Engineering Electromagnetics Essentials

Chapter 6

Wave equation and its solution for a wave propagating through an unbounded medium

Objective

Study of propagation of uniform plane electromagnetic waves through unbounded media such as free-space and conducting media

Topics dealt with

Representation of a wave

Establishment of wave equations in electric and magnetic fields

Wave propagation through an unbounded free-space medium

Wave propagation through a resistive medium

Skin depth, surface resistance and ac resistance

Wave propagation through sea water and choice of operating frequency vis-à-vis attenuation

Wave propagation through a medium of charge particles

Background

Basic concepts of time-varying fields developed in Chapter 5 including Maxwell's equations

Representation of a quantity associated with a wave

A quantity that varies periodically with space and time is said to be associated with a wave. Let us see how a mathematical function can represent a propagating wave.

Let us examine the function

$$f_+(t,z) = f_+\left(t - \frac{z}{v}\right)$$

← Subscript + refers to a forward wave propagating in positive z direction

↓

$$f_+(t+\Delta t, z+\Delta z) = f_+\left(t + \Delta t - \frac{z+\Delta z}{v}\right)$$

$$= f_+\left(t - \frac{z}{v} + (\Delta t - \frac{\Delta z}{v})\right) = f_+(t - \frac{z}{v})$$

$$\text{provided } \Delta t - \frac{\Delta z}{v} = 0 \ \text{(condition)} \longrightarrow v = \frac{\Delta z}{\Delta t}$$

↓

$$v = \underset{\Delta t \to 0}{Lt} \frac{\Delta z}{\Delta t} = \frac{dz}{dt} \ \text{(condition)}$$

$$f_+(t,z) = f_+\left(t - \frac{z}{v}\right)$$

$$f_+(t+\Delta t, z+\Delta z) = f_+\left(t - \frac{z}{v}\right)$$

provided $v = \underset{\Delta t \to 0}{Lt} \dfrac{\Delta z}{\Delta t} = \dfrac{dz}{dt}$ (condition)

Rewritten

v is identified as the velocity of wave.

Thus the function

$$f_+(t,z) = f_+\left(t - \frac{z}{v}\right)$$

represents a forward wave propagating in positive z direction

Similarly, now let us examine the function

$$f_-(t,z) = f_-\left(t + \frac{z}{v}\right)$$

Subscript - refers to a backward wave propagating in negative z direction

$$f_-(t+\Delta t, z-\Delta z) = f_-\left(t + \Delta t + \frac{z-\Delta z}{v}\right)$$

$$= f_-\left(t - \frac{z}{v} + (\Delta t - \frac{\Delta z}{v})\right) = f_-(t + \frac{z}{v})$$

$$\text{provided } \Delta t - \frac{\Delta z}{v} = 0 \text{ (condition)}$$

$$\rightarrow \quad v = \frac{\Delta z}{\Delta t}$$

$$v = \underset{\Delta t \to 0}{Lt} \frac{\Delta z}{\Delta t} = \frac{dz}{dt} \text{ (condition)}$$

v is identified as the velocity of wave.

Thus the function

$$f_-(t,z) = f_-\left(t + \frac{z}{v}\right)$$

represents a backward wave propagating in negative z direction

We have thus seen that

the function

$$f_+(t,z) = f_+\left(t - \frac{z}{v}\right)$$

represents a forward wave propagating in positive z direction

and that

the function

$$f_-(t,z) = f_-\left(t + \frac{z}{v}\right)$$

represents a backward wave propagating in negative z direction

How to choose the functions so that they can represent the periodicity of the wave with space and tome coordinates?

Choose the functions as proportional to exp $j(\omega t\text{-}z/v)$ for a forward wave and to exp $j(\omega t+z/v)$ for a backward wave in phasor notation?

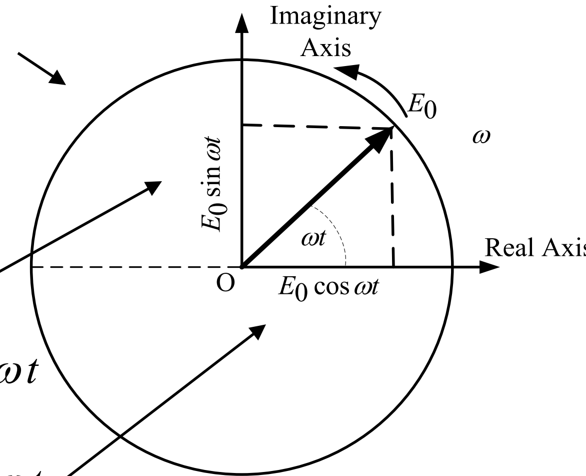
Choose the functions as proportional to exp $j(\omega t\text{-}z/v)$ for a forward wave and to exp $j(\omega t+z/v)$ for a backward wave in phasor notation.

$$f_+(t,z) = f_+\left(t - \frac{z}{v}\right) \quad \text{(forward wave)}$$

$$f_-(t,z) = f_-\left(t + \frac{z}{v}\right) \quad \text{(backward wave)}$$

Phasor is a rotating vector

Phasor diagram of a time-periodic quantity, say magnitude of electric field E of amplitude E_0 rotating with angular velocity ω



Imaginary part $E_0 \sin \omega t$

Real part $E_0 \cos \omega t$

$$E = E_0 \exp j\omega t = E_0(\cos\omega t + j\sin\omega t)$$
$$= E_0 \cos\omega t + jE_0 \sin\omega t$$

With reference to physical quantity, say, electric field for forward and backward waves, we may take the functions taking into regard their periodicity as follows:

$$E_+ = E_0 \exp(j\omega t') = E_0 \exp j\omega(t - z/v)$$

interpreting t in phasor notation t' as

$t' = t - z/v$ (forward wave)

and

$$E_- = E_0 \exp j\omega t'' = E_0 \exp j\omega(t + z/v)$$

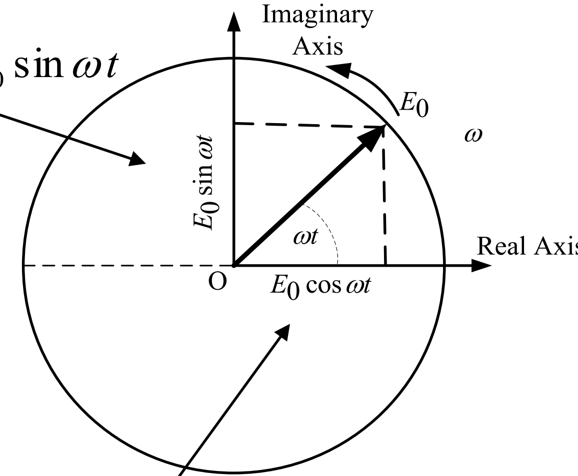
interpreting t in phasor notation t'' as

$t'' = t + z/v$ (backward wave)

$$E_+ = E_0 \exp(j\omega t') = E_0 \exp j\omega(t - z/v)$$
$$= E_0[\cos\omega(t - z/v) + j\sin\omega(t - z/v)]$$
(forward wave)

$$E_- = E_0 \exp(j\omega t'') = E_0 \exp j\omega(t + z/v)$$
$$= E_0[\cos\omega(t + z/v) + j\sin\omega(t + z/v)]$$
(backward wave)

Imaginary part $E_0 \sin\omega t$



Real part $E_0 \cos\omega t$

interpreting *t* as *t'* and *t"* in phasor notation

$t' = t - z/v$ (forward wave) and $t'' = t + z/v$ (backward wave)

In an alternative approach, we can choose either the real or imaginary part of the function to represent the wave. Taking the real part:

$$E_+ = E_0 \cos\omega(t - z/v) \text{ (forward wave)}$$
$$E_- = E_0 \cos\omega(t + z/v) \text{ (backward wave)}$$

← $\beta = \omega / v =$ phase propagation constant

$$\omega(t - z/v) = \omega t - (\omega/v) z = \omega t - \beta z$$
$= $ phase of the forward wave ←

$$\omega(t + z/v) = \omega t + (\omega/v) z = \omega t + \beta z$$
$= $ phase of the backward wave ←

Phase of the wave determines (i) the value of the quantity (here, typically, E_- or E_+) with which the wave is associated, and (ii) the state of variation of this value.

Wave phase velocity, wavelength and frequency and the relation between them

$$\omega t - \beta z = \text{phase of the forward wave (recalled)}$$

Put the phase of the wave as constant:

$$\text{phase} = \omega t - \beta z = \text{constant}$$

↓ ← Differentiating

$$\frac{dz}{dt} = \frac{\omega}{\beta}$$ ← velocity of the constant phase of the wave

↓

$$v_{\text{ph}} = \frac{\omega}{\beta}$$ (phase velocity of the wave)

Variation of the quantity (here, electric field magnitude) representing a wave, typically, forward wave, with space coordinate z at a fixed time (snap-shot), typically $t = 0$:

$$E_+ = E_0 \cos(\omega t - \beta z)$$

↓ ← $t = 0$ (at a fixed time)

$$E_+ = E_0 \cos(-\beta z) = E_0 \cos \beta z$$

$$E_+ = E_0 \cos(\omega t - \beta z)$$

$\downarrow$ $\leftarrow$ $t = 0$ (at a fixed time) (recalled)

$$E_+ = E_0 \cos(-\beta z) = E_0 \cos\beta z$$ $\longleftarrow$ Positive maximum $\rightarrow \cos\beta z = 1$

$\downarrow$

$$E_+ = E_0$$

$\downarrow$

$$\beta z = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$$ (corresponding to the same phase: here, referring to positive maximum value of the quantity)

$\downarrow$

Interval of βz between two consecutive same phases at a fixed time

$\downarrow$

$$\beta z = 2\pi$$

$\downarrow$

$$z = \frac{2\pi}{\beta}$$ Distance between two consecutive same phases at a fixed time called wavelength λ

$\downarrow$

$$\lambda = \frac{2\pi}{\beta}$$ (wavelength)

$\downarrow$

$$\beta = \frac{2\pi}{\lambda}$$ (wave propagation constant)

$$E_+ = E_0 \cos(\omega t - \beta z)$$

$\downarrow \quad \leftarrow \quad z = 0$ (at a fixed distance)

$$E_+ = E_0 \cos \omega t$$

$\downarrow \quad \leftarrow \quad$ Positive maximum $\rightarrow \cos\omega t = 1$

$$E_+ = E_0$$

$\downarrow$

$\omega t = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots$ (corresponding to the same phase: here, referring to positive maximum value of the quantity)

$\downarrow$

Interval of ωt between two consecutive same phases at a fixed time

$\downarrow$

$\omega t = 2\pi$ Time interval between two consecutive same phases at a fixed distance called time period T

$\downarrow$

$t = \dfrac{2\pi}{\omega}$ Time interval between two consecutive same phases at a fixed distance called time period T

$\downarrow$

$$T = \frac{2\pi}{\omega}$$

$$T = \frac{2\pi}{\omega}$$

$$\downarrow$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$ (wave circular frequency or simply frequency)

$$\downarrow$$

$$\omega = 2\pi f$$ (wave circular frequency or simply frequency)

$$\downarrow$$

$$v_{\text{ph}} = \frac{2\pi f}{\beta} \quad \longleftarrow \quad \beta = \frac{2\pi}{\lambda}$$

$$\downarrow$$

$$v_{\text{ph}} = f\lambda$$ (relation between phase velocity, frequency and wavelength)

Wave equations in electric and magnetic fields

Wave equation in electric field

Electric and magnetic fields are coupled in the following two Maxwell's equations:

$$\left.\begin{aligned} \nabla\times\vec{E} &= -\mu\frac{\partial\vec{H}}{\partial t} \\ \nabla\times\vec{H} &= \vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t} \end{aligned}\right\}$$

(Maxwell's equations recalled)

$$\left.\begin{aligned} \vec{D} &= \varepsilon\,\vec{E} \\ \vec{B} &= \mu\,\vec{H} \end{aligned}\right\}$$

In order to decouple these two equations to obtain a single equation, namely wave equation in electric field, take the curl operation on the first of these equations and read it using the second of these equations.

$$\nabla\times\nabla\times\vec{E} = -\mu\nabla\times\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\nabla\times\vec{E}=-\mu\nabla\times\frac{\partial\vec{H}}{\partial t}$$ (rewritten) ← Partial time derivative and curl involving partial derivatives in space coordinates are interchangeable

↓

$$\nabla\times\nabla\times\vec{E}=-\mu\frac{\partial}{\partial t}(\nabla\times\vec{H})$$ ← $$\nabla\times\vec{H}=\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}$$ (Maxwell's equation)

↓

$$\nabla\times\nabla\times\vec{E}=-\mu\frac{\partial}{\partial t}(\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t})=-\mu\frac{\partial\vec{J}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$ ← $$\nabla\times\nabla\times\vec{E}=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}$$ (vector identity)

↓

$$\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\mu\frac{\partial\vec{J}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

↓

$$\nabla\left(\frac{\rho}{\varepsilon}\right)-\nabla^2\vec{E}=-\mu\frac{\partial\vec{J}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$ ← $$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon}$$ (Maxwell's equation)

↓ ← Rearranged

$$\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=\nabla\left(\frac{\rho}{\varepsilon}\right)+\mu\frac{\partial\vec{J}}{\partial t}$$ (wave equation in electric field)

Wave equation in magnetic field

$$\left.\begin{aligned}\nabla\times\vec{E} &= -\mu\frac{\partial\vec{H}}{\partial t}\\ \nabla\times\vec{H} &= \vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}\end{aligned}\right\}$$

(Maxwell's equations recalled)

In order to decouple these two equations to obtain a single equation, namely wave equation in magnetic field, take the curl operation on the second of these equations and read it using the first of these equations.

$$\nabla\times\nabla\times\vec{H} = \nabla\times(\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}) = \nabla\times\vec{J}+\varepsilon\nabla\times\frac{\partial\vec{E}}{\partial t})$$

↓ ← Partial time derivative and curl involving partial derivatives in space coordinates are interchangeable

$$\nabla\times\nabla\times\vec{H} = \nabla\times\vec{J}+\varepsilon\frac{\partial}{\partial t}(\nabla\times\vec{E})$$

← $\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$ (Maxwell's equation)

↓

$$\nabla\times\nabla\times\vec{H} = \nabla\times\vec{J}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\nabla\times\nabla\times\vec{H}=\nabla\times\vec{J}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \longleftarrow \nabla\times\nabla\times\vec{H}=\nabla(\nabla\cdot\vec{H})-\nabla^2\vec{H}$$

(vector identity)

$$\downarrow$$

$$\nabla(\nabla\cdot\vec{H})-\nabla^2\vec{H}=\nabla\times\vec{J}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \longleftarrow \nabla\cdot\vec{H}=0$$

(Maxwell's equation)

$$\downarrow$$

$$\nabla^2\vec{H}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}=-\nabla\times\vec{J}$$

(wave equation in magnetic field)

Wave propagation through a Free-space medium

$$\left.\begin{aligned}\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}&=\nabla\left(\frac{\rho}{\varepsilon}\right)+\mu\frac{\partial\vec{J}}{\partial t}\\ \nabla^2\vec{H}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}&=-\nabla\times\vec{J}\end{aligned}\right\} \longleftarrow \rho=\vec{J}=0$$

$$\downarrow$$

$$\left.\begin{aligned}\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}&=0\\ \nabla^2\vec{H}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}&=0\end{aligned}\right\}$$

(source-free wave equations in electric and magnetic fields) $(\rho=\vec{J}=0)$

Consider a uniform plane wave propagating through an unbounded free-space medium.

$$\left.\begin{aligned} \nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0 \\ \nabla^2\vec{H} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} = 0 \end{aligned}\right\}$$

(source-free wave equations in electric and magnetic fields) $(\rho = \vec{J} = 0)$

A plane wave has a plane wavefront. A wave front is an equi-phase surface over which the phase of the quantity associated with the wave remains constant. For a uniform plane wave, in addition to the phase, the amplitude of the quantity associated with the wave also remains constant over the plane wavefront.

Formulate the problem by considering wave propagation along z in the rectangular coordinate system.

$$\left.\begin{aligned} \nabla^2 E_x = \mu_0\varepsilon_0\frac{\partial^2 E_x}{\partial t^2} \\ \nabla^2 E_y = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2} \\ \nabla^2 E_z = \mu_0\varepsilon_0\frac{\partial^2 E_z}{\partial t^2} \end{aligned}\right\} \longleftarrow \quad \nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\left.\begin{aligned} \nabla^2 H_x = \mu_0\varepsilon_0\frac{\partial^2 H_x}{\partial t^2} \\ \nabla^2 H_y = \mu_0\varepsilon_0\frac{\partial^2 H_y}{\partial t^2} \\ \nabla^2 H_z = \mu_0\varepsilon_0\frac{\partial^2 H_z}{\partial t^2} \end{aligned}\right\} \longleftarrow \quad \nabla^2\vec{H} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

$$\left.\begin{aligned}\nabla^2 E_x &= \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}\\ \nabla^2 E_y &= \mu_0\varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}\\ \nabla^2 E_z &= \mu_0\varepsilon_0 \frac{\partial^2 E_z}{\partial t^2}\end{aligned}\right\}$$

$$\left.\begin{aligned}\nabla^2 H_x &= \mu_0\varepsilon_0 \frac{\partial^2 H_x}{\partial t^2}\\ \nabla^2 H_y &= \mu_0\varepsilon_0 \frac{\partial^2 H_y}{\partial t^2}\\ \nabla^2 H_z &= \mu_0\varepsilon_0 \frac{\partial^2 H_z}{\partial t^2}\end{aligned}\right\}$$

$$\downarrow$$

$$\left.\begin{aligned}\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}\\ \frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}\\ \frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_z}{\partial t^2}\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial^2 H_x}{\partial x^2}+\frac{\partial^2 H_x}{\partial y^2}+\frac{\partial^2 H_x}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_x}{\partial t^2}\\ \frac{\partial^2 H_y}{\partial x^2}+\frac{\partial^2 H_y}{\partial y^2}+\frac{\partial^2 H_y}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_y}{\partial t^2}\\ \frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+\frac{\partial^2 H_z}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_z}{\partial t^2}\end{aligned}\right\}$$

For a uniform plane wave propagating along z

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

$$\left.\begin{aligned}
\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \\
\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \\
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_z}{\partial t^2}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_x}{\partial t^2} \\
\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_y}{\partial t^2} \\
\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_z}{\partial t^2}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{\partial^2 E_x}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \\
\frac{\partial^2 E_y}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \\
\frac{\partial^2 E_z}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 E_z}{\partial t^2}
\end{aligned}\right\}$$

(wave equation in electric field for a uniform plane wave in free space)

$$\left.\begin{aligned}
\frac{\partial^2 H_x}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_x}{\partial t^2} \\
\frac{\partial^2 H_y}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_y}{\partial t^2} \\
\frac{\partial^2 H_z}{\partial z^2} &= \mu_0\varepsilon_0 \frac{\partial^2 H_z}{\partial t^2}
\end{aligned}\right\}$$

(wave equation in magnetic field for a uniform plane wave in free space)

$\nabla \cdot \vec{E} = 0$ ← $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$ (Maxwell's equation)

$\rho = 0$ for a free-space medium

$\nabla \cdot \vec{H} = 0$ (Maxwell's equation)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad \leftarrow \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \quad \rightarrow \quad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

(for a uniform plane wave propagating along z)

$$\frac{\partial E_z}{\partial z} = 0 \qquad \frac{\partial H_z}{\partial z} = 0$$

E_z and H_z are each constant with z

$$\frac{\partial^2 E_z}{\partial z^2} = 0 \rightarrow \frac{\partial^2 E_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2} \qquad \frac{\partial^2 H_z}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2} \leftarrow \frac{\partial^2 H_z}{\partial z^2} = 0$$

$$\frac{\partial^2 E_z}{\partial t^2} = 0 \qquad \frac{\partial^2 H_z}{\partial t^2} = 0$$

$$\frac{\partial^2 E_z}{\partial t^2} = 0 \quad \longleftarrow \text{Integrating} \longrightarrow \quad \frac{\partial^2 H_z}{\partial t^2} = 0$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\frac{\partial E_z}{\partial t} = A \quad \longleftarrow \text{Integrating} \longrightarrow \quad \frac{\partial H_z}{\partial t} = C$$

A, B, C, D are constants with t

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$E_z = At + B \qquad\qquad H_z = Ct + D$$

(linearly dependent on *t*) (linearly dependent on *t*)

E_z and H_z are each constant with *z* and that they are linearly dependent on *t*.

$\downarrow$

Contradicts with the dependence of these quantities as exp $j(\omega t-z/v)$ (for a forward wave in free space considered) with non-zero values of the constants *A*, *B*, *C* and *D*.

$\downarrow$ ⟵ Avoidance of contradiction

$$A = B = C = D = 0$$

$$E_z = At + B \qquad H_z = Ct + D$$

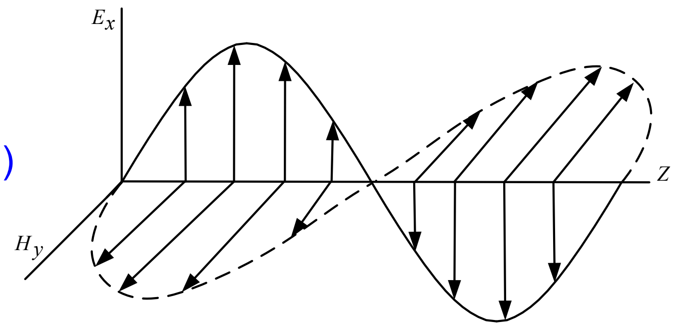
$$\begin{Bmatrix} E_z = 0 \\ H_z = 0 \end{Bmatrix}$$

$\downarrow$

A uniform plane wave propagating through an unbounded free-space medium is essentially a transverse electromagnetic wave (TEM) with no electric and magnetic field components in the direction of propagation (along z)

$$\left.\begin{aligned}\nabla\times\vec{E} &= -\mu\frac{\partial\vec{H}}{\partial t}\\ \nabla\times\vec{H} &= \vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}\end{aligned}\right\}\longleftarrow\ \vec{J}=0\ \text{(for free space)}$$

(Maxwell's equations recalled)



$$\left.\begin{aligned}\nabla\times\vec{E} &= -\mu\frac{\partial\vec{H}}{\partial t}\\ \nabla\times\vec{H} &= \varepsilon\frac{\partial\vec{E}}{\partial t}\\ E_z &= 0\\ H_z &= 0\end{aligned}\right\}$$

A uniform plane wave propagating through an unbounded free-space medium is essentially a transverse electromagnetic wave (TEM) with no electric and magnetic field components in the direction of propagation (along z)

$$-\frac{\partial E_y}{\partial z}\vec{a}_x+\frac{\partial E_x}{\partial z}\vec{a}_y=-\mu_0\frac{\partial}{\partial t}(H_x\vec{a}_x+H_y\vec{a}_y)$$

$$-\frac{\partial H_y}{\partial z}\vec{a}_x+\frac{\partial H_x}{\partial z}\vec{a}_y=\varepsilon_0\frac{\partial}{\partial t}(E_x\vec{a}_x+E_y\vec{a}_y)$$

Equating the x-components and y-components of the left- and right-hand sides

$$\left.\begin{aligned}\frac{\partial E_y}{\partial z} &= \mu_0\frac{\partial H_x}{\partial t}\\ \frac{\partial E_x}{\partial z} &= -\mu_0\frac{\partial H_y}{\partial t}\end{aligned}\right\}$$

$$\left.\begin{aligned}-\frac{\partial H_y}{\partial z} &= \varepsilon_0\frac{\partial E_x}{\partial t}\\ \frac{\partial H_x}{\partial z} &= \varepsilon_0\frac{\partial E_y}{\partial t}\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial E_y}{\partial z} &= \mu_0 \frac{\partial H_x}{\partial t}\\ \frac{\partial E_x}{\partial z} &= -\mu_0 \frac{\partial H_y}{\partial t}\end{aligned}\right\}$$

Field quantities vary as $\exp j(\omega t - \beta z)$

$$\left.\begin{aligned}-\frac{\partial H_y}{\partial z} &= \varepsilon_0 \frac{\partial E_x}{\partial t}\\ \frac{\partial H_x}{\partial z} &= \varepsilon_0 \frac{\partial E_y}{\partial t}\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial}{\partial t} &\equiv j\omega\\ \frac{\partial}{\partial z} &\equiv -j\beta\end{aligned}\right\}$$

$$\left.\begin{aligned}\beta E_y &= -\omega\mu_0 H_x\\ \beta E_x &= \omega\mu_0 H_y\end{aligned}\right\}$$

$$\left.\begin{aligned}\beta H_y &= \omega\varepsilon_0 E_x\\ \beta H_x &= -\omega\varepsilon_0 E_y\end{aligned}\right\}$$

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu_0}{\beta}$$

$$E_x H_x + E_y H_y = 0$$

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta}{\omega\varepsilon_0}$$

$$\vec{E} \cdot \vec{H} = 0$$

Electric and magnetic fields are at right angle to each other.

$$\frac{\omega\mu_0}{\beta} = \frac{\beta}{\omega\varepsilon_0}$$

$$\beta^2 = \omega^2 \mu_0 \varepsilon_0 \longrightarrow \frac{\omega}{\beta} = v_{\text{ph}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$$

$$\vec{E}\cdot\vec{H}=0$$

$$\frac{\omega}{\beta}=v_{\text{ph}}=\frac{1}{\sqrt{\mu_0\varepsilon_0}}=c$$

Unbounded free-space medium supports transverse electromagnetic (TEM) wave, such that the directions of the electric field, magnetic field and propagation are mutually perpendicular to one another, propagating with phase velocity equal to the speed of light *c:*

$$\mu_0=4\pi\times10^{-7}\ \text{H/m}\ \rightarrow\ c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}=3\times10^{8}\ \text{m/s}\ \leftarrow\ \varepsilon_0=1/(36\pi)\times10^{-9}=8.854\times19^{-12}\ \text{F/m}$$

10

Intrinsic impedance η_0 of the free-space medium is

$$\frac{E_x}{H_y}=-\frac{E_y}{H_x}=\frac{\omega\mu_0}{\beta}\ \rightarrow\ \eta_0=\frac{E}{H}=\frac{(E_x{}^2+E_y{}^2)^{1/2}}{(H_x{}^2+H_y{}^2)^{1/2}}$$

$$\eta_0=\frac{(E_x{}^2+E_y{}^2)^{1/2}}{(H_x{}^2+H_y{}^2)^{1/2}}=\frac{\left[\left(\frac{\omega\mu_0}{\beta}\right)^2(H_x{}^2+H_y{}^2)\right]^{1/2}}{(H_x{}^2+H_y{}^2)^{1/2}}=\frac{\omega\mu_0}{\beta}$$

$$\eta_0 = \frac{\omega\mu_0}{\beta} \text{ (rewritten)} \longleftarrow \beta = \omega\sqrt{\mu_0\varepsilon_0} \longleftarrow \beta^2 = \omega^2\mu_0\varepsilon_0$$

$\downarrow$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$\swarrow$ $\mu_0 = 4\pi\times 10^{-7}$ H/m

$\nwarrow$ $\varepsilon_0 = 1/(36\pi)\times 10^{-9} = 8.854\times 19^{-12}$ F/m

$\downarrow$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi$$

(intrinsic impedance of a free-space medium)

Wave propagation through a conducting medium

Let us recall wave equation in electric field:

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla\left(\frac{\rho}{\varepsilon}\right) + \mu \frac{\partial \vec{J}}{\partial t}$$ (recalled)

(wave equation in electric field)

$$\left.\begin{aligned} \rho &= 0 \\ \vec{J} &= \sigma \vec{E} \end{aligned}\right\} \text{(conducting medium)}$$

↓

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$ ←

$$\left.\begin{aligned} \frac{\partial}{\partial t} &\equiv j\omega \\ \frac{\partial^2}{\partial t^2} &\equiv (j\omega)(j\omega) = -\omega^2 \end{aligned}\right\}$$

↓

$$\nabla^2 \vec{E} = j\omega\mu\,(\sigma + j\omega\varepsilon)\vec{E}$$ ← $\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$

↓

$$\nabla^2 \vec{E} = \gamma^2 \vec{E}$$

$$\nabla^2 \vec{E} = \gamma^2 \vec{E}$$

Expanding Laplacian ∇^2

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

(for a uniform plane wave propagating along z)

$$\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$$

$$\left.\begin{aligned} \frac{\partial^2 E_x}{\partial z^2} &= \gamma^2 E_x = j\omega\mu(\sigma + j\omega\varepsilon)E_x \\ \frac{\partial^2 E_y}{\partial z^2} &= \gamma^2 E_y = j\omega\mu(\sigma + j\omega\varepsilon)E_y \end{aligned}\right\}$$

γ is called the propagation constant of the wave

$$\left.\begin{aligned} E_x &= E_{x0}\exp(-\gamma z) \\ E_y &= E_{y0}\exp(-\gamma z) \end{aligned}\right\}$$

(solution for a forward wave) (subscript 0 referring to field amplitudes)

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

$$\left.\begin{aligned} E_x &= E_{x0}\exp[-(\alpha + j\beta)z] = E_{x0}\exp(-\alpha z)\exp(-j\beta z) \\ E_y &= E_{y0}\exp[-(\alpha + j\beta)z] = E_{y0}\exp(-\alpha z)\exp(-j\beta z) \end{aligned}\right\}$$

$$\left.\begin{aligned} \alpha &= \text{attenuation constant of the wave} \\ \beta &= \text{phase propagation constant of the wave} \end{aligned}\right\}$$

Skin depth

$$\left.\begin{aligned} E_x &= E_{x0}\exp[-(\alpha + j\beta)z] = E_{x0}\exp(-\alpha z)\exp(-j\beta z) \\ E_y &= E_{y0}\exp[-(\alpha + j\beta)z] = E_{y0}\exp(-\alpha z)\exp(-j\beta z) \end{aligned}\right\}$$

$$\left.\begin{aligned} \alpha &= \text{attenuation constant of the wave} \\ \beta &= \text{phase propagation constnat of the wave} \end{aligned}\right\}$$

↓ ← Invoking time dependence exp($j\omega t$)

$$\left.\begin{aligned} E_x &= E_{x0}\exp[-(\alpha + j\beta)z] = E_{x0}\exp(-\alpha z)\exp(-j\beta z)\exp(j\omega t) \\ E_y &= E_{y0}\exp[-(\alpha + j\beta)z] = E_{y0}\exp(-\alpha z)\exp(-j\beta z)\exp(j\omega t) \end{aligned}\right\}$$

Amplitude exponentially attenuates as $\exp(-\alpha z)$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$ (recalled)

For a good conductor: $\sigma >> \omega\varepsilon$

$$\gamma = \sqrt{j\omega\mu\sigma} = \alpha + j\beta$$

$$\gamma = \sqrt{j\omega\mu\sigma} = \alpha + j\beta$$ (for a good conductor) $(\sigma >> \omega\varepsilon)$

Squaring

Separate real and imaginary parts

$$j\omega\mu_0\,\sigma = (\alpha + j\beta)^2 = \alpha^2 + (j\beta)^2 + j2\alpha\beta = \alpha^2 - \beta^2 + j2\alpha\beta$$

Equating real part

Equating imaginary part

$$\alpha^2 - \beta^2 = 0$$

$$\omega\mu_0\,\sigma = 2\alpha\beta$$

$$\alpha = \beta$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

(for a good conductor) $(\sigma >> \omega\varepsilon)$

$$\alpha = \beta = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

(for a good conductor) $(\sigma >> \omega\varepsilon)$

$$\left.\begin{aligned} E_x &= E_{x0}\exp[-(\alpha + j\beta)z] = E_{x0}\exp(-\alpha\, z)\exp(-j\beta\, z)\exp(j\omega t) \\ E_y &= E_{y0}\exp[-(\alpha + j\beta)z] = E_{y0}\exp(-\alpha\, z)\exp(-j\beta\, z)\exp(j\omega t) \end{aligned}\right\}$$

Amplitude exponentially attenuates as $\exp(-\alpha\, z)$

Taking $\alpha = 1/\delta$, the exponential attenuating factor of the field amplitude may be put as

$$\exp(-\alpha\, z) = \exp(-z/\delta)$$

Putting $z = \delta$, we appreciate that the field amplitude at the surface (skin) of a good conductor at $z = 0$ attenuates by a factor of exp (-1) = 1/*e* at a depth equal to z = δ called the skin depth of the conductor.

$$\alpha = 1/\delta$$

$$\downarrow$$

$$\delta = 1/\alpha \quad \longleftarrow \quad \alpha = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

$$\downarrow$$

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

(skin depth)

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$ (skin depth) $\leftarrow$ $\omega = 2\pi f$

$\downarrow$

$$\delta = \sqrt{\frac{1}{\pi f \mu_0 \sigma}}$$

Skin depth decreases with the increase of the conductivity of the medium and operating frequency

$\nwarrow$

$$\left.\begin{aligned} &\alpha = \beta = \sqrt{\frac{\omega\mu_0\sigma}{2}} \quad (\sigma >> \omega\varepsilon) \\ &\beta = 2\pi / \lambda \end{aligned}\right\}$$

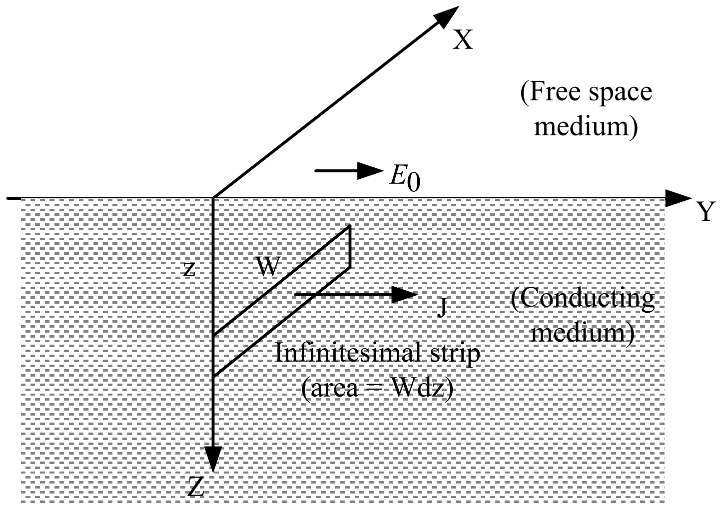
$\downarrow$

$$\delta = \frac{1}{\beta} = \frac{\lambda}{2\pi}$$

Skin depth is directly proportional to the wavelength in the conducting medium.

Typically, for copper (σ = 5.8x10^7 mho/m), at operating frequency 50 Hz, the value of the skin depth is ~ 9.4 mm.
The skin depths are ~ 2.1 mm, ~ 66.1 μm, and ~ 2.1 μm, for frequencies 1 kHz, 1 MHz, and 1 GHz, respectively. For 10 GHz frequency, we find the skin depth as 6.61x10^{-7} m, which happens to be in the range of wavelength of visible light.

Surface resistance and ac resistance



Let us recall the expression for electric field intensity at depth *z* from the surface of the conductor:

$$E_y = E_{y0} \exp -(\alpha + j\beta)z \quad \text{(recalled)}$$

(RF time dependence $j\omega t$ understood)

$$E_y = E_{y0} \exp -(\alpha + j\beta)z \exp(j\omega t)$$

E_{y0} = Electric field intensity amplitude at the surface of the conductor

Current density along y at depth z from the surface of the conductor

$$= \sigma E_{y0} \exp - (\alpha + j\beta) z$$

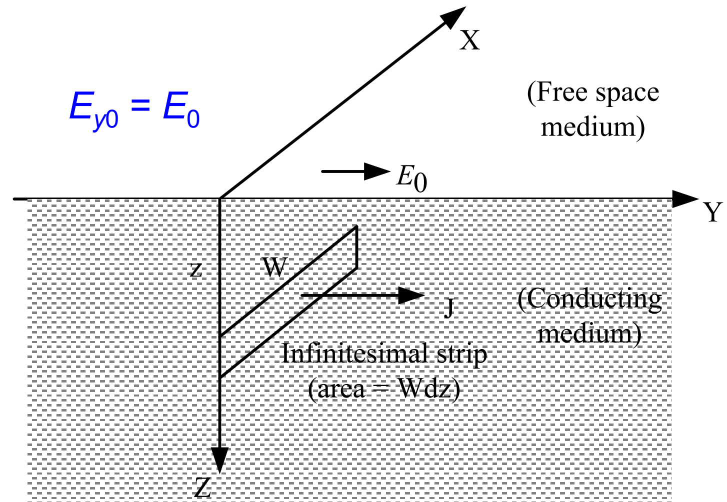
Current through the strip of infinitesimal thickness dz and width W

$$= [\sigma E_{y0} \exp - (\alpha + j\beta) z][W dz]$$

Current through the entire conductor of width W

$$= \int_0^{\infty} \sigma W E_{y0} \exp - (\alpha + j\beta) z dz$$

$$= \sigma W E_{y0} \left[\frac{\exp - (\alpha + j\beta) z}{-(\alpha + j\beta)} \right]_0^{\infty} = \frac{\sigma W E_{y0}}{\alpha + j\beta}$$



Current through the entire conductor of unit width ($W = 1$)

$$= \frac{\sigma E_{y0}}{\alpha + j\beta}$$

Surface impedance Z_s is defined as

$$Z_s = \frac{\text{Potential drop per 'unit' length at the surface of the conductor}}{\text{Current through the entire conductor of 'unit' width}}$$

E_{y0}

$\dfrac{\sigma E_{y0}}{\alpha + j\beta}$

$\alpha = 1/\delta$

$$= \frac{E_{y0}}{\dfrac{\sigma E_{y0}}{\alpha + j\beta}} = \frac{\alpha + j\beta}{\sigma} = \frac{\alpha}{\sigma} + j\frac{\alpha}{\sigma} = \frac{1}{\sigma\delta} + j\frac{1}{\sigma\delta} = R_s + jX_s$$

R_s = Surface resistance (Ω or Ω/)

X_s = Surface reactance (Ω or Ω/)

Comparing the real and imaginary parts

$$R_s = X_s = \frac{1}{\sigma\delta}$$

$$\delta = \sqrt{\frac{1}{\pi f \mu_0 \sigma}}$$

$$R_S = X_S = \sqrt{\frac{\pi f \mu_0}{\sigma}}$$

ac resistance of a straight conducting wire:

We can find the ac resistance of a straight round wire (that is, of circular cross section) made of a good conductor at high frequencies and compare it with its dc resistance to show that the ac-to-DC resistance ratio of the wire is $a/2\delta$, where *a* is the radius of the wire and δ is the skin depth of the conductor making the wire.

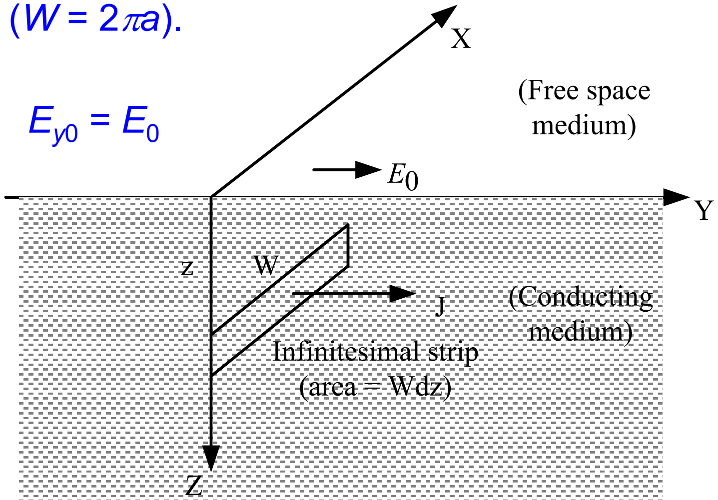
Expression for the dc resistance R_{dc} of a wire of length *l* and radius *a* is well known:

$$R_{\text{dc}} = \frac{1}{\sigma}\frac{l}{\pi a^2}$$

Let us next proceed to find an expression for the ac resistance R_{ac} of the wire using the same approach as used to find the surface resistance.

For a round wire of radius *a* large compared to the skin depth δ, a point inside the wire where the electric field is significant will not 'see' the curvature of the round wire and therefore we can take the surface of the wire as a planar surface.

Therefore, following exactly the same approach as used to find the surface resistance of a planar conductor we can find the resistance of the round wire interpreting the width as the circumference of the wire ($W = 2\pi a$).



Current through the wire of width $W = 2\pi a$

$$= \frac{\sigma E_{y0}}{\alpha + j\beta}(2\pi a)$$

Potential drop across length *l* at the surface of the wire

$$= E_{y0} l$$

$$\text{Impedance of the wire } Z_{ac} = \frac{\text{Potential drop across length } l \text{ at the surface of the wire}}{\text{Current through the wire of width } W = 2\pi a}$$

$$Z_{ac} = \frac{E_{y0} l}{\dfrac{\sigma E_{y0}}{\alpha + j\beta}(2\pi a)} = \frac{(\alpha + j\beta) l}{\sigma(2\pi a)} = \frac{\alpha l}{\sigma(2\pi a)} + \frac{j\beta l}{\sigma(2\pi a)} = R_{ac} + jX_{ac}$$

Separated in the real part R_{ac}, which is the ac resistance of the wire, and the imaginary part X_{ac}, which is the ac reactance of the wire

$$Z_{ac} = \frac{\alpha l}{\sigma(2\pi a)} + \frac{j\beta l}{\sigma(2\pi a)} = R_{ac} + jX_{ac}$$

$$\alpha = \beta = \frac{1}{\delta}$$

$$Z_{ac} = \frac{l}{\sigma\delta(2\pi a)} + \frac{jl}{\sigma\delta(2\pi a)} = R_{ac} + jX_{ac}$$

$$R_{ac} = X_{ac} = \frac{1}{\sigma\delta}\frac{l}{2\pi a}$$

$$R_{dc} = \frac{1}{\sigma}\frac{l}{\pi a^2}$$

$$\frac{R_{ac}}{R_{dc}} = \frac{\frac{1}{\sigma\delta}\frac{l}{2\pi a}}{\frac{1}{\sigma}\frac{l}{\pi a^2}} = \frac{a}{2\delta}$$

Since the wire radius $a \gg \delta$, skin depth, the wire ac resistance $R_{ac} \gg R_{dc}$, the wire dc resistance

ac resistance per unit length of a coaxial cable:

We can find the ac resistance of a coaxial cable made of a central solid round conductor and a coaxial annular conductor surrounding it with a dielectric medium filling the region between the conductors by extending the analysis presented for a straight round wire. For a good conductor making the coaxial cable and at high frequencies, as in the case of a straight wire, we can treat the inner and outer conductors behave as a planar surface.

$$R_{\mathrm{ac}} = \frac{1}{\sigma\delta}\frac{l}{2\pi a}$$

(recalled expression for the ac resistance of length *l* of a straight round wire)

Concept of finding the ac resistance of a straight wire extended to the inner conductor of radius *a* and to the outer conductor of radius *b*

$$R_{\mathrm{ac}} = \frac{1}{\sigma\delta}\frac{l}{2\pi a}$$

(ac resistance contributed by the inner conductor of length *l*)

$$R_{\mathrm{ac}} = \frac{1}{\sigma\delta}\frac{l}{2\pi b}$$

(ac resistance contributed by the outer conductor of length *l*)

$$R_{\mathrm{ac}} = \frac{1}{\sigma\delta}\frac{l}{2\pi a} + \frac{1}{\sigma\delta}\frac{l}{2\pi b} = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)l$$

(ac resistance of a coaxial cable of length *l*)

$$\text{Resistance per unit length } (l = 1) \text{ of a coaxial cable} = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

Wave propagation through sea water

The objective of the study is to find out which frequencies of operation (lower ~ 10 kHz or higher ~ 10 GHz) should be preferred from the standpoint of a lower attenuation of the wave propagating through sea water.

Maxwell's equations (recalled):

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \longleftarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} \longleftarrow \vec{J} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\downarrow$$

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$ (Maxwell's equations)

Partial time derivative and curl involving partial derivatives with respect to space coordinates are interchangeable

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \nabla \times \frac{\partial \vec{H}}{\partial t} = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \vec{H})$$

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$ (Maxwell's equations)

RF quantities vary as $\exp j(\omega t - \beta z)$

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \frac{\partial}{\partial t}(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t})$$

$$\frac{\partial}{\partial t} = j\omega$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -j\omega\mu_0 \sigma \vec{E} - \mu_0 \varepsilon (j\omega)(j\omega)\vec{E}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -j\omega\mu_0 (\sigma + j\omega\varepsilon)\vec{E}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla^2 \vec{E} = j\omega\mu_0 (\sigma + j\omega\varepsilon)\vec{E} = \gamma^2 \vec{E}$$

$$\gamma = \sqrt{j\omega\mu_0(\sigma + j\omega\varepsilon_0)} = \alpha + j\beta$$

$$\nabla^2 \vec{E} = j\omega\mu_0(\sigma + j\omega\varepsilon)\vec{E} = \gamma^2 \vec{E}$$

$$\gamma = \sqrt{j\omega\mu_0(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

$$\frac{\partial^2 E_{x,y}}{\partial x^2} + \frac{\partial^2 E_{x,y}}{\partial y^2} + \frac{\partial^2 E_{x,y}}{\partial z^2} = \gamma^2 E_{x,y}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

(uniform plane wave supposedly propagating along z)

$$\frac{\partial^2 E_{x,y}}{\partial x^2} = \gamma^2 E_{x,y}$$

$$\gamma = \sqrt{j\omega\mu_0(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

$$E_{x,y} = \hat{E}_{x,y} \exp(-\gamma z) = \hat{E}_{x,y} \exp[-(\alpha + j\beta)]z = \hat{E}_{x,y} \exp(-\alpha z)\exp(-j\beta z)$$

Let us examine two situations: one for $\sigma >> \omega\varepsilon$ and the other for $\sigma << \omega\varepsilon$ with reference to sea-water communication.

$$\left.\begin{aligned} \varepsilon &= 81\varepsilon_0 = 81 \times 8.854 \times 10^{-12}\ \mathrm{F/m} \\ \sigma &= 4\ \mathrm{mho/m} \end{aligned}\right\} \text{(sea water)}$$

Two alternative ways of putting the propagation constant γ

$$\gamma = \sqrt{j\omega\mu_0(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

$$\gamma = \sqrt{j\omega\mu_0\sigma(1 + \frac{j\omega\varepsilon}{\sigma})} = \alpha + j\beta$$

$$\gamma = j\omega\sqrt{\mu_0\varepsilon}(1 + \frac{\sigma}{j\omega\varepsilon})^{1/2}$$

Propagation at lower frequencies $f = 10\,\text{kHz} = 10\times10^3\ \text{Hz}\ \ (\text{typically})$

$$E_{x,y} = \hat{E}_{x,y}\exp(-\gamma z) = \hat{E}_{x,y}\exp[-(\alpha + j\beta)]z = \hat{E}_{x,y}\exp(-\alpha z)\exp(-j\beta z)$$

$$\frac{\omega\varepsilon}{\sigma} = \frac{2\pi f\varepsilon}{\sigma} \cong 1.1\times10^{-5}$$

$$f = 10\,\text{kHz} = 10\times10^3\ \text{Hz}\ \ (\text{typically})$$

$$\left.\begin{aligned} \varepsilon &= 81\varepsilon_0 = 81\times8.854\times10^{-12}\ \text{F/m} \\ \sigma &= 4\ \text{mho/m} \end{aligned}\right\}\ (\text{sea water})$$

Let us recall the following expression at lower frequencies:

$$\gamma = \sqrt{j\omega\mu_0\sigma(1 + \frac{j\omega\varepsilon}{\sigma})} = \alpha + j\beta$$

$$\gamma = \sqrt{j\omega\mu_0\sigma(1+\frac{j\omega\varepsilon}{\sigma})} = \alpha + j\beta$$ (recalled at lower frequencies)

$f = 10\,\text{kHz} = 10\times 10^3\,\text{Hz}$ (typically)

$\varepsilon = 81\varepsilon_0 = 81\times 8.854\times 10^{-12}\,\text{F/m}$
$\sigma = 4\,\text{mho/m}$
(sea water)

$$\frac{\omega\varepsilon}{\sigma} = \frac{2\pi f\varepsilon}{\sigma} \cong 1.1\times 10^{-5}$$ (recalled)

Ignoring the second term under the radical)

$$\gamma = \sqrt{j\omega\mu_0\sigma} = \alpha + j\beta$$ (separating the real and imaginary parts)

Separating the real and imaginary parts

$$\alpha_{\text{lf}} = \beta_{\text{lf}} = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

(recalled)
(obtained by separating the real and imaginary parts)
(subscript 'lf' referring to lower frequencies)

Propagation at higher frequencies $f = 10\,\text{GHz} = 10\times10^{9}\,\text{Hz}$ (typically)

Let us recall the following expression at higher frequencies:

$$\gamma = j\omega\sqrt{\mu_0\varepsilon}\left(1+\frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

$$\frac{\sigma}{\omega\varepsilon} = 10^{-1} \longleftarrow \begin{rcases} f = 10\,\text{GHz} = 10\times10^{9}\,\text{Hz} \\ \varepsilon = 81\varepsilon_0 = 81\times8.854\times10^{-12}\,\text{F/m} \\ \sigma = 4\,\text{mho/m} \end{rcases} \text{(sea water)}$$

Expanding binomially and ignoring higher order terms

$$\gamma = j\omega\sqrt{\mu_0\varepsilon}\left(1+\frac{1}{2}\frac{\sigma}{j\omega\varepsilon}\right) = \alpha + j\beta$$

$$j\omega\sqrt{\mu_0\varepsilon} + \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}} = \alpha_{\text{hf}} + j\beta_{\text{hf}}$$

(subscript 'hf' referring to quantities at higher frequencies)

$$j\omega\sqrt{\mu_0\varepsilon}+\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}}=\alpha_{\mathrm{hf}}+j\beta_{\mathrm{hf}}$$

$$\alpha_{\mathrm{hf}}=\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}}$$

$$\beta_{\mathrm{hf}}=\omega\sqrt{\mu_0\varepsilon}$$

$$\alpha_{\mathrm{lf}}=\sqrt{\frac{\omega\mu_0\sigma}{2}}$$

$$\frac{\alpha_{\mathrm{lf}}}{\alpha_{\mathrm{hf}}}=\frac{\sqrt{\frac{\omega\mu_0\sigma}{2}}}{\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}}}=\sqrt{\frac{2\omega\varepsilon}{\sigma}}=\sqrt{\frac{2\omega_{\mathrm{lf}}\varepsilon}{\sigma}}$$

(ω referring to lower frequencies)

$(\omega=2\pi f)$

$f=10\ \mathrm{kHz}=10\times10^{3}\ \mathrm{Hz}$ (lower frequency, typically)

$$\left.\begin{aligned}\varepsilon&=81\varepsilon_0=81\times8.854\times10^{-12}\ \mathrm{F/m}\\ \sigma&=4\ \mathrm{mho/m}\end{aligned}\right\}\ \text{(sea water)}$$

$$\frac{\alpha_{\mathrm{lf}}}{\alpha_{\mathrm{hf}}}\approx10^{-3}$$

Lower attenuation at lower frequencies than at higher frequencies

$$\beta_{\mathrm{lf}}=\sqrt{\frac{\omega\mu_0\sigma}{2}}$$

$$\lambda=\frac{2\pi}{\beta_{\mathrm{lf}}}=\frac{2\pi}{\sqrt{\frac{\omega\mu_0\sigma}{2}}}=2\pi\sqrt{\frac{2}{\omega\mu_0\sigma}}(=2\pi\delta_{\mathrm{skin}})\ \approx15\ \mathrm{m}$$

Lower frequencies are preferred to higher frequencies for sea-water communication for lower attenuation as well as for a reasonable antenna size typically of the order of half wavelength in sea water (as compared to the corresponding antenna size in free space).

Wave propagation in a medium of charged particles

Plasma oscillation

Refractive index of ionosphere

Space-charge waves on an electron beam

Cyclotron waves on an electron beam

Plasma oscillation

Consider an ensemble of electrons and positive ions maintaining overall charge neutrality

Consider the physical displacement (perturbation) of electrons (which are much more mobile than positive ions) from their equilibrium position to a small extent

Space-charge electric field in the direction of the displacement of negatively charged electrons $\Rightarrow$ providing a restoring force

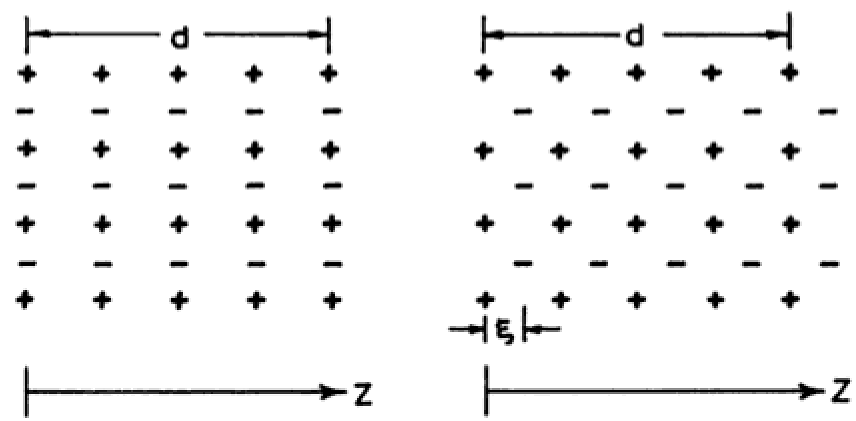
Overshoot of electrons

Restoring force again coming into play

$\Rightarrow$

Oscillation of electrons about their mean position at the natural angular frequency— electron plasma frequency.

Displacement of electron layers by an amount ξ and the resulting space-charge restoring force



$$\vec{E} = \left(\frac{|\rho_s|}{2\varepsilon_0}\right)(\vec{a}_z) + \left(\frac{-|\rho_s|}{2\varepsilon_0}\right)(-\vec{a}_z) = \frac{|\rho_s|}{\varepsilon_0}\vec{a}_z$$

(electric field due to positive-end layer)

(electric field due to negative-end layer)

ρ_s = surface charge density

ρ_0 = volume charge density

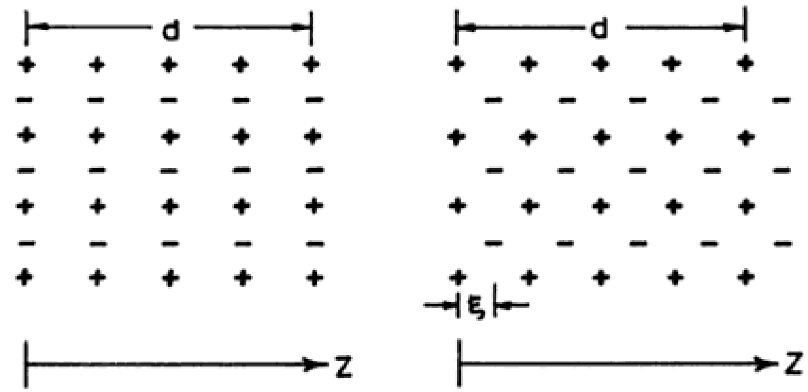
α = cross-sectional area perpendicular to displacement

$$|\rho_s|\alpha = \rho_0 \xi \alpha$$

$$|\rho_s| = \rho_0 \xi$$

$$\vec{E} = \frac{|\rho_s|}{\varepsilon_0}\vec{a}_z$$

Displacement of electron layers by an amount ξ and the resulting space-charge restoring force



(before perturbation) (after perturbation)

$$E_z = \frac{|\rho_0|\xi}{\varepsilon_0}$$

$$\eta = e/m$$

$$m\frac{d^2\xi}{dt^2} = eE_z \quad \text{(force equation)}$$

$$\frac{d^2\xi}{dt^2} = \eta E_z = \frac{\eta|\rho_0|}{\varepsilon_0}\xi = -\frac{|\eta||\rho_0|}{\varepsilon_0}\xi$$

$$\omega_p = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$

$$\frac{d^2\xi}{dt^2} = -\omega_p^2\xi$$

$$\frac{d^2\xi}{dt^2} = -\omega_p^2\,\xi \qquad \left(\omega_p = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}\right)$$

↓ Solving

$$\xi = A\exp(j\omega_p t) + B\exp(-j\omega_p t)$$ (*A* and *B* are constants)

↓

$$\xi = A(\cos\omega_p t + j\sin\omega_p t) + B(\cos\omega_p t - j\sin\omega_p t) = (A+B)\cos\omega_p t + j(A-B)\sin\omega_p t$$

↓

$$\xi = C\cos\omega_p t + D\sin\omega_p t$$

(*C* = *A*+*B* and *D* = *j(A-B)* are constants)

Set

$\xi = 0$ at $t = 0$ → $C = 0$

↓

$$\xi = D\sin\omega_p t$$

↓

Solution indicating that the electrons oscillate about their mean position with an angular frequency of oscillation ω_p called the plasma frequency (electron):

$$\omega_p = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$

Refractive index of ionosphere

The ionosphere is located at the heights of 50-300 km from the earth.
The ionosphere consists of electrons and ions
The ionization in the ionosphere is caused by solar radiation (ultraviolet, soft X-ray, α-particle (helium atoms from which the electrons are knocked off), etc.)
The sky-wave propagation utilizes the phenomenon of total internal reflection from the ionosphere, which, in turn, is dependent on the value of its refractive index.
We can find value of the **refractive index of the ionosphere** by studying the interaction of electromagnetic waves with the ionosphere.

The current density in a medium constituted by charged particles in a free- space medium consists of the convection current density (unlike the conduction current density in a conducting medium) and the displacement current density.

$$\vec{J} = \text{Convection current density} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J} = \rho \vec{v} + \frac{\partial \vec{D}}{\partial t}$$

The relation between the convection current density *J*, volume charge density ρ and velocity *v* of charged particles (electrons) has been obtained earlier (in Chapter 3 while studying Child-Langmuir's law) as:

$$J = \rho v.$$

We can write the relation in vector form as

$$\vec{J} = \rho \vec{v}$$

Current density equation:

Perturbed electron current density in the ionosphere under the influence of an interactive electromagnetic wave

$$\vec{D} = \varepsilon_0 \vec{E}$$

Time dependence as $\exp j\omega t$

$$\frac{\partial}{\partial t} \equiv j\omega$$

$$\vec{J} = \rho_0 \vec{v} + \frac{\partial \vec{D}}{\partial t} = \rho_0 \vec{v} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Perturbed convection current density obtainable from the relation $J = \rho v$ (electrons not having a dc beam velocity unlike in an electron beam)

Displacement current density

$$\vec{J} = \rho_0 \vec{v} + j\omega\varepsilon_0 \vec{E}$$ (current density equation)

Force equation:

$$m\frac{d\vec{v}}{dt} = e\vec{E}$$

$$m\frac{\partial \vec{v}}{\partial t} = e\vec{E}$$

$$\frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + \frac{dz}{dt}\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + v\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + (v_0 + v_1)\frac{\partial v_1}{\partial z}$$

$$= \frac{\partial v_1}{\partial t} + (v_0 + v_1)\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial z} + v_1\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial z}$$

$$= \frac{\partial v_1}{\partial t}$$

(ignoring $v_1 \frac{\partial v_1}{\partial z}$ which is of second order of importance)

($v_0 = 0$ for electrons of the ionosphere)

$$m\frac{\partial \vec{v}}{\partial t} = e\vec{E}$$ (force density equation) (recalled)

$$\frac{\partial}{\partial t} \equiv j\omega$$

$$j\omega m\vec{v} = e\vec{E}$$

$$\eta = \frac{e}{m}$$ (charge-to-mass ratio of an electron)

$$\vec{v} = \frac{\eta\vec{E}}{j\omega}$$

$$\vec{J} = \rho_0\vec{v} + j\omega\varepsilon_0\vec{E}$$ (current density equation) (recalled)

$$\vec{J} = j\omega\varepsilon_0\vec{E}[\rho_0\frac{\eta}{j\omega}\frac{1}{j\omega} + 1] = j\omega\varepsilon_0\vec{E}(1 - \frac{\eta\rho_0/\varepsilon_0}{\omega^2})$$

$$\omega_p = \sqrt{\frac{\eta\rho_0}{\varepsilon_0}} = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$ (recalled)

$$\vec{J} = j\omega\varepsilon_0(1 - \frac{\omega_p{}^2}{\omega^2})\vec{E}$$

$$\vec{J} = j\omega\varepsilon_{\text{effective}}\vec{E} \longrightarrow \varepsilon_{\text{effective}} = \varepsilon_0(1 - \frac{\omega_p{}^2}{\omega^2}) \longrightarrow \varepsilon_{r,\text{effective}} = \varepsilon_{\text{effective}}/\varepsilon_0 = 1 - \frac{\omega_p{}^2}{\omega^2}$$

Refractive index *n* of the ionosphere

$$n(=\sqrt{\varepsilon_{r,\text{effective}}}) = 1 - \frac{\omega_p{}^2}{\omega^2}$$ (refractive index of the ionosphere)

Space-charge waves on an electron beam

It is of interest to study the interaction of electromagnetic waves with a charge flow, for instance, a beam of electrons, which is a constituent of many practical electron devices such as vacuum electron devices like microwave tubes and charge accelerators. We are going to show that an electron beam supports two space-charge waves. Coupling of space-charge waves with electromagnetic waves is of relevance to understanding the behaviour of practical electron devices. Our study will be restricted here to finding the phase velocities of space-charge waves supported by an electron beam.

Current density equation

$J = \rho v$ (convection current density) ← One-dimensional beam flow

Let us consider a large cross-sectional area of the electron beam perpendicular to the beam flow along z over which the beam velocity *v*, volume charge density ρ and current density *J* remain constant:

$$\left.\begin{array}{l} \partial / \partial x = \partial / \partial y = 0 \\ \partial / \partial z \neq 0 \end{array}\right\}$$

$$\left.\begin{array}{l} J = J_0 + J_1 \\ \rho = \rho_0 + \rho_1 \\ v = v_0 + v_1 \\ J_0 = \rho_0 v_0 \end{array}\right\}$$

Subscript 0 refers to unperturbed quantities

Subscript 1 refers to perturbed quantities

$$\cancel{J_0} + J_1 = (\rho_0 + \rho_1)(v_0 + v_1) = \cancel{\rho_0 v_0} + \rho_0 v_1 + \rho_1 v_0 + \rho_1 v_1$$

← $J_0 = \rho_0 v_1$

$$J_1 = \rho_0 v_1 + \rho_1 v_0 + \rho_1 v_1$$

$$J_1 = \rho_0 v_1 + \rho_1 v_0 + \rho_1 v_1$$

Taking perturbed quantities much smaller than unperturbed quantities so that we can ignore the product of perturbed quantities $\rho_1 v_1$

$$J_1 = \rho_0 v_1 + v_0 \rho_1$$

Taking partial derivative with respect to *z*

$$(\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0)$$ (continuity equation)

$$\partial / \partial x = \partial / \partial y = 0, \quad \partial / \partial z \neq 0$$

$$\frac{\partial J_1}{\partial z} + \frac{\partial \rho_1}{\partial t} = 0$$

$$\frac{\partial J_1}{\partial z} = \rho_0 \frac{\partial v_1}{\partial z} + v_0 \frac{\partial \rho_1}{\partial z}$$

$$-\frac{\partial \rho_1}{\partial t} = \rho_0 \frac{\partial v_1}{\partial z} + v_0 \frac{\partial \rho_1}{\partial z}$$

$$\frac{\partial \rho_1}{\partial t} + v_0 \frac{\partial \rho_1}{\partial z} = -\rho_0 \frac{\partial v_1}{\partial z}$$

$$\mathrm{D} = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}$$

$$\mathrm{D}\rho_1 = -\rho_0 \frac{\partial v_1}{\partial z}$$ (to be recalled later in the analysis)

Force equation

Space-charge electric field was introduced in course of the deduction of electron plasma frequency of oscillation in a space-charge neutralised medium consisting of electrons and relatively immobile positive ions. In an ideal vacuum, it is not possible for electrons to move to form an electron beam because of mutual repulsive force between the advancing and the following electrons. However, in a practical vacuum electron beam device like a microwave tube where the vacuum is not ideal, the presence of charge neutralising positive ions makes it possible for the electrons to move and form an electron beam. The space-charge field E_s comes into play for any perturbation in the position of electrons from their equilibrium position.

$$m\frac{dv_1}{dt} = eE_s$$

$$\frac{dv_1}{dt} = \frac{e}{m}E_s = \eta E_s$$

$$\frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + \frac{dz}{dt}\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + v\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + (v_0 + v_1)\frac{\partial v_1}{\partial z}$$

$$= \frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial z} + v_1\frac{\partial v_1}{\partial z} = \frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial z}$$

(ignoring higher order term, being the product of perturbed quantities: $v_1 \partial v_1 / \partial z$

$$\frac{\partial v_1}{\partial t} + v_0\frac{\partial v_1}{\partial z} = \eta E_s \quad \longleftarrow \quad \mathrm{D} = \frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}$$

$\mathrm{D}v_1 = \eta E_s$ (to be recalled later in the analysis)

$$D\rho_1 = -\rho_0 \frac{\partial v_1}{\partial z}$$ (recalled)

Performing D operation

$$D = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}$$

$$Dv_1 = \eta E_s$$ (recalled)

$$D^2\rho_1 = -\rho_0 D\frac{\partial v_1}{\partial z} = -\rho_0 \frac{\partial}{\partial z} Dv_1 = -\rho_0 \frac{\partial}{\partial z}\eta E_s = -\eta\rho_0 \frac{\partial E_s}{\partial z} \qquad [\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z} = D]$$

(∂/z, partial derivative with respect to *z*, and D involving ∂/z. partial derivative with respect to *z*, and ∂/*t*, partial derivative with respect to *t*, being interchangeable)

(for a uniform plane wave propagating along *z:* $\partial/\partial x = \partial/\partial y = 0$)

$$D^2\rho_1 = -\eta\rho_0 \frac{\partial E_s}{\partial z} \longleftarrow \frac{\partial E_s}{\partial z} = \frac{\rho_1}{\varepsilon_0}$$ (following from Maxwell's equation: $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$)

$$D^2\rho_1 = -\eta\rho_0 \frac{\rho_1}{\varepsilon_0} = \frac{-\eta\rho_0}{\varepsilon_0}\rho_1 = \frac{-|\eta||\rho_0|}{\varepsilon_0}\rho_1 = -\omega_p{}^2\rho_1$$

$$\frac{|\eta||\rho_0|}{\varepsilon_0} = \omega_p{}^2 \longleftarrow \omega_p = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$

$$D^2 = -\omega_p{}^2$$

$$D = \pm j\omega_p$$

$\mathrm{D} = \pm j\omega_p$ (recalled)

Perturbed quantities vary as $\exp j(\omega t - \beta z)$

$$\mathrm{D} = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z} = j\omega - j\beta v_0 = j(\omega - \beta v_0)$$

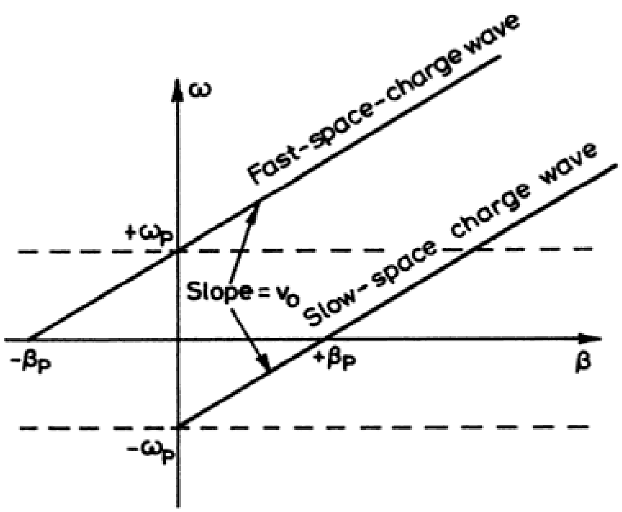
$$\left.\begin{array}{l} \dfrac{\partial}{\partial t} = j\omega \\ \dfrac{\partial}{\partial z} = -j\beta \end{array}\right\}$$

$\mathrm{D} = \pm j\omega_p$

$$\pm j\omega_p = j(\omega - \beta v_0)$$

$$\omega - \beta v_0 = \pm\omega_p$$

(dispersion relation of space-charge waves)

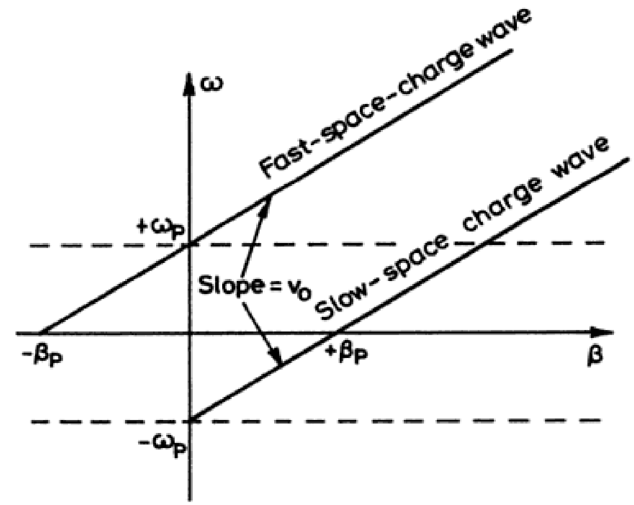


$$\omega - \beta v_0 = \pm\omega_p$$ (dispersion relation of space-charge waves) (recalled)

$$\frac{\omega}{v_0} = \beta_e \text{ (beam propagation constant)}$$

$$\beta = \frac{\omega \mp \omega_p}{v_0} = \beta_e \mp \beta_p$$

$$\frac{\omega_p}{v_0} = \beta_p \text{ (plasma propagation constant)}$$



$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \mp \omega_p} v_0$$

The upper sign ⇒ Fast space-charge wave

The lower sign ⇒ Slow space-charge wave

$$v_p = \left(\frac{\omega}{\omega - \omega_p}\right) v_0 > v_0$$

(phase velocity of fast space-charge wave)

$$v_p = \left(\frac{\omega}{\omega + \omega_p}\right) v_0 < v_0$$

(phase velocity of slow space-charge wave)

Cyclotron waves on an electron beam

An electron beam supports cyclotron waves in the presence of a dc magnetic field in the direction of beam flow, provided there also exist the components of beam velocity transverse to the magnetic field. The interaction of electromagnetic waves with an electron beam supporting cyclotron waves forms the basis of understanding the behaviour of electron beam devices such as the gyrotron for the generation of electromagnetic waves in the millimetre-wave frequency range, which finds application in industrial heating, material processing, plasma heating for thermonuclear power generation as well as in domestic microwave ovens.

Let us consider the electron beam along z and a uniform dc magnetic field present in the region of beam flow, also along z. The components of Lorentz forces exist transverse to the longitudinal magnetic field. Let us treat the problem in rectangular system of coordinates for a large cross-sectional area of the beam perpendicular to the axis of the beam and magnetic field (that is perpendicular to z direction)

$\eta = e/m$ carrying the negative sign of the electronic charge

$$\left.\begin{aligned} \frac{dv_{1x}}{dt} &= \frac{e}{m}(\vec{v}_1 \times \vec{B})_x = \eta(\vec{v}_1 \times \vec{B})_x \\ \frac{dv_{1y}}{dt} &= \frac{e}{m}(\vec{v}_1 \times \vec{B})_y = \eta(\vec{v}_1 \times \vec{B})_y \end{aligned}\right\}$$

subject to Lorentz force instead of force due to space-charge field E_s

$$\frac{dv_1}{dt} = \frac{e}{m}E_s = \eta E_s \quad \text{(recalled)}$$

(force equation for an electron subject to space-charge electric field)

$$\text{Lorentz force} = e\vec{v} \times \vec{B}$$

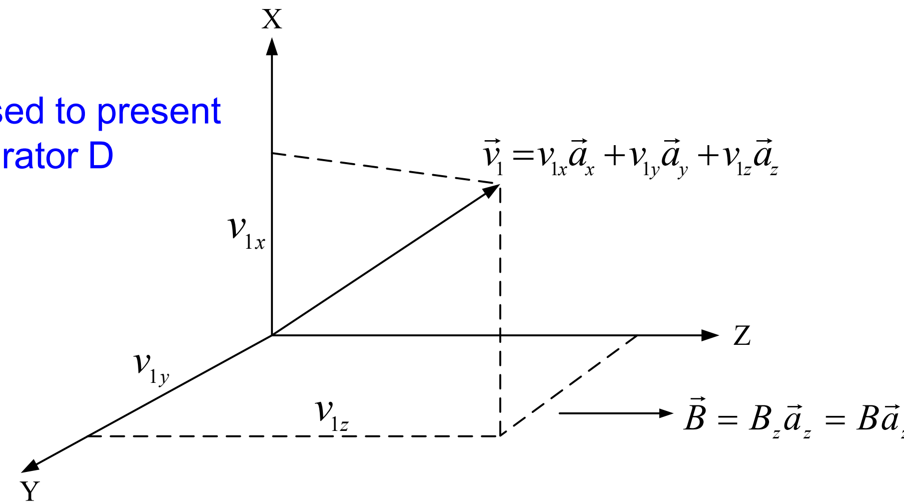
$$\left.\begin{aligned}\frac{dv_{1x}}{dt} &= \frac{e}{m}(\vec{v}_1 \times \vec{B})_x = \eta(\vec{v}_1 \times \vec{B})_x \\ \frac{dv_{1y}}{dt} &= \frac{e}{m}(\vec{v}_1 \times \vec{B})_y = \eta(\vec{v}_1 \times \vec{B})_y\end{aligned}\right\}$$

(recalled)

Recalling the same approach as used to present the expressions in terms of the operator D

$$\left.\begin{aligned}\mathrm{D}v_{1x} &= \eta(\vec{v}_1 \times \vec{B})_x \\ \mathrm{D}v_{1y} &= \eta(\vec{v}_1 \times \vec{B})_y\end{aligned}\right\}$$

$$\mathrm{D} = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}$$



$$\vec{v}_1 \times \vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ v_{1x} & v_{1y} & v_{1z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$B_x = 0,\ B_y = 0,\ B_z = B$$

(magnetic field *B* is along z)

$$\left.\begin{aligned}\mathrm{D}v_{1x} &= \eta B v_{1y} = -\omega_c v_{1y} \\ \mathrm{D}v_{1y} &= -\eta B v_{1x} = \omega_c v_{1x}\end{aligned}\right\}$$

$$\omega_c = -\eta B = |\eta| B$$

(electron cyclotron frequency introduced in Chapter 4)

$$\left.\begin{aligned} \mathrm{D}v_{1x} &= -\omega_c v_{1y} \\ \mathrm{D}v_{1y} &= \omega_c v_{1x} \end{aligned}\right\}$$ (recalled)

$$\mathrm{D} = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}$$

$$\mathrm{D}v_{1x} = -\omega_c v_{1y}$$

$$\mathrm{D}v_{1y} = \omega_c v_{1x}$$

Performing D operation

$$\mathrm{D}^2 v_{1x} = -\omega_c \mathrm{D}v_{1y} = -\omega_c(\omega_c v_{1x}) = -\omega_c^{\ 2} v_{1x}$$

$\mathrm{D}v_{1y} = \omega_c v_{1x}$ (recalled)

$$\mathrm{D}^2 = -\omega_c^{\ 2}$$

Compare it with

$\mathrm{D}^2 = -\omega_p^{\ 2}$ (recalled)

Led earlier to the derivation of the dispersion relation for space-charge waves

$$\omega - \beta v_0 = \pm\omega_p$$

(Dispersion relation for space-charge waves)

Following the same approach we can write the dispersion relation of cyclotron waves simply by replacing ω_p with ω_c

$$\omega - \beta v_0 \mp \omega_c = 0$$

(Dispersion relation for cyclotron waves)

$$\omega - \beta v_0 \mp \omega_c = 0$$

(Dispersion relation for cyclotron waves)

$$\omega - \beta v_0 = \pm \omega_p$$

(Dispersion relation for space-charge waves)

Led earlier to the derivation of the phase velocities of fast and slow space-charge waves

$$v_p = (\frac{\omega}{\omega - \omega_p}) v_0 > v_0$$

(phase velocity of fast space-charge wave)

and

$$v_p = (\frac{\omega}{\omega + \omega_p}) v_0 < v_0$$

(phase velocity of slow space-charge wave)

Following the same approach we can write the phase velocities of cyclotron waves simply by replacing ω_p with ω_c

$$v_p = (\frac{\omega}{\omega - \omega_c}) v_0 > v_0$$

(phase velocity of fast cyclotron wave)

and

$$v_p = (\frac{\omega}{\omega + \omega_c}) v_0 < v_0$$

(phase velocity of slow cyclotron wave)

Space-charge and cyclotron waves

Space-charge waves

$$\omega - \beta v_0 \mp \omega_p = 0 \qquad \omega_p = \sqrt{\frac{|\eta||\rho_0|}{\varepsilon_0}}$$

$$\beta = \frac{\omega}{v_p} = \frac{\omega}{v_0} \mp \frac{\omega_p}{v_0} = \frac{\omega \mp \omega_p}{v_0}$$

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \mp \omega_p} v_0$$

Cyclotron waves

$$\omega - \beta v_0 \mp \omega_c = 0 \qquad \omega_c = |\eta| B$$

$$\beta = \frac{\omega}{v_p} = \frac{\omega}{v_0} \mp \frac{\omega_c}{v_0} = \frac{\omega \mp \omega_c}{v_0}$$

$$v_p = \frac{\omega}{\omega \mp \omega_c} v_0$$

Upper sign for the fast wave and lower sign for the slow wave

Summarising Notes

✓Wave equations—one in electric field and the other in magnetic field—have been derived with the help of those two Maxwell's equations in which these field quantities are coupled.

✓Solutions to the wave equations in electric and magnetic fields have been obtained for uniform, plane electromagnetic waves propagating through an unbounded free-space medium.

✓With the help of Maxwell's equations and the solutions to wave equations in electric and magnetic fields, it has been established, with reference to uniform, plane electromagnetic waves propagating through an unbounded free-space medium, that

◊ there exists no components of electric and magnetic fields in the direction of propagation;

◊ directions of the electric field, magnetic field and wave propagation are mutually perpendicular to one another;

◊ transverse electromagnetic (TEM) mode of propagation is supported by the unbounded medium;

◊ intrinsic impedance and phase velocity of the wave are each related to the permeability and the permittivity of free space, the wave phase velocity being the speed of light *c*.

✓Concepts of the skin depth, surface resistance and ac or RF resistance of a medium have been developed by studying the propagation of uniform, plane electromagnetic waves propagating through a semi-infinite conducting medium, considering such waves to be incident from a free-space region to a planar conducting surface extending to infinity.

✓Ratio of the ac or RF resistance to dc resistance of a conducting wire of circular cross section becomes equal to the ratio of wire radius to twice skin depth, considering the wire to be of high conductivity and/or wave frequencies to be very high such that the skin depth of the conductor becomes small enough compared to the wire radius to make the planar conductor approximation for the conducting wire of circular cross section.

✓ Lower frequencies, say, ~10 kHz is preferred to higher frequencies, say, ~10 GHz, in view of comparatively lower attenuation of waves at such lower frequencies for sea-water communication as revealed by studying propagation through unbounded sea-water of finite conductivity and permittivity.

✓ Study of wave propagation through a medium of charged particles give the concepts of sky-wave propagations through ionosphere as well as those of space-charge waves and cyclotron waves on an electron beam.

Readers are encouraged to go through Chapter 6 of the book for more topics and more worked-out examples and review questions.