$$A = 2\pi al = \text{Area over the length } l \text{ of the inner conductor}$$

$$\text{Charge over the length } l \text{ of the inner conductor} = \rho_s A = \varepsilon\frac{V_0}{\ln\frac{b}{a}}\frac{1}{a}A = \varepsilon\frac{V_0}{\ln\frac{b}{a}}\frac{1}{a}2\pi al$$

$$\downarrow$$

$$C = \frac{\rho_s A}{V_0} = \frac{\varepsilon\frac{V_0}{\ln\frac{b}{a}}\frac{1}{a}}{V_0}2\pi al = \varepsilon\frac{1}{\ln\frac{b}{a}}\frac{1}{a}2\pi al = \frac{2\pi\varepsilon l}{\ln\frac{b}{a}}$$ (Capacitance over the length *l*)

Capacitance per unit length $= \dfrac{2\pi\varepsilon}{\ln\frac{b}{a}}$

In yet another illustration let us calculate the peak surface current density developed at the surface of a conducting medium forming an interface with a dielectric medium of relative permittivity 4 if the electric field in the dielectric medium at the conducting surface is given by

$$\vec{E} = 60\pi \sin \beta z \sin \omega t \, \vec{a}_x \text{ V/m.}$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$ (Maxwell's equation recalled)

$$\downarrow$$

$$\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\mu_0 \frac{\partial E_x}{\partial t}\vec{a}_x - \mu_0 \frac{\partial E_y}{\partial t}\vec{a}_y - \mu_0 \frac{\partial E_z}{\partial t}\vec{a}_z = -\mu_0 \left(\frac{\partial H_x}{\partial t}\vec{a}_x + \frac{\partial H_y}{\partial t}\vec{a}_y + \frac{\partial H_z}{\partial t}\vec{a}_z\right)$$

y-component

$$\downarrow$$

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t}$$

$$\vec{E} = 60\pi \sin\beta z \sin\omega t\, \vec{a}_x \text{ V/m.}$$

(given)

$$E_x = 60\pi \sin\beta z \sin\omega t$$

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial t} = -\frac{60\pi\beta \cos\beta z \sin\omega t}{\mu_0}$$

Integrating

$$H_y = -\frac{60\pi\beta \cos\beta z \sin\omega t}{\mu_0} \int (\sin\omega t)\, dt$$

$$= -\frac{60\pi\beta \cos\beta z}{\mu_0} \frac{-\cos\omega t}{\omega} + \text{constant}$$

Choosing the boundary phase such that $H_y = 0$ at $\omega t = \pi/2$ that makes constant = 0

$$H_y = \frac{60\pi \cos\beta z \cos\omega t}{\frac{\omega\mu_0}{\beta}}$$

$$\left.\begin{aligned} \mu_0 &= 4\pi \times 10^{-7} \text{ Henry/metre} \\ \varepsilon_0 &= \frac{1}{36\pi} \times 10^{-9} \text{ Farad / metre} \end{aligned}\right\}$$

$$\varepsilon_r = 4 \text{ (given)}$$

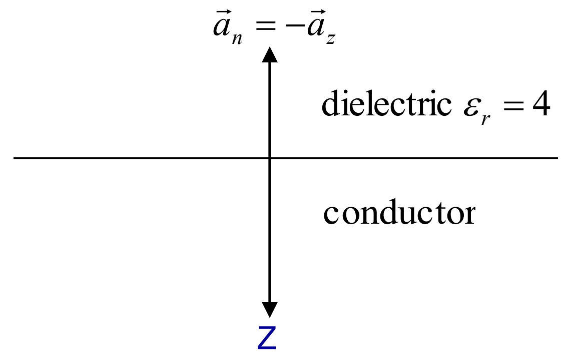
$$\frac{\omega\mu_0}{\beta} = \frac{\omega\mu_0}{\omega\sqrt{\mu_0\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}} = \sqrt{\frac{\mu_0}{4\varepsilon_0}}$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{2}(120) = 60\pi\ \Omega$$

$$H_y = \frac{60\pi \cos\beta z \cos\omega t}{60\pi} = \cos\beta z \cos\omega t$$

$H_y = \cos\beta z \cos\omega t$ (recalled)

$\vec{a}_n \times \vec{H}_2 = \vec{J}_S$ (boundary condition)



$$\downarrow$$

$$-\vec{a}_z \times H_y \vec{a}_y = \vec{J}_S$$

$$\downarrow \leftarrow \vec{a}_z \times \vec{a}_y = -\vec{a}_x$$

$$\vec{J}_S = H_y \vec{a}_x \longleftarrow H_y = \cos\beta z \cos\omega t \text{ (recalled)}$$

$$\downarrow$$

$$\vec{J}_S = \cos\beta z \cos\omega t \vec{a}_x$$

$$\downarrow \leftarrow z = 0$$

$\vec{J}_S = \cos\omega t \vec{a}_x$ (surface current density at conductor-dielectric interface at z = 0)

$$\downarrow$$

$$\vec{J}_S = \vec{a}_x \longrightarrow \text{Peak surface current density magnitude} = 1 \text{ A/m}^2$$

General electromagnetic boundary conditions

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_S \quad \longleftarrow \quad \rho_s = \underset{dh \to 0}{Lt}\,\rho\, dh$$

$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s \quad \longleftarrow \quad \vec{J}_S = \underset{dh \to 0}{Lt}\,\vec{J} dh$$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

Dielectric (1)-dielectric (2) interface

For both time-dependent and time-independent situations

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = 0$$

$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = 0$$

$$\vec{a}_n \times (\vec{E}_2 - \vec{E}_1) = 0$$

Conductor (1)-dielectric (2) interface

For time-independent situations

$$\vec{D}_2.\vec{a}_n = \rho_S$$

$$(\vec{B}_2 - \vec{B}_1).\vec{a}_n = 0$$

$$\vec{a}_n \times (\vec{H}_2 - \vec{H}_1) = 0$$

$$\vec{a}_n \times \vec{E}_2 = 0$$

For time-dependent situations

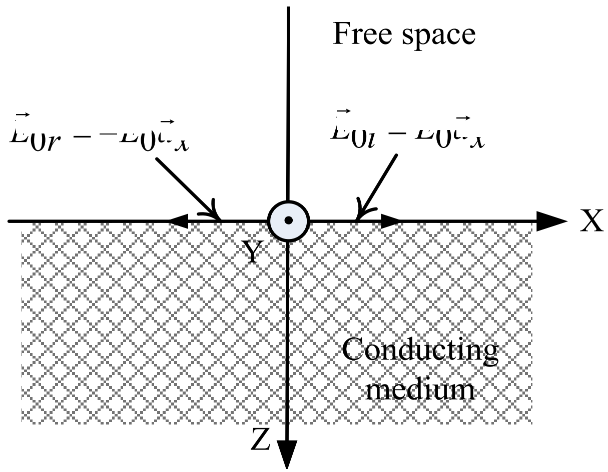
$$\vec{D}_2.\vec{a}_n = \rho_S$$

$$\vec{B}_2.\vec{a}_n = 0$$

$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$$

$$\vec{a}_n \times \vec{E}_2 = 0$$

Reflection from a good conductor



Let a uniform plane electromagnetic wave ($\partial / \partial x = \partial/\partial y = 0$) propagating in free space along positive z direction be incident on a conducting surface and reflected from the surface in the negative z direction. Field quantities in incident wave will vary as exp $j(\omega t - \beta z)$ and those in reflected wave as exp $j(\omega t + \beta z)$.

$$\frac{E_x}{H_y} = \eta_0 \text{ (for incident forward wave)}$$

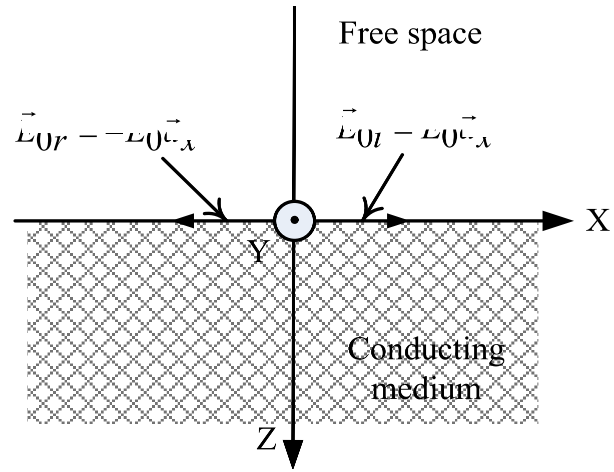
$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \text{free-space intrinsic impedance}$$

$$\frac{E_x}{H_y} = -\eta_0 \text{ (for reflected backward wave)}$$

(as explained in Chapter 6)

$$\vec{E}_{0i} + \vec{E}_{0r} = 0 \leftarrow \begin{cases} \vec{E}_{0i} = E_{0i}\vec{a}_x \\ \vec{E}_{0r} = E_{0r}\vec{a}_x \end{cases}$$

Subscript 0 represents the amplitude. Subscripts *i* and *r* refer to incident and reflected components.



Tangential component of electric field is continuous at the interface (boundary condition)

$$E_{0i}\vec{a}_x + E_{0r}\vec{a}_x = 0$$

$$\begin{Bmatrix} \vec{E}_{0i} = E_{0i}\vec{a}_x \\ \vec{E}_{0r} = E_{0r}\vec{a}_x \end{Bmatrix}$$ (recalled)

$$E_{0r}\vec{a}_x = -E_{0i}\vec{a}_x$$

$$\vec{E}\Big|_z = E_{0i}\exp j(\omega t - \beta z)\vec{a}_x + E_{0r}\exp j(\omega t + \beta z)\vec{a}_x$$

$$E_{0i} = E_0 \text{ (say)}$$

$$\vec{E}\Big|_z = E_{0i}\exp j(\omega t - \beta z)\vec{a}_x - E_{0i}\exp j(\omega t + \beta z)\vec{a}_x$$

[electric field in free space at *z* (perpendicular distance from the interface *z* = 0)]

$$\vec{E}\Big|_z = E_0[\exp j(\omega t - \beta z) - \exp j(\omega t + \beta z)]\vec{a}_x$$

$$\exp(\pm j\beta z) = \cos\beta z \pm j\sin\beta z$$

$$\vec{E}\Big|_z = -2jE_0 \sin\beta z \exp(j\omega t)\vec{a}_x$$

Similarly,

$$\vec{H}\Big|_z = \vec{H}_{0i}\exp j(\omega t - \beta z) + \vec{H}_{0r}\exp j(\omega t + \beta z)$$

[magnetic field in free space at *z* (perpendicular distance from the interface *z* = 0)]

$$\left\{\begin{array}{l}\vec{H}_{0i} = H_{0i}\vec{a}_y \\ \vec{H}_{0r} = H_{0r}\vec{a}_y\end{array}\right\}$$

(magnetic field directions along positive y conforming to incident and reflected waves propagating in positive and negative z directions respectively) (see Chapter 6)

$$\vec{H}\Big|_z = H_{0i}\vec{a}_y\exp j(\omega t - \beta z) + H_{0r}\vec{a}_y\exp j(\omega t + \beta z)$$
$$= [H_{0i}\exp j(\omega t - \beta z) + H_{0r}\exp j(\omega t + \beta z)]\vec{a}_y$$

$$\frac{E_x}{H_y} = \eta_0 \text{ (for incident forward wave)}$$

$$\frac{E_x}{H_y} = -\eta_0 \text{ (for reflected backward wave)}$$

$E_{0r}\vec{a}_x = -E_{0i}\vec{a}_x$ (recalled)

$$E_{0r} = -E_{0i} = -E_0$$

$$\vec{H}\Big|_z = \frac{E_0}{\eta_0}[\exp(-\beta z) + \exp(\beta z)]j\omega t\,\vec{a}_y$$

$$\exp(\pm j\beta z)$$
$$= \cos\beta z \pm j\sin\beta z$$

$$\vec{H}\Big|_z = \frac{2E_0}{\eta_0}\cos\beta z\exp(j\omega t)\,\vec{a}_y]$$

$$\vec{H}\Big|_z = \frac{2E_0}{\eta_0}\cos\beta z \exp(j\omega t)\ \vec{a}_y]$$

[magnetic field in free space at *z* (perpendicular distance from the interface *z* = 0)]

$$\vec{E}\Big|_z = -2jE_0 \sin\beta z \exp(j\omega t)\vec{a}_x$$

$$\exp(-j\pi/2) = \cos(\pi/2) - j\sin(\pi/2) = 0 - (j)(1) = -j$$

[electric field in free space at *z* (perpendicular distance from the interface *z* = 0)]

$$\vec{E}\Big|_z = 2E_0 \sin\beta z \exp j(\omega t - j\pi/2)\vec{a}_x$$

$$\vec{H}\Big|_z = \frac{2E_0}{\eta_0}\cos\beta z \exp(j\omega t)\ \vec{a}_y]$$

Electric field at z lags behind magnetic field in phase by $\pi/2$

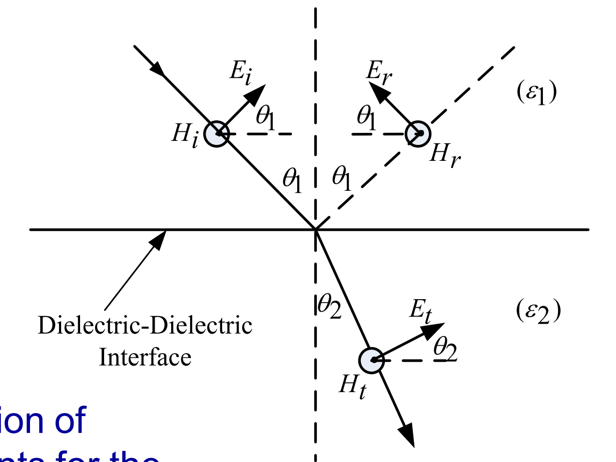
Absence of the factor exp $j(\omega t - \beta z)$ or exp $j(\omega t + \beta z)$ in electric and magnetic fields at z indicates absence of forward or backward wave and presence of standing wave when incident and reflected waves combine at z.

Presence of the factor sinβz in electric field amplitude and that of cosβz in magnetic field amplitude also indicate that the maxima of electric field coincide with the minima of magnetic field in the standing-wave pattern and vice versa.

Reflection and refraction at dielectric-dielectric interface

Parallel polarisation:

Electric field is parallel to the plane of incidence (that contains incident ray and normal to the medium interface) while magnetic field is perpendicular to this plane



$$E_i \cos\theta_1 - E_r \cos\theta_1 = E_t \cos\theta_2$$

$$H_i + H_r = H_t$$

Tangential electric and magnetic field components both are continuous at dielectric-dielectric interface (boundary condition.

On solving

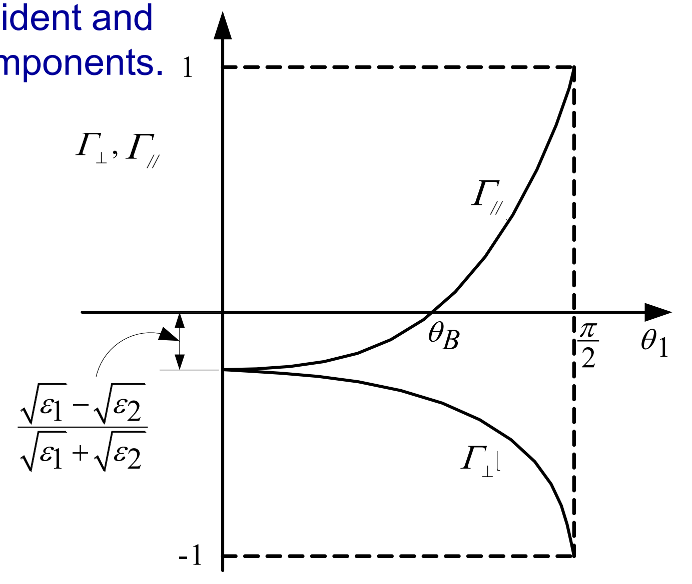
$$\frac{E_i}{\eta_1} + \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}} \qquad \eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}}$$

Negative sign in the definition of reflection coefficient accounts for the opposite directions of incident and reflected electric field components.

$$\Gamma_{//} = -\frac{E_r}{E_i} = \frac{\eta_2 \cos\theta_2 - \eta_1 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} = \frac{\sqrt{\varepsilon_1}\cos\theta_2 - \sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1}$$

(reflection coefficient for parallel polarisation)



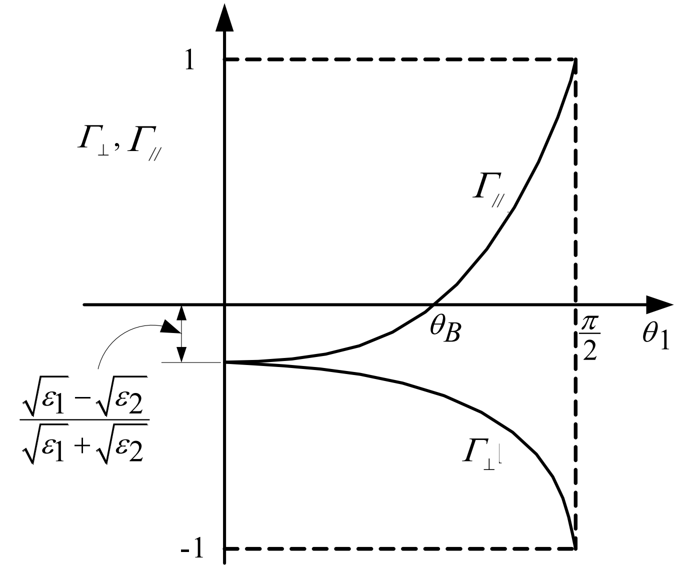
$$T_{//} = \frac{E_t}{E_i} = \frac{2\eta_2 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_2} = \frac{2\sqrt{\varepsilon_1}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1}$$

(transmission coefficient for parallel polarisation)

Brewster's law: Condition for $\Gamma_{//} = 0$

$$\Gamma_{//} = -\frac{E_r}{E_i} = \frac{\eta_2\cos\theta_2 - \eta_1\cos\theta_1}{\eta_1\cos\theta_1 + \eta_2\cos\theta_2} = \frac{\sqrt{\varepsilon_1}\cos\theta_2 - \sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1} = 0$$

$$(\Gamma_{//} = 0)$$



$$\downarrow$$

$$\sqrt{\varepsilon_1}\cos\theta_2 = \sqrt{\varepsilon_2}\cos\theta_1$$

On multiplication with

$$\sqrt{\varepsilon_1}\sin\theta_1 = \sqrt{\varepsilon_2}\sin\theta_2$$ (Snell's law)

$$\downarrow$$

$$\sqrt{\varepsilon_1}\sqrt{\varepsilon_2}\sin\theta_1\cos\theta_1 = \sqrt{\varepsilon_1}\sqrt{\varepsilon_2}\sin\theta_2\cos\theta_2$$

$$\downarrow$$

$$2\sin\theta_1\cos\theta_1 = 2\sin\theta_2\cos\theta_2$$

$$\downarrow$$

$$\sin 2\theta_1 = \sin 2\theta_2$$

$$\downarrow$$

$$2\theta_2 = \pi - 2\theta_1$$

$$\downarrow$$

$$\theta_2 = \frac{\pi}{2} - \theta_1$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$ (Snell's law)

$$\downarrow$$

$$\frac{\sin\theta_1}{\sin(\frac{\pi}{2} - \theta_1)} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\frac{\sin\theta_1}{\cos\theta_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\downarrow$$

$$\tan\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\downarrow$$

Angle of incidence corresponding to nil refection coefficient for parallel polarisation $(\Gamma_{//} = 0)$ $\rightarrow$

$$\theta_1 = \theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

(Brewster's angle)

Calculate Brewster's angle for glass of refractive index of 1.5.

$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \longleftarrow \left.\begin{aligned} n_1 &= \frac{c}{v_{\text{ph},1}} \\ n_2 &= \frac{c}{v_{\text{ph},2}} \end{aligned}\right\} \longleftarrow \left.\begin{aligned} v_{\text{ph},1} &= \frac{1}{\sqrt{\mu_0 \varepsilon_1}} \\ v_{\text{ph},2} &= \frac{1}{\sqrt{\mu_0 \varepsilon_2}} \end{aligned}\right\}$$

(refractive indices of media 1 and 2)

(phase velocities of media 1 and 2)

$$\theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

(recalled)

(Brewster's angle)

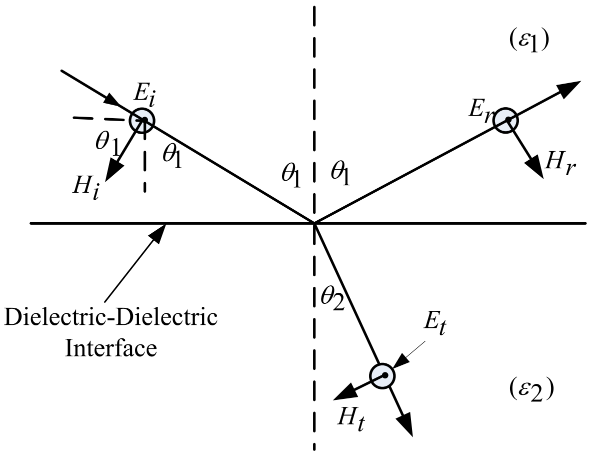
Incidence from free-space medium to glass

$$\theta_B = \tan^{-1}\frac{n_2}{n_1} \longleftarrow n_2 = 1.5\,(\text{given});\ n_1 = 1\,(\text{free space})$$

$$\theta_B = \tan^{-1} 1.5 = 56.3^0$$

Perpendicular polarization:

Electric field is perpendicular to the plane of incidence (that contains incident ray and normal to the medium interface) while magnetic field is parallel to this plane.



Tangential electric and magnetic field components both are continuous at dielectric-dielectric interface (boundary condition.

$$E_i + E_r = E_t$$

$$H_i \cos\theta_1 - H_r \cos\theta_1 = H_t \cos\theta_2$$

$$\downarrow$$

$$\frac{E_i}{\eta_1}\cos\theta_1 - \frac{E_r}{\eta_1}\cos\theta_1 = \frac{E_t}{\eta_2}\cos\theta_2$$

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}} \qquad \eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}}$$

$$\downarrow$$

$$E_i - E_r = \frac{\cos\theta_2}{\cos\theta_1}\frac{\eta_1}{\eta_2}E_t$$

Positive sign in the definition of reflection coefficient accounts for the same directions of incident and reflected electric field components unlike in parallel polarisation.

$$\Gamma_\perp = \frac{E_r}{E_i} = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2} = \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2}\cos\theta_2}$$

(reflection coefficient for perpendicular polarisation)

$$T_\perp = \frac{E_t}{E_i} = \frac{2\eta_2\cos\theta_1}{\eta_1\cos\theta_1 + \eta_2\cos\theta_2} = \frac{2\sqrt{\varepsilon_1}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1}$$

(transmission coefficient for perpendicular polarisation)

Total internal reflection:

Irrespective of whether the polarisation of an electromagnetic wave is parallel or perpendicular, the wave incident from a denser medium to a rarer medium undergoes reflection at the interface between these two media with a magnitude of the reflection coefficient equal to unity when the angle of incidence is greater than a value called the ‘critical angle’. The phenomenon is called total internal reflection.

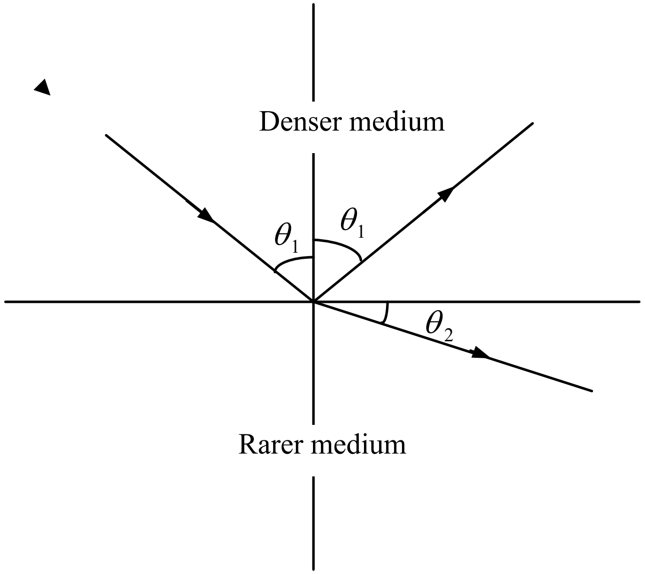
$$\theta_2 = \pi/2$$

$$\sqrt{\varepsilon_1}\sin\theta_1 = \sqrt{\varepsilon_2}\sin\theta_2 \quad \text{(Snell's law)} \longrightarrow \sqrt{\varepsilon_1}\sin\theta_1 = \sqrt{\varepsilon_1}\sin\theta_c = \sqrt{\varepsilon_2}\sin(\pi/2) = \sqrt{\varepsilon_2}$$

$$\sin\theta_1 = \sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

(critical angle)



Let us next study the behaviour of the reflection coefficient for angle of incidence greater than the critical angle.

$$\cos\theta_2 = \sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1} \longleftarrow \cos\theta_2 = \sqrt{1 - \sin^2\theta_2} \longleftarrow \sin\theta_2 = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_1 \longleftarrow \sqrt{\varepsilon_1}\sin\theta_1 = \sqrt{\varepsilon_2}\sin\theta_2$$

(Snell's law)

$$\sin^2\theta_c = \frac{\varepsilon_2}{\varepsilon_1} \longleftarrow \theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

(Critical angle)

$$\cos\theta_2 = \sqrt{1 - \frac{\sin^2\theta_1}{\frac{\varepsilon_2}{\varepsilon_1}}} = \sqrt{1 - \frac{\sin^2\theta_1}{\sin^2\theta_c}}$$

For angle of incidence θ_1 greater than the critical angle θ_c , sin θ_1 > sin θ_c .

$\cos\theta_2$ becomes imaginary for $\theta_1 > \theta_c$

$$\Gamma_{//} = \frac{\sqrt{\varepsilon_1}\cos\theta_2 - \sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1}$$

$\sqrt{\varepsilon_1}\cos\theta_2 = jB$ where B is real

$\cos\theta_2$ becomes imaginary for $\theta_1 > \theta_c$

(recalled)

$A = \sqrt{\varepsilon_2}\cos\theta_1$ where A is real

(reflection coefficient for parallel polarisation)

$C = \sqrt{\varepsilon_1}\cos\theta_1$

$jD = \sqrt{\varepsilon_2}\cos\theta_2$

Similarly

$$\Gamma_{//} = \frac{\sqrt{\varepsilon_1}\cos\theta_2 - \sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2 + \sqrt{\varepsilon_2}\cos\theta_1} = \frac{jB - A}{jB + A}$$

$$\Gamma_{\perp} = \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2}\cos\theta_2} = \frac{C - jD}{C + jD}$$

$$\left|\Gamma_{//}\right| = \left|\frac{jB - A}{jB + A}\right| = 1 \quad (\theta_1 > \theta_c)$$

$$\left|\Gamma_{\perp}\right| = \left|\frac{C - jD}{C + jD}\right| = 1 \quad (\theta_1 > \theta_c)$$

Thus, there is total internal reflection for rays incident from the rarer to denser medium, irrespective of whether the polarisation is parallel or perpendicular.

Refraction of current at conductor-conductor interface

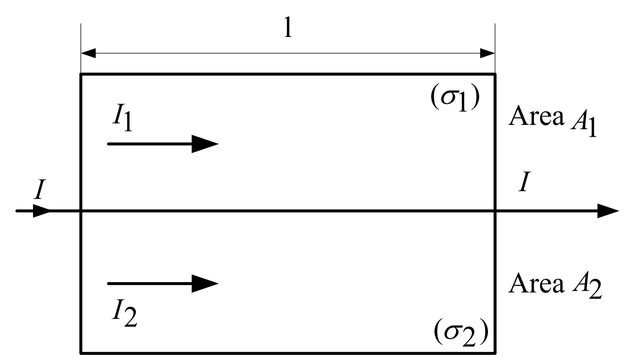
Let us begin the study with the appreciation of the circuit law of parallel resistances with the help of the boundary condition that the tangential component of electric field is continuous at the interface between two media.

For this purpose, the said boundary condition is applied at the interface between two rectangular conducting slabs in contact of the same length l, conductivities σ_1 and σ_2 and cross-sectional areas A_1 and A_2 respectively.

Current densities J_1 and J_2 are related to electric fields E_1 and E_2 in the slabs through Ohm's law while the current I fed into the slabs in contact is divided in currents I_1 and I_2 through the slabs.

$$J_1 = \sigma_1 E_1 \qquad J_2 = \sigma_2 E_2$$

$$E_1 = \frac{J_1}{\sigma_1} \qquad E_2 = \frac{J_2}{\sigma_2}$$



E_1 and E_2, which are tangential at the interface, are continuous at the interface:

$$E_1 = E_2 \text{ (boundary condition)}$$

$$\downarrow$$

$$\frac{J_1}{\sigma_1} = \frac{J_2}{\sigma_2}$$

$$\downarrow$$

$$\frac{I_1}{A_1\sigma_1} = \frac{I_2}{A_2\sigma_2}$$

$$\downarrow$$ Multiplying by l

$$I_1 \frac{l}{A_1\sigma_1} = I_2 \frac{l}{A_2\sigma_2}$$

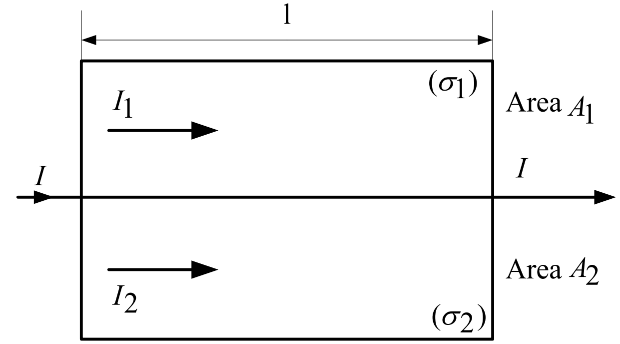
$$\downarrow \longleftarrow$$

$$R_1 = \frac{1}{\sigma_1}\frac{l}{A_1}$$

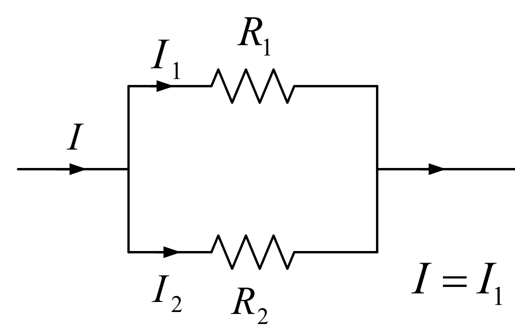
$$R_2 = \frac{1}{\sigma_2}\frac{l}{A_2}$$

$$I_1 R_1 = I_2 R_2$$

$$I = I_1 + I_2 \longrightarrow$$



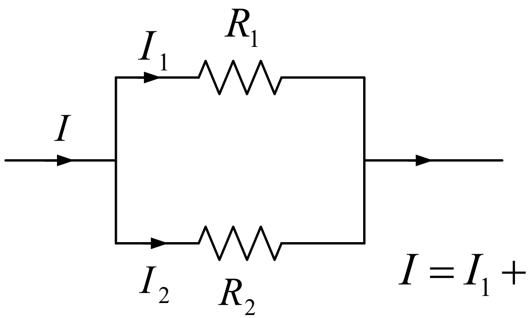
$$I = I_1 + I_2 \longrightarrow$$



$$I = I_1 + I_2$$

$$I_1R_1 = I_2R_2 \longrightarrow \frac{I_1}{I_2} = \frac{R_2}{R_1}$$

$$\downarrow$$



$$I = I_1 + I_2$$

$$\frac{I_1}{I_1 + I_2} = \frac{R_2}{R_2 + R_1} = \frac{R_2}{R_1 + R_2}$$

$$\downarrow$$

$$I = I_1 + I_2$$

$$\frac{I_1}{I} = \frac{R_2}{R_1 + R_2}$$

$$\downarrow$$

Multiplying by R_1

$$I_1 = \frac{R_2}{R_1 + R_2} I \longrightarrow I_1R_1 = I\frac{R_1R_2}{R_1 + R_2} = IR_{\text{equivalent}}$$

$$\downarrow$$

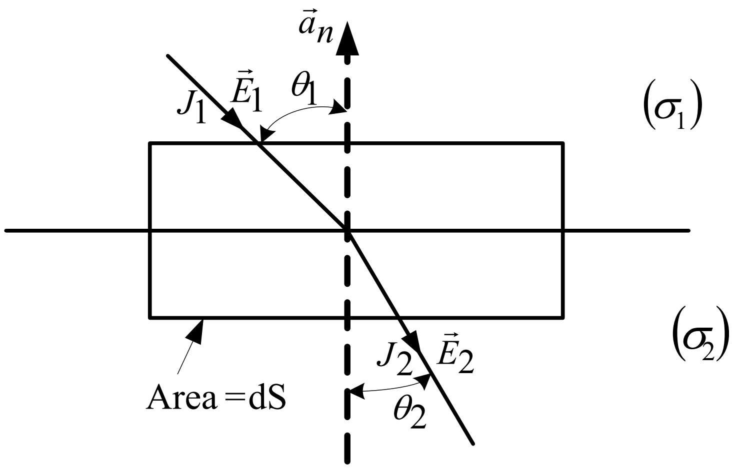
$$R_{\text{equivalent}} = \frac{R_1R_2}{R_1 + R_2}$$

$$\downarrow$$

$$\frac{1}{R_{\text{equivalent}}} = \frac{1}{R_1} + \frac{1}{R_2}$$

(Law of parallel resistances)

Let us now consider dc current passing through two conductors of different conductivities σ_1 and σ_2 and see how the current refracts through the conductors with a definite relation between the angles of incidence and refraction θ_1 and θ_2 respectively in terms of the conductivities σ_1 and σ_2 .



$$\int_S \vec{J}.\vec{a}_n dS = 0$$

(under dc conditions)

Apply to a pill-box (rectangular) volume element of infinitesimal thickness at the interface

Ignoring side contribution to the surface integral in view of infinitesimal thickness of the volume element

$$J_{n1} = J_{n2}$$

$$\sigma_1 E_{n1} = \sigma_2 E_{n2}$$

$$\sigma_1 E_1 \cos\theta_1 = \sigma_2 E_2 \cos\theta_2$$

Subscript *n* refers to the normal component

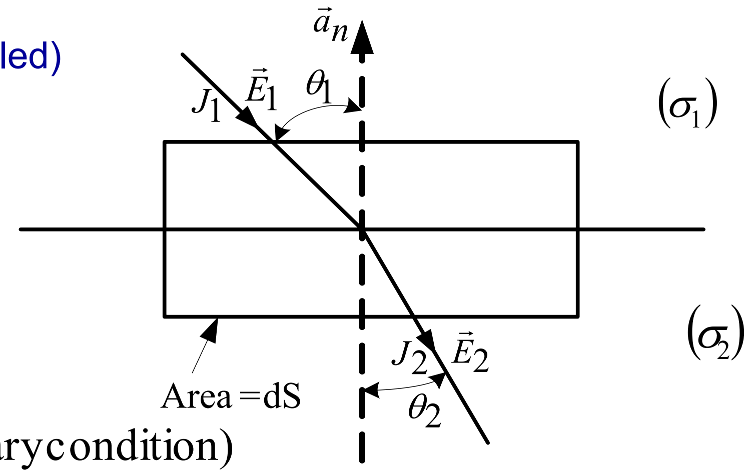
$$E_{t1} = E_{t2}$$

Tangential component

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

$$\sigma_1 E_1 \cos\theta_1 = \sigma_2 E_2 \cos\theta_2$$ (recalled)

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$ (recalled)



By division

$$\frac{E_1 \sin\theta_1}{\sigma_1 E_1 \cos\theta_1} = \frac{E_2 \sin\theta_2}{\sigma_2 E_2 \cos\theta_2}$$

$$\frac{\tan\theta_1}{\tan\theta_1} = \frac{\sigma_1}{\sigma_2}$$

(refraction of dc current through conductors)

$$(\vec{D}_2 - \vec{D}_1).\vec{a}_n = \rho_S \text{ (boundarycondition)}$$

$$D_{n2} - D_{n1} = \rho_S \rightarrow \varepsilon_0 (E_{n2} - E_{n1}) = \rho_S$$

$$\varepsilon_0 \left(\frac{J_{n2}}{\sigma_2} - \frac{J_{n1}}{\sigma_1}\right) = \rho_S$$

$$\varepsilon_0 \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) J_n = \rho_S$$

$$\left(\frac{\varepsilon_0}{\sigma_{1,2}} = \tau_{1,2}\right)$$ (relaxation time)

$$(\tau_2 - \tau_1) J_n = \rho_S$$

(surface charge density developed at the interface)

Summarising Notes

√ Electric and magnetic field quantities get modified due to an abrupt change of the medium properties at the interface between two media such that these quantities satisfy a set of electromagnetic boundary conditions at a point on the interface between the media.

√ General electromagnetic boundary conditions at a point on the interface between two media have been deduced in vector form with the help of Maxwell's equations in integral form in terms of the field quantities and a unit vector directed from one medium to another.

√ Surface charge density is defined as the product of the volume charge density and the infinitesimal thickness over which the charge is spread at the interface, in the limit of the infinitesimal thickness tending to zero. This definition emerges in course of the deduction of general electromagnetic boundary conditions.

√ Surface current density is defined as the product of the current density and the infinitesimal thickness at the interface over which the current density is significant, in the limit of the infinitesimal thickness tending to zero. This definition emerges in course of the deduction of general electromagnetic boundary conditions.

√ General boundary conditions can be interpreted for dielectric-dielectric and conductor-dielectric interfaces on the following findings.

◊ Relaxation time of a dielectric being quite large, the electric charge can stay longer inside the bulk of a dielectric and the bulk of a dielectric can be electrically charged with a finite volume charge density that makes the surface charge density nil at the dielectric surface which can be appreciated from the definition of surface charge density. The finding is valid for both time-independent and time-dependent situations.

◊ Relaxation time of a good conductor being quite small, the charge inside the bulk of a good conductor decays very fast to appear with quite a large volume charge density at the conductor surface to get concentrated over a layer of infinitesimal thickness, which in turn renders a finite value of surface charge density according to its definition. The finding is valid for both time-independent and time-dependent situations.

◊ Electric field or electric displacement is absent in a good conductor since the bulk of the conductor cannot be electrically charged. The finding is valid for both time-independent and time-dependent situations.

◊ Finite magnetic field or magnetic flux density can be established inside a dielectric independently of electric field, for both time-independent and time-dependent situations.

◊ Magnetic field and the magnetic flux density each become zero inside a conductor for time-dependent situations since for such situations the magnetic field is coupled to the electric field which is absent in a conductor.

◊ Finite magnetic field or magnetic flux density can be established independently of an electric field in a conductor for time-independent situations.

◊ Surface current density at a dielectric surface is nil since a dielectric supposedly perfect does not conduct current. The finding is valid for both time-independent and time-dependent situations.

◊ Surface current density at a conductor surface is nil for time-independent situations according to the definition of surface density, since a finite current can be made to flow through the bulk of the conductor for such situations.

◊ Finite surface current density according to its definition can be established at the surface of a good conductor for time-dependent situations since quite a large current density can be concentrated over a layer of infinitesimal thickness at the surface of the conductor.

√ Understanding of the phenomenon of formation of a standing wave when a uniform plane electromagnetic wave is incident from the free-space region on the surface of a conducting medium has been developed with the help of the relevant electromagnetic boundary condition.

√ Reflection and refraction of electromagnetic waves at a dielectric-dielectric interface have been studied with the help of the relevant electromagnetic boundary conditions for parallel and perpendicular polarisations.

√ Brewster's phenomenon for parallel polarisation can be understood with the help of the relevant electromagnetic boundary conditions.

√ Brewster's angle of incidence that corresponds to no reflection at a dielectric-dielectric interface has been deduced for parallel polarisation.

√Total internal reflection at a dielectric-dielectric interface for the angle of incidence greater than the critical angle has been understood for both parallel and perpendicular polarisations.

√ Circuit law of parallel resistances can be appreciated from the electromagnetic boundary condition that the tangential component of electric field is continuous at the interface between two media.

√ Boundary conditions at the interface between two conducting media yield the law of refraction of current for time-independent situations.

Readers are encouraged to go through Chapter 7 of the book for more topics and more worked-out examples and review questions.