Engineering Electromagnetics Essentials

Chapter 3

Basic concepts of static electric fields

Objective

Recapitulation of basic concepts of static electric fields or electrostatics

Topics dealt with

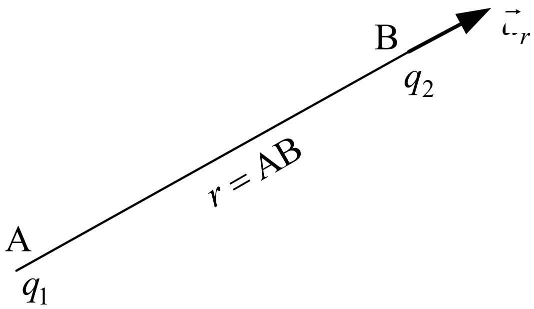
Coulomb's law
Gauss's law
Electric potential
Poisson's and Laplace's equations

Background

Vector calculus expressions developed in Chapter 2

Coulomb's law

$$\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon\, r^2}\,\vec{a}_r$$ [Newton or N]



q_1 : Point charge at A [Coulomb or C]

q_2 : Point charge at B [Coulomb or C]

ε : Permittivity of a medium [C^2/N-m^2 or F/m]

A: x_1, y_1, z_1

B: x_2, y_2, z_2

(That the unit of permittivity is F/m will become clear from the expression for the capacitance of a parallel-plate capacitor to be derived later, the unit of capacitance being Farad or F).

$$\vec{a}_r = \frac{\vec{r}}{r}$$

$$\vec{r} = (x_2 - x_1)\vec{a}_x + (y_2 - y_1)\vec{a}_y + (z_2 - z_1)\vec{a}_z$$ [metre or m]

$$r = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$ [metre or m]

$$\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon\, r^2}\,\vec{a}_r$$

Rationalized MKS system of unit is followed.

The rationalization factor 4π introduced in Coulomb force expression removes the appearance of this factor from many extensively used expressions derived from Coulomb force expression.

Relative permittivity or dielectric constant ε_r

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$$

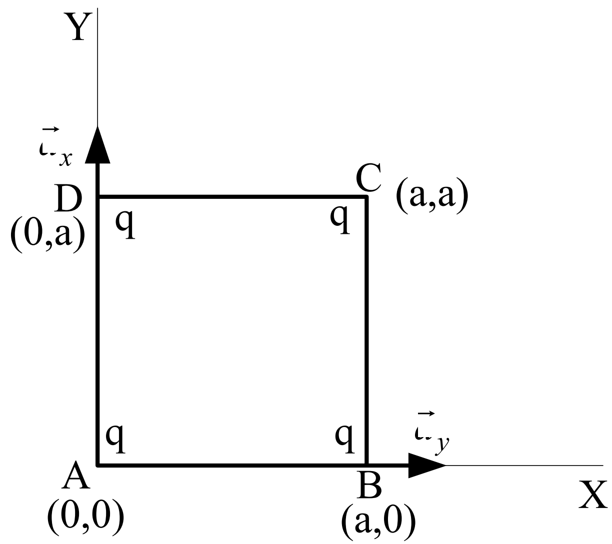
ε_0 : Permittivity of a free-space medium

$$\varepsilon_0 = 1/(36\pi)\times 10^{-9} = 8.854\times 10^{-12}$$ $C^2/N\text{-}m^2$ or F/m

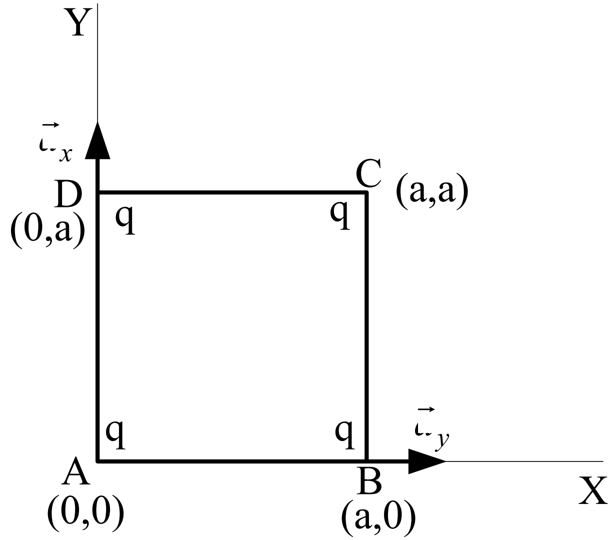
Coulomb force is maximum in a free-space medium.

Use Coulomb's law to find electrostatic force

The problem is to find the force on a point charge q placed at a corner A $(0,0)$ of a square on a plane due to three identical point charges each placed at the remaining three corners B $(a,0)$, C (a,a) and D $(0,a)$ of the square, respectively, located in rectangular system of coordinates, one of the corners, namely, A being at the origin of the coordinates



$\vec{F}_{\mathrm{A}}$ = Force on the point charge at A due to the point charges at B, C and D



$$\vec{F}_{\mathrm{A}} = \vec{F}_{\mathrm{BA}} + \vec{F}_{\mathrm{CA}} + \vec{F}_{\mathrm{DA}}$$

$$\left.\begin{aligned}
\vec{F}_{\mathrm{BA}} &= -\frac{(q)(q)}{4\pi\varepsilon_0 (\mathrm{a})^2}\vec{a}_x \\
\vec{F}_{\mathrm{CA}} &= -\frac{(q)(q)}{4\pi\varepsilon_0 (\sqrt{2}\mathrm{a})^2}\frac{(\vec{a}_x + \vec{a}_y)}{\sqrt{2}} \\
\vec{F}_{\mathrm{DA}} &= -\frac{(q)(q)}{4\pi\varepsilon_0 (\mathrm{a})^2}\vec{a}_y
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\vec{a}_{\mathrm{BA}} &= \frac{(0-a)\vec{a}_x + (0-0)a_y}{\mathrm{BA}} = \frac{-\mathrm{a}\vec{a}_x}{\mathrm{a}} = -\vec{a}_x \\
\vec{a}_{\mathrm{CA}} &= \frac{(0-a)\vec{a}_x + (0-a)a_y}{\mathrm{CA}} = \frac{-\mathrm{a}(\vec{a}_x + a_y)}{\sqrt{2}\mathrm{a}} = \frac{-(\vec{a}_x + \vec{a}_y)}{\sqrt{2}} \\
\vec{a}_{\mathrm{DA}} &= \frac{(0-0)\vec{a}_x + (0-a)a_y}{\mathrm{DA}} = \frac{-\mathrm{a}\vec{a}_y}{\mathrm{a}} = -\vec{a}_y
\end{aligned}\right\}$$

$$\vec{F}_{\mathrm{BA}} = -\frac{(q)(q)}{4\pi\varepsilon_0 (\mathrm{a})^2}\vec{a}_x;\ \vec{F}_{\mathrm{CA}} = -\frac{(q)(q)}{4\pi\varepsilon_0 (\sqrt{2}\mathrm{a})^2}\frac{(\vec{a}_x + \vec{a}_y)}{\sqrt{2}};\ \vec{F}_{\mathrm{DA}} = -\frac{(q)(q)}{4\pi\varepsilon_0 (\mathrm{a})^2}\vec{a}_y$$

$$\vec{F}_{\mathrm{BA}} + \vec{F}_{\mathrm{CA}} + \vec{F}_{\mathrm{DA}} = -\frac{q^2}{4\pi\varepsilon_0 \mathrm{a}^2}(1 + \frac{1}{(2\sqrt{2})})(\vec{a}_x + \vec{a}_y)$$

$$\vec{F}_{\mathrm{A}}(= \vec{F}_{\mathrm{BA}} + \vec{F}_{\mathrm{CA}} + \vec{F}_{\mathrm{DA}}) = \frac{q^2}{4\pi\varepsilon_0 \mathrm{a}^2}(\sqrt{2} + \frac{1}{2})[\frac{-(\vec{a}_x + \vec{a}_y)}{\sqrt{2}}]$$

Unit vector directed from C to A

$$\vec{F}_{\mathrm{A}} = \left[\frac{q^2}{4\pi\varepsilon_0 \mathrm{a}^2}(\sqrt{2} + \frac{1}{2})\right]\ \left[\vec{a}_{\mathrm{CA}}\right] \qquad (3.10)$$

(magnitude) (unit vector)

Magnitude

Unit vector directed from C to A

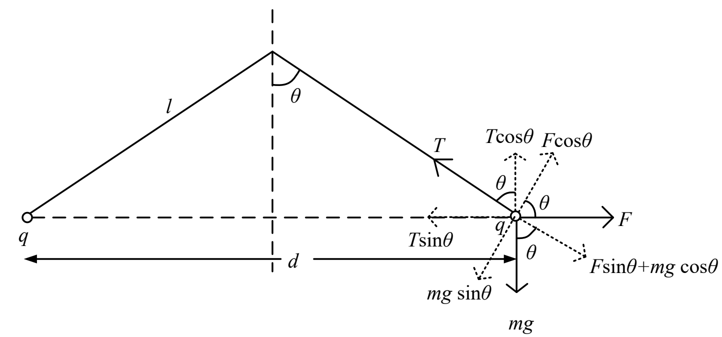
$\vec{a}_{\mathrm{CA}}$

Take up another problem for the use of Coulomb's law

Two small identical metallic charged balls, each of charge q and mass m = 1 gm, hung by a long thread of length l = 1 m from a common point (hook), get separated from each other by a distance d = 2 cm due to electrostatic repulsion and is set to an equilibrium on the vertical plane. Find the value of the charge q.

At equilibrium, the tension T in the string balances a component of the Coulomb force and a component of the gravitational force put together as follows:

$$T = F\sin\theta + mg\cos\theta.$$



Also, at equilibrium, a component of the gravitation force will balance a component of the Coulomb force to staify the following relation:

$$F\cos\theta = mg\sin\theta$$

$$\downarrow$$

$$\tan\theta = \frac{q^2}{4\pi\varepsilon_0 d^2 mg}$$

From geometry

$$\downarrow$$

$$\tan\theta = \frac{d/2}{\sqrt{(d/2)^2 + l^2}}$$

$$\tan\theta = \frac{q^2}{4\pi\varepsilon_0 d^2 mg}$$

$$\tan\theta = \frac{d/2}{\sqrt{(d/2)^2 + l^2}}$$

Equating the right hand sides

$$\frac{q^2}{4\pi\varepsilon_0 d^2 mg} = \frac{d/2}{\sqrt{(d/2)^2 + l^2}}$$

$$q = \left(\frac{2\pi\varepsilon_0 d^3 mg}{\sqrt{d^2/4 + l^2}}\right)^{1/2}$$

$$l >> d$$

$$q = \left(\frac{2\pi\varepsilon_0 d^3 mg}{l}\right)^{1/2}$$

Remembering $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

and putting *m* = 1 gm , *l* = 1 m and *d* = 2 cm, you can calculate each charge *q* as

$q = 2.088 \times 10^{-9}$ C $= 2.088$ nC.

Electric field due to a point charge and electric field due to a distribution of charges

Electric field due to a point charge

The force on a point charge q_2 due to the another point charge q_1, given by Coulomb's law, can be interpreted as the electric field caused by the point charge q_1.

$$\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon\, r^2}\vec{a}_r \quad \text{(Coulomb's law)}$$

$$\downarrow$$

$$\vec{F} = q_2 \vec{E}$$

$$\vec{E} = \frac{q_1}{4\pi\varepsilon\, r^2}\vec{a}_r \quad \longleftarrow$$

Electric field due to the point charge q_1

In general, the electric field due to point charge q is then given by

$$\vec{E} = \frac{q}{4\pi\varepsilon\, r^2}\vec{a}_r$$

In general, the electric field due to the point charge q is then given by

$$\vec{F} = q\vec{E}$$

Take up the following simple problem

An electron moves along x with a velocity 2 cm/s in a region of uniform electric field of 5 kV/cm along z. What is the force on the electron?

Answer:

Electron carries a negative charge and it may be regarded as a point charge *q:*

$$q = -|e| = -1.6\times10^{-19}\ \mathrm{C}$$

Electric field magnitude is given as 5 kV/cm along z:

$$E = 5\,\mathrm{kV/cm} = 5\times10^{3}\times10^{2} = 5\times10^{5}\ \mathrm{V/m}$$

$$\vec{E} = |e|E\vec{a}_z$$

Force on point charge q due to electric field is given by

$$\vec{F} = q\vec{E} = -|e|\vec{E} = -|e|E\vec{a}_z = -5\times10^{5}\times1.6\times10^{-19}\,\vec{a}_z = -8\times10^{-14}\,\vec{a}_z\ \mathrm{N}$$

Electric field due to a long uniform line-charge distribution

Charge is uniformly sprayed over a long line along z axis

Medium is a free-space medium

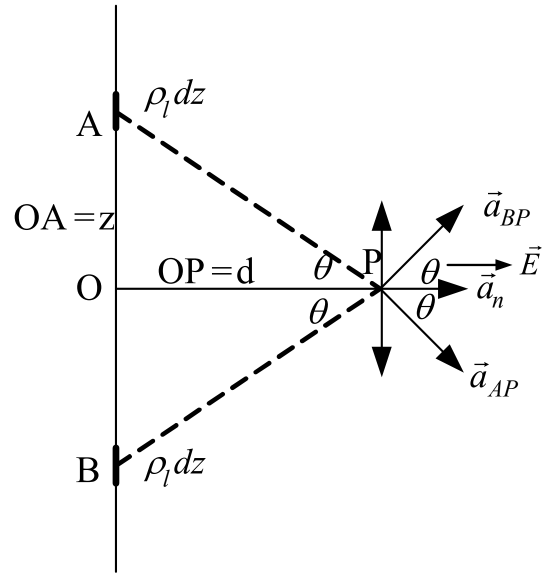
The length of the line is very large compared to the distance of the point

P is the point where the electric field is sought.

OP = *d* is the perpendicular distance of P from the long line charge distribution.

O is the foot of perpendicular drawn from P to the long line charge distribution.

A and B are the two points symmetrically located with respect to O on each of which equal amounts of line charge elements are considered (OA = OB).



$\rho_l dz =$ Charge element each on A and B to be regarded as point charges

$\rho_l =$ Charge per unit length or line charge density of the line charge distribution

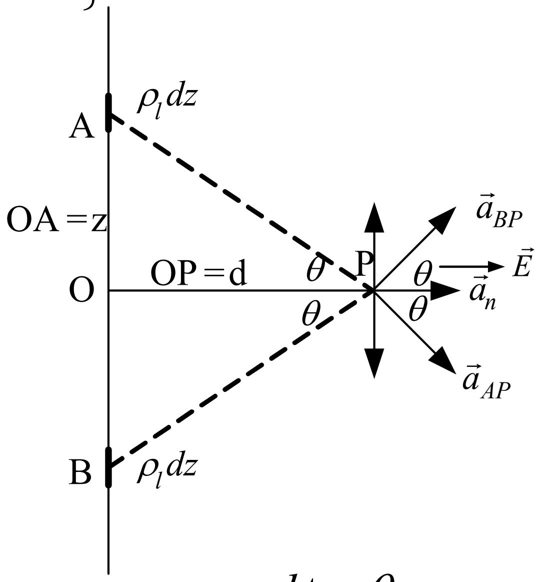
$$d\vec{E}_{\mathrm{AP}} = \frac{\rho_l dz}{4\pi\varepsilon_0 (\mathrm{AP})^2}\vec{a}_{\mathrm{AP}}$$

(Element of electric field at P due to charge element at A)

$$\left.\begin{aligned} z &= d\tan\theta \\ (\mathrm{AP})^2 &= (\mathrm{BP})^2 = d^2\sec^2\theta \end{aligned}\right\}$$

$$d\vec{E}_{\mathrm{BP}} = \frac{\rho_l dz}{4\pi\varepsilon_0 (\mathrm{BP})^2}\vec{a}_{\mathrm{BP}}$$

(Element of electric field at P due to charge element at B)



$\vec{a}_{\mathrm{AP}}$ = Unit vector directed from A to P

$\vec{a}_{\mathrm{BP}}$ = Unit vector directed from B to P

$$\mathrm{AP} = \mathrm{BP} = d\sec\theta$$

$$z = d\tan\theta$$

$$\downarrow$$

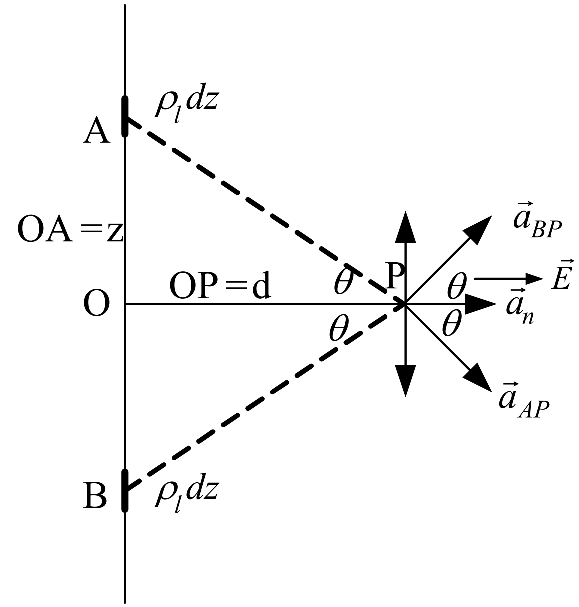
$$dz = d\sec^2\theta\, d\theta$$

$$dE_{\mathrm{AP}} = dE_{\mathrm{BP}} = \frac{\rho_l dz}{4\pi\varepsilon_0 d^2\sec^2\theta}$$

Find out the components of element of electric field, due to a charge element on the line charge distribution, in directions parallel and perpendicular to the line.

Parallel components of element of electric field at P due to the charge elements at A and B — equal and in opposite directions —cancel out.

Normal components of element of electric field at P due to the charge elements at A and B will add up in the direction of $\vec{a}_n$.



$$(dE_{\mathrm{AP}})_\perp = dE_{\mathrm{AP}}\cos\theta = \frac{\rho_l dz}{4\pi\varepsilon_0(\mathrm{AP})^2}\cos\theta$$

$$(dE_{\mathrm{BP}})_\perp = dE_{\mathrm{BP}}\cos\theta = \frac{\rho_l dz}{4\pi\varepsilon_0(\mathrm{AP})^2}\cos\theta$$

$$(dE_{\mathrm{AP}})_\perp = (dE_{\mathrm{BP}})_\perp$$

Elements of electric field component will add up in the direction of $\vec{a}_n$.

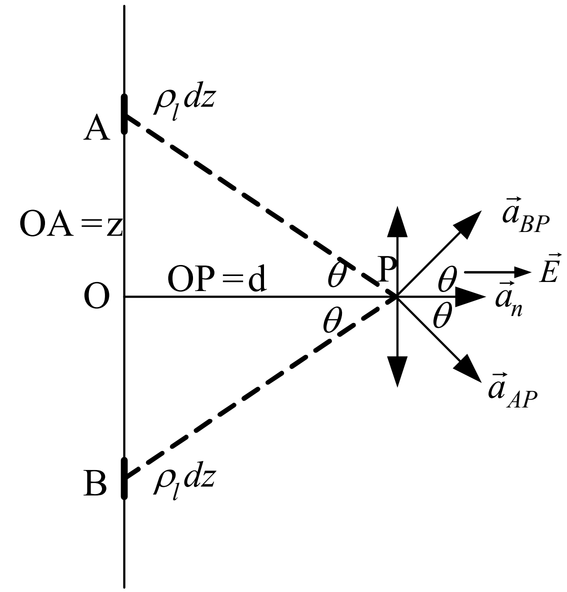
Elements of electric field component will add up in the direction of $\vec{a}_n$.

$$z = d\tan\theta$$
$$dz = d\sec^2\theta\, d\theta$$
$$(\mathrm{AP})^2 = d^2\sec^2\theta$$

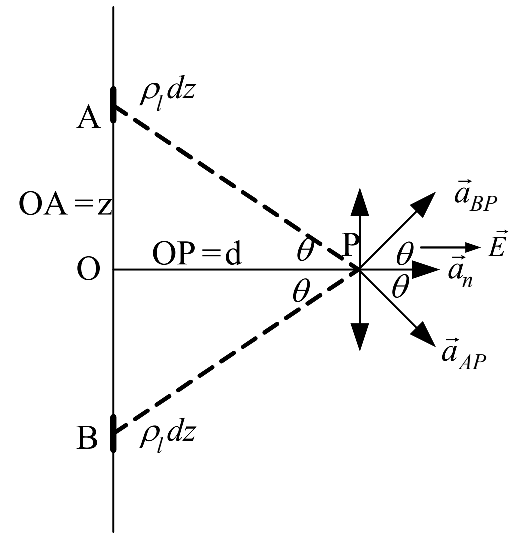
$$(dE_{\mathrm{AP}})_\perp = \frac{\rho_l dz}{4\pi\varepsilon_0(\mathrm{AP})^2}\cos\theta = \frac{\rho_l dz}{4\pi\varepsilon_0 d^2\sec^2\theta}\cos\theta$$

$$(dE_{\mathrm{AP}})_\perp = \frac{\rho_l d\sec^2\theta\, d\theta}{4\pi\varepsilon_0 d^2\sec^2\theta}\cos\theta = \frac{\rho_l}{4\pi\varepsilon_0 d}\cos\theta d\theta$$

For the length of charge distribution very large compared to the distance *d* of the point P, you can integrate the above expression between the limits *z* = 0 and ∞ and take twice its value to find the magnitude of the electric field in the direction of $\vec{a}_n$. The limits *z* = 0 corresponds to $\theta = 0$ and z = ∞ corresponds to $\theta = \pi/2$



$$(dE_{AP})_{\perp} = \frac{\rho_l d \sec^2\theta \, d\theta}{4\pi\varepsilon_0 d^2 \sec^2\theta} \cos\theta = \frac{\rho_l}{4\pi\varepsilon_0 d} \cos\theta d\theta$$



For the length of charge distribution very large compared to the distance *d* of the point P, you can integrate the above expression between the limits $z = 0$ and ∞ and take twice its value to find the magnitude of the electric field in the direction of $\vec{a}_n$. The limits $z = 0$ corresponds to $\theta = 0$ and $z = \infty$ corresponds to $\theta = \pi/2$

$$E = 2 \times \int_0^{\pi/2} \frac{\rho_l}{4\pi\varepsilon_0 d} \cos\theta d\theta = 2 \times \frac{\rho_l}{4\pi\varepsilon_0 d} \int_0^{\pi/2} \cos\theta d\theta$$

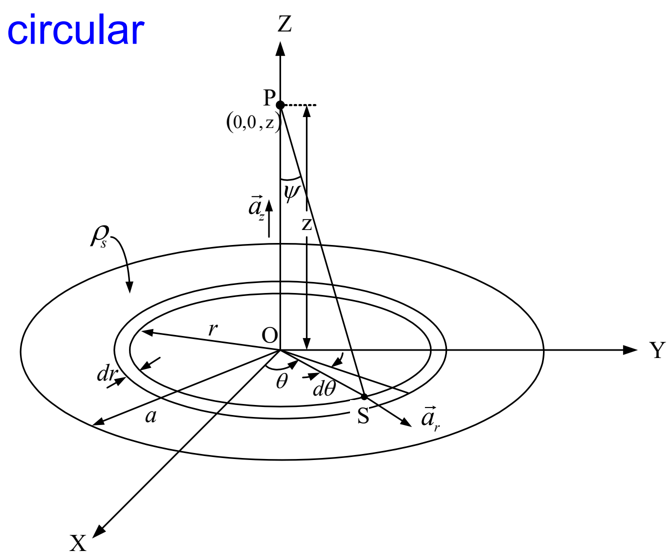
$$E = \frac{\rho_l}{2\pi\varepsilon_0 d} [\sin\theta]_0^{\pi/2} = \frac{\rho_l}{2\pi\varepsilon_0 d} (\sin\pi/2 - \sin 0) = \frac{\rho_l}{2\pi\varepsilon_0 d} (1 - 0) = \frac{\rho_l}{2\pi\varepsilon_0 d}$$

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0 d} \vec{a}_n$$ (Electric field near a long line charge distribution)

Alternatively, the above expression can be very easily derived by **Gauss's law** yet to be introduced.

Electric field due to a uniform planar surface-charge distribution with a circular boundary at a perpendicular distance from the centre of the circle

Charge is uniformly sprayed over XY plane within a circular boundary



$a =$ Radius of the circular boundary of charge distribution

$\rho_s =$ Uniform surface charge density

$r =$ Radius of an annular charge ring

$z =$ Perpendicular distance of $P(0,0,z)$ where to find the electric field

$= d\vec{E}'$

Approach:

Step 1: Find the sub-element of electric field at the point P due the sub-element of charge on an annular ring sub-element of infinitesimal radial thickness.

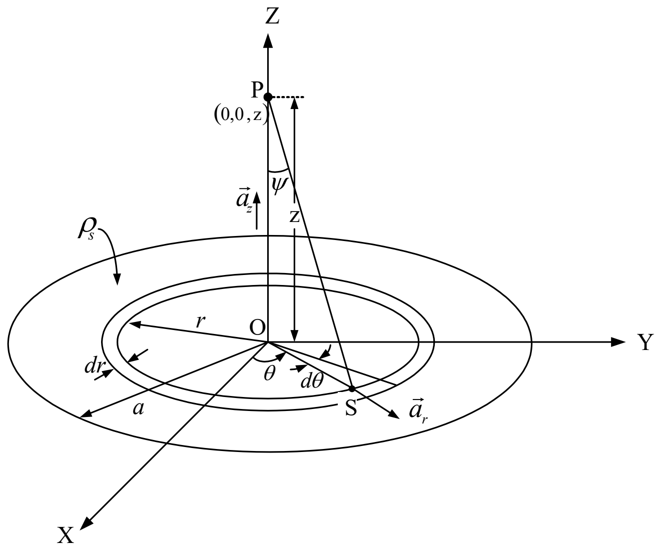
$$\Rightarrow d\vec{E}'$$

Step 2: Find the element of electric field at the point P due to the charge on the entire annular charge ring by integrating the sub-element of electric field obtained in Step 1.

$$\Rightarrow d\vec{E}$$

Step 3: Integrate the element of electric field obtained in Step 2 to find the electric field at P due to the entire circular charge distribution.

$$\Rightarrow \vec{E}$$

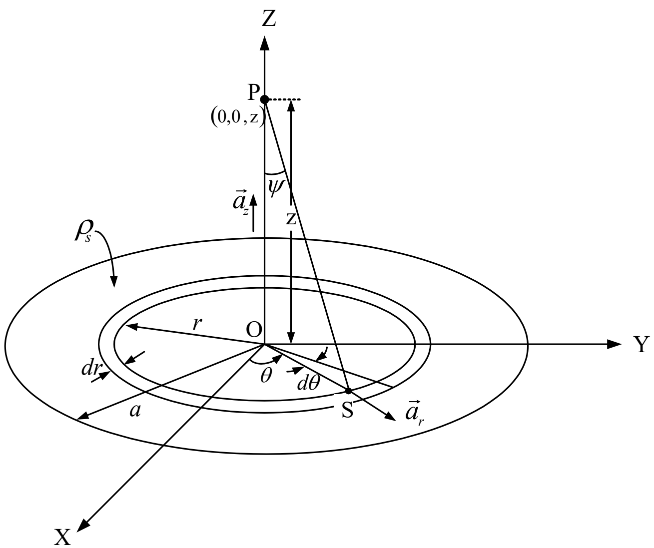


Step 1: $\Rightarrow d\vec{E}'$

ρ_s = Surface charge density

$rd\theta dr$ = Area of the ring sub-element at S

$\rho_s rd\theta dr$ = Charge on the ring sub-element at S



$$d\vec{E}' = \frac{\rho_s rd\theta dr}{4\pi\varepsilon_0 s^2}\vec{a}_{SP} \longleftarrow \vec{E} = \frac{q}{4\pi\varepsilon r^2}\vec{a}_r$$

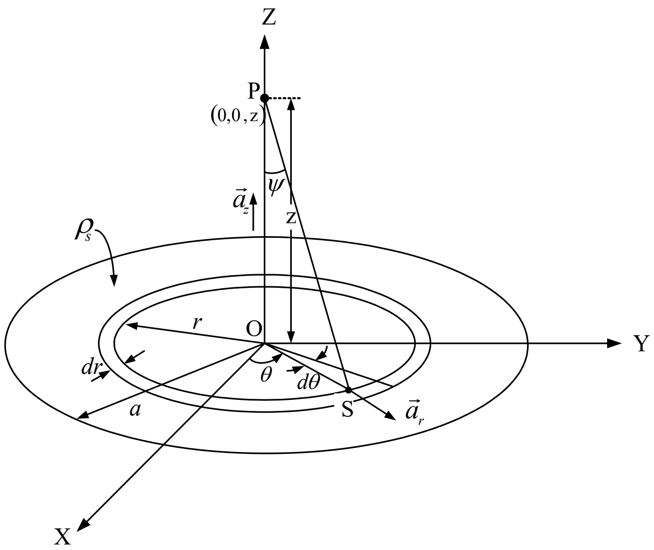
$q = \rho_s rd\theta dr$

$r = s = \mathrm{SP}$

$\vec{a}_r = \vec{a}_{SP}$

$$d\vec{E}' = \frac{\rho_s r d\theta dr}{4\pi\varepsilon_0 s^2}\vec{a}_{SP} \longleftarrow \vec{a}_{SP} = \frac{\overrightarrow{SP}}{SP} = \frac{\overrightarrow{SO}+\overrightarrow{OP}}{SP} = \frac{-r\vec{a}_r + z\vec{a}_z}{s}$$

$$d\vec{E}' = \left(\frac{\rho_s r d\theta dr}{4\pi\varepsilon_0 s^2}\right)\left(\frac{-r\vec{a}_r + z\vec{a}_z}{s}\right) = \left(\frac{\rho_s r^2 d\theta dr}{4\pi\varepsilon_0 s^3}\right)\vec{a}_r + \left(\frac{\rho_s rzd\theta dr}{4\pi\varepsilon_0 s^3}\right)\vec{a}_z$$



Step 2: $\Rightarrow d\vec{E}$

$$d\vec{E} = \int d\vec{E}' = \int\left(\frac{\rho_s r^2 d\theta dr}{4\pi\varepsilon_0 s^3}\right)\vec{a}_r + \left(\frac{\rho_s rzd\theta dr}{4\pi\varepsilon_0 s^3}\right)\vec{a}_z$$

$$\downarrow$$

$$d\vec{E} = \left(\frac{\rho_s r^2 dr}{4\pi\varepsilon_0 s^3}\right)\int_0^{2\pi} d\theta\vec{a}_r + \left(\frac{\rho_s rzdr}{4\pi\varepsilon_0 s^3}\right)\int_0^{2\pi} d\theta\vec{a}_z$$

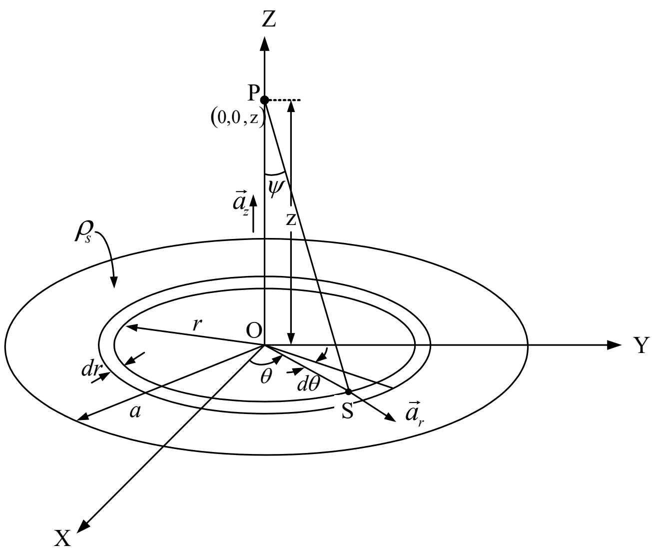
$$d\vec{E} = \left(\frac{\rho_s r^2 dr}{4\pi\varepsilon_0 s^3}\right)\int_0^{2\pi} d\theta \vec{a}_r + \left(\frac{\rho_s rzdr}{4\pi\varepsilon_0 s^3}\right)\int_0^{2\pi} d\theta \vec{a}_z$$

$\vec{a}_r$ changes with the position of S, that is, from one sub-element of charge to another on the annular charge ring since their directions change even though their magnitudes remain the same as unity. On the contrary $\vec{a}_z$ does not change so. Therefore,

$\vec{a}_z$ can be taken outside the integration while $\vec{a}_r$ cannot be so done.

$$d\vec{E} = \left(\frac{\rho_s r^2 dr}{4\pi\varepsilon_0 s^3}\right)\int_0^{2\pi} d\theta \vec{a}_r + \left(\frac{\rho_s rzdr}{4\pi\varepsilon_0 s^3}\right)\vec{a}_z \int_0^{2\pi} d\theta$$

The magnitudes of $\vec{a}_r$ are the same as unity but their directions are opposite at two diametrically opposite charge sub-elements. That makes the first integral vanish.



$$d\vec{E} = \left(\frac{\rho_s rzdr}{4\pi\varepsilon_0 s^3}\right)\vec{a}_z \int_0^{2\pi} d\theta$$

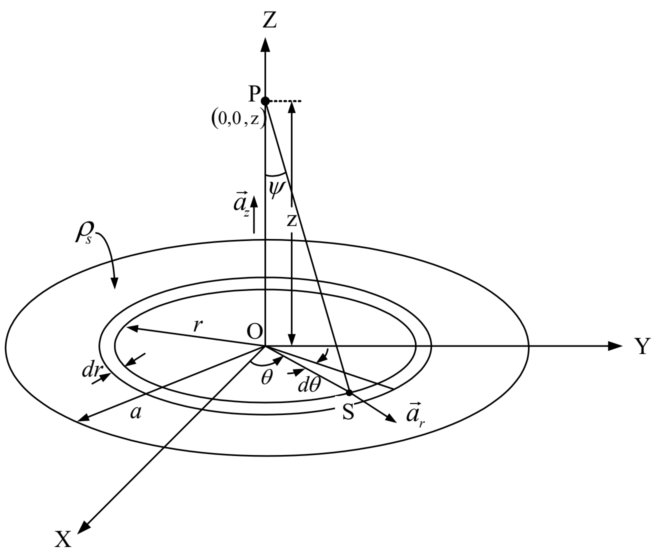
$$d\vec{E} = \left(\frac{\rho_s rzdr}{4\pi\varepsilon_0 s^3}\right)\vec{a}_z \times (2\pi - 0) =$$

$$\left(\frac{\rho_s rzdr}{4\pi\varepsilon_0 s^3}\right)\vec{a}_z \times 2\pi = \left(\frac{\rho_s rzdr}{2\varepsilon_0 s^3}\right)\vec{a}_z$$

$$d\vec{E} = \left(\frac{\rho_s rzdr}{2\varepsilon_0 s^3}\right)\vec{a}_z = \left(\frac{\rho_s z}{2\varepsilon_0}\right)\left(\frac{rdr}{s^3}\right)\vec{a}_z$$

$$s = z\sec\psi$$
$$r = z\tan\psi$$
$$dr = z\sec^2\psi d\psi$$

$$d\vec{E} = \left(\frac{\rho_s}{2\varepsilon_0}\right)\sin\psi\, d\psi\, \vec{a}_z$$



Step 3: $\Rightarrow \vec{E}$

The upper limit of integration corresponds to the annular charge ring at the periphery of circular charge distribution at $r = a$.

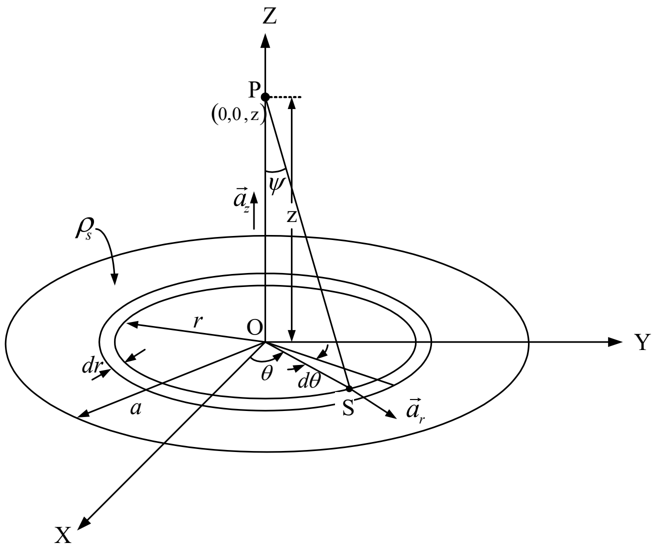
The lower limit of integration corresponds to the annular charge ring at the centre of circular charge distribution at $r = 0$.

$$\vec{E} = \int d\vec{E} = \left(\frac{\rho_s}{2\varepsilon_0}\right)\int_0^{\tan^{-1} a/z}\sin\psi\, d\psi\, \vec{a}_z$$

$$\vec{E} = \left(\frac{\rho_s}{2\varepsilon_0}\right)[-\cos\psi]_0^{\tan^{-1} a/z}\vec{a}_z = -\left(\frac{\rho_s}{2\varepsilon_0}\right)\left[\cos\left(\tan^{-1}\frac{a}{z}\right) - \cos 0\right]\vec{a}_z$$

$$\vec{E} = \left(\frac{\rho_s}{2\varepsilon_0}\right)\left[-\cos\psi\right]_0^{\tan^{-1} a/z}\vec{a}_z = -\left(\frac{\rho_s}{2\varepsilon_0}\right)\left[\cos\left(\tan^{-1}\frac{a}{z}\right) - \cos 0\right]\vec{a}_z$$

$$\vec{E} = -\left(\frac{\rho_s}{2\varepsilon_0}\right)\left(\frac{z}{\sqrt{a^2+z^2}} - 1\right)\vec{a}_z = \left(\frac{\rho_s}{2\varepsilon_0}\right)\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)\vec{a}_z$$



$z << a$

The electric field near a large area of charge distribution—large sheet of charge— becomes

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\vec{a}_z \quad (z << a)$$

Interpreting $\vec{a}_z = \vec{a}_n$

Electric field near a large sheet of charge becomes

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\vec{a}_n$$

Alternatively, the above expression can be very easily derived by **Gauss's law** yet to be introduced.

Gauss's law

Life becomes simple while finding electric field in problems, if you can do away with the integrations involved in the application of Coulomb's law to such problems!

For problems that enjoy geometrical symmetry—rectangular, cylindrical or spherical —one can apply Gauss's law to make the solution to such problem very simple. You can avoid carrying out complex integrations to find electric field which would be otherwise necessary if you would apply Coulomb's law to such problems.

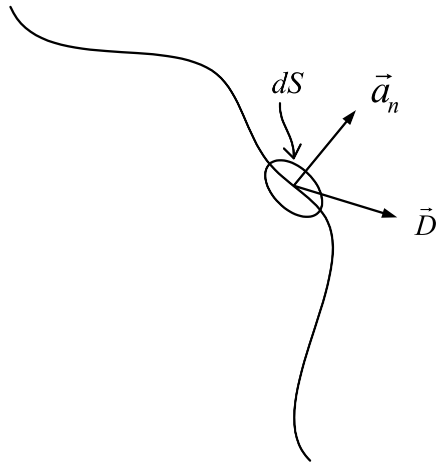
Let us state Gauss's law and prove it starting from Coulomb's law.

Gauss's law can be stated in terms of electric field and electric displacement.

Electric displacement $\vec{D}$ is related to electric field $\vec{E}$.

$$\vec{D} = \varepsilon \vec{E}$$

Element of flux of electric displacement $= d\phi_D$



$d\phi_D =$ Element of flux of electric displacement

$dS =$ Element of area

$\vec{a}_n =$ Unit vector normal to the element of area

$$d\phi_D = \vec{D} \cdot d\vec{S} = \vec{D} \cdot \vec{a}_n dS$$

Flux of electric displacement $= \phi_D$

$$\phi_D = \int_S d\phi_D = \int_S \vec{D} \cdot d\vec{S} = \int_S \vec{D} \cdot \vec{a}_n dS$$

Flux of electric displacement through a closed volume $= \phi_D$

$$\phi_D = \oint_S \vec{D} \cdot d\vec{S} = \oint_S \vec{D} \cdot \vec{a}_n dS \longleftarrow$$ Involves closed surface integral

Flux of electric displacement through a closed volume $= \phi_D$

$$\phi_D = \oint_S \vec{D} \cdot d\vec{S} = \oint_S \vec{D} \cdot \vec{a}_n dS \longleftarrow$$ Involves closed surface integral

Gauss's law can be stated in terms of ϕ_D

Gauss's law states that the outward flux of electric displacement $\vec{D}$ through the surface of a volume enclosing a charge Q —estimated by a closed surface integral of $\vec{D}$ over the area of the volume enclosure—is equal to the charge Q enclosed, .

Gauss's law

$$\phi_D = \oint_S \vec{D} \cdot d\vec{S} = \oint_S \vec{D} \cdot \vec{a}_n dS = Q \longleftarrow$$ Outward flux of electric displacement $= \phi_D$

$\vec{D} = \varepsilon \vec{E}$ $(\varepsilon = \text{medium permittivity})$

$$\phi_E = \oint_S \vec{E} \cdot d\vec{S} = \oint_S \vec{E} \cdot \vec{a}_n dS = \frac{Q}{\varepsilon} \longleftarrow$$ Outward flux of electric displacement $= \phi_E$

$Q =$ Charge enclosed

- *Let us prove Gauss's law*

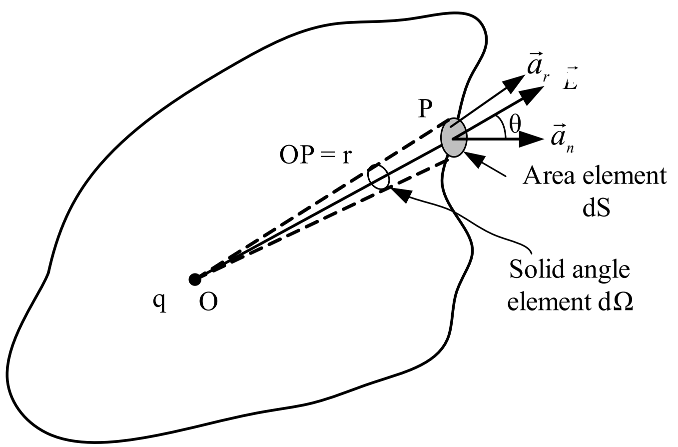
- *Let us apply the law to problems that enjoy geometrical symmetry—rectangular, cylindrical or spherical*

Proof of Gauss's law

$$\vec{D} = \varepsilon \vec{E}$$

Take a point charge q inside a volume enclosure

$$\vec{E} = \frac{q}{4\pi\varepsilon\, r^2}\vec{a}_r$$



$$\vec{D} = \varepsilon \vec{E} = \varepsilon \frac{q}{4\pi\varepsilon\, r^2}\vec{a}_r = \frac{q}{4\pi\, r^2}\vec{a}_r$$

$$\phi_D = \oint_S \vec{D}\cdot d\vec{S} = \oint_S \vec{D}\cdot \vec{a}_n dS$$

$$\phi_D = \oint_S \frac{q}{4\pi\, r^2}\vec{a}_r \cdot \vec{a}_n dS$$

$$\vec{a}_r \cdot \vec{a}_n = \cos\theta$$

$$\phi_D = \oint_S \vec{D}\cdot d\vec{S} = \oint_S \frac{q}{4\pi\, r^2}\cos\theta dS$$

$\vec{a}_r$ is directed from O to P.

$$\phi_D = \oint_S \vec{D}\cdot d\vec{S} = \oint_S \vec{D}\cdot\vec{a}_n dS$$

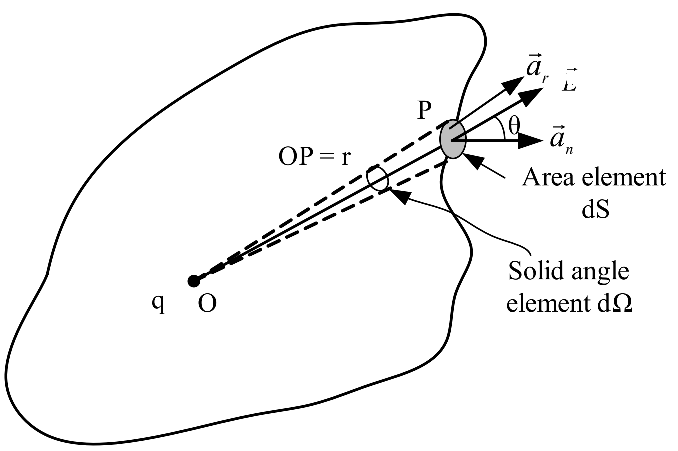
$$\frac{dS\cos\theta}{r^2} = d\Omega$$

(Solid angle element subtended by area element at P)

$$\phi_D = \oint_S \vec{D}\cdot\vec{a}_n dS = \oint_S \frac{q}{4\pi r^2}\cos\theta dS = \frac{q}{4\pi}\oint_S \frac{dS\cos\theta}{r^2}$$

$$\phi_D = \oint_S \vec{D}\cdot\vec{a}_n dS = \frac{q}{4\pi}\oint_S d\Omega = \left(\frac{q}{4\pi}\right)(4\pi) = q$$

$$\oint_S \vec{D}\cdot\vec{a}_n dS = q$$ (Gauss's law)



Charge q inside a volume enclosure

What will be the outward flux of electric displacement if a number of point charges, instead of a single point charge *q*, are present within the volume enclosure?

Outward flux ϕ_D of electric displacement if a number of point charges q_1, q_2, q_3, ……q_n,, instead of a single point charge q, are present within the volume enclosure

$$\phi_D = \frac{q}{4\pi}\oint_S d\Omega = (\frac{q}{4\pi})(4\pi) = q$$

$$\phi_D = \oint_S \vec{D}\cdot\vec{a}_n dS = \frac{q_1}{4\pi}\oint_S d\Omega_1 + \frac{q_2}{4\pi}\oint_S d\Omega_2 + \frac{q_3}{4\pi}\oint_S d\Omega_3 + \ldots\ldots\frac{q_n}{4\pi}\oint_S d\Omega_n$$

$$\oint_S d\Omega_1 = \oint_S d\Omega_2 = \oint_S d\Omega_3 = \ldots\ldots = \oint_S d\Omega_n = 4\pi$$

$d\Omega_1, d\Omega_2 d\Omega_3, \ldots.. d\Omega_n$ are elements of solid angle subtended by point charges $q_1 + q_2 + q_3 + \ldots\ldots q_n$ respectively.

$$\phi_D = \oint_S \vec{D}\cdot\vec{a}_n dS = \frac{q_1}{4\pi}4\pi + \frac{q_2}{4\pi}4\pi + \frac{q_3}{4\pi}4\pi + \ldots\ldots\frac{q_n}{4\pi}4\pi$$

$$= q_1 + q_2 + q_3 + \ldots\ldots q_n = Q$$

$$\phi_D = \oint_S \vec{D}\cdot d\vec{S} = \oint_S \vec{D}\cdot\vec{a}_n dS$$

$$\oint_S \vec{D}\cdot\vec{a}_n dS = Q$$ (Gauss's law)

$$\vec{D} = \varepsilon\vec{E}$$

$$\oint_S \vec{E}\cdot\vec{a}_n dS = \frac{Q}{\varepsilon}$$ (Gauss's law)

You have thus proved Gauss's law)!

Gauss's in terms of volume charge density

$$\phi_D = \oint_S \vec{D}\cdot\vec{a}_n dS \;= Q = q_1 + q_2 + q_3 + \ldots\ldots q_n$$

$$Q = q_1 + q_2 + q_3 + \ldots\ldots q_n$$

ρ is regarded constant over the volume element $d\tau$

$$Q = q_1 + q_2 + q_3 + \ldots\ldots q_n == \int_\tau \rho d\tau$$

$$\phi_D = \oint_S \vec{D}\cdot\vec{a}_n dS \;= Q = q_1 + q_2 + q_3 + \ldots\ldots q_n = \int_\tau \rho d\tau$$

$$\oint_S \vec{D}\cdot\vec{a}_n dS \;= \int_\tau \rho d\tau$$

$$\oint_S \vec{E}\cdot\vec{a}_n dS \;= \frac{\int_\tau \rho d\tau}{\varepsilon}$$

(Gauss's law in terms of volume charge density)

Application of Gauss's law to a problem that enjoys rectangular symmetry

Find the electric field at a point near a large planar sheet of charge of uniform surface charge density.

Approach:

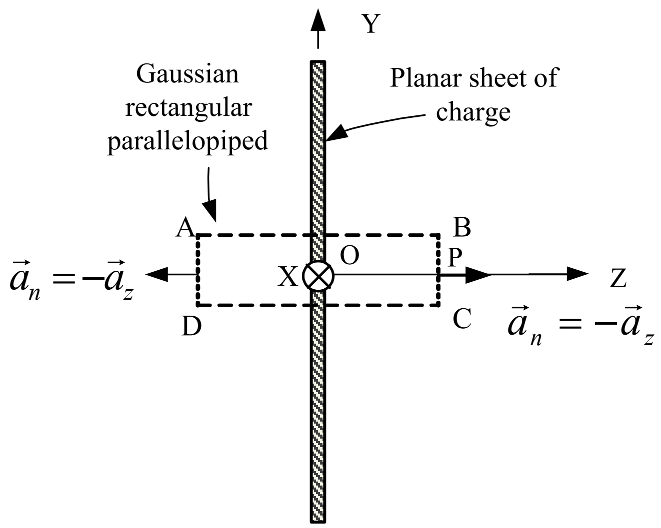
Step 1: Appreciate that the direction of electric field at a point P near the sheet of charge is in the direction perpendicular to the the plane of charge in the direction of $\vec{a}_n$.

In other words the electric field is along Z axis: $\vec{a}_z = \vec{a}_n$.

Step 2: Find the magnitude of electric field *E* at P using Gauss's law.

Srep 3: $\vec{E} = E\vec{a}_n$.

.



(b)

Cross section of planar charge sprayed over a very large area on XY plane. X is perpendicular to the plane of the paper directed away from the reader. Gaussian volume (rectangular parallelepiped is also shown.

Consider on the planar sheet of charge, supposedly positive, a pair of identical charge elements each treated as a point charge symmetrically placed with respect to the point P.

Appreciate the following with the help of Coulomb's law,

(i) The magnitudes of electric field at P due to the pair of charge elements are equal.

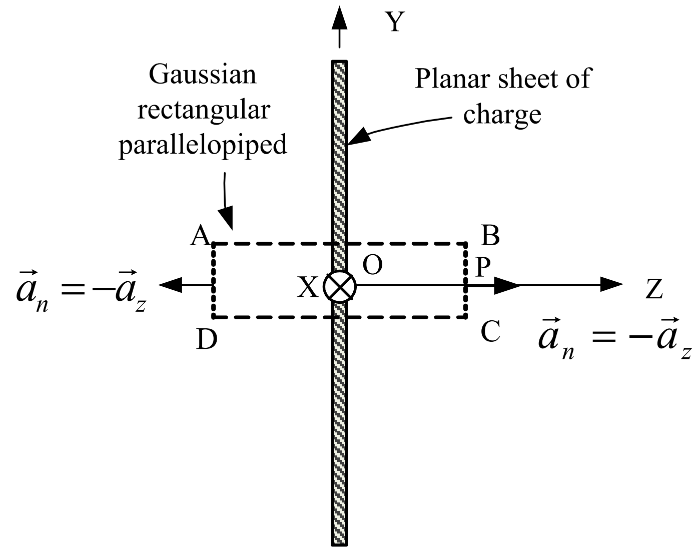
(ii) The direction of electric field at P due to a charge element of the pair of charge elements is from the charge element to the point.

(iii) The components of electric field due to the charge elements of the pair parallel to the plane of the sheet of charge are equal in magnitude and opposite in direction and they cancel out.

(iv) The components of electric field due to the charge elements of the pair perpendicular to the plane of the sheet of charge add up in the perpendicular direction.

$$\downarrow$$

$$\vec{E} = E\vec{a}_n \quad \text{(Step 1)}$$



(b)

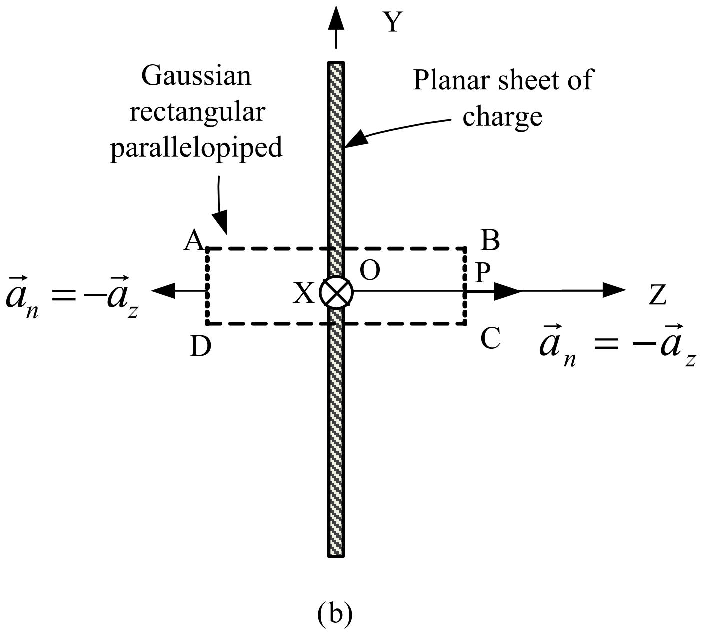
Cross section of a planar sheet of charge and Gaussian volume (rectangular parallelepiped)

The areas of the faces AD (left) and BC (right) are equal and each much smaller than the area of the large sheet of charge considered. Therefore, the magnitude of electric field can be regarded as constant over each of these faces.

The faces AD (left) and BC (right) are equidistant from the sheet of charge. Following step 1

$$\text{Electric field} = E\vec{a}_n \text{ (right face BC)}$$

$$\text{Electric field} = -E\vec{a}_n \text{ (left face AD)}$$

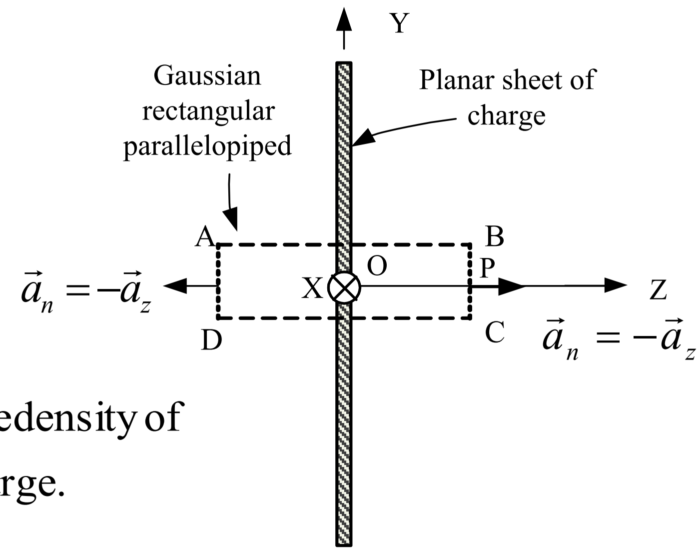


Cross section of a planar sheet of charge and Gaussian volume (rectangular parallelepiped)

Left face AD and right face BC of closed rectangular parallelepiped Gaussian volume are equidistant from the planar charge distribution. Right face BC passes through P.

$\vec{a}_n$ = unit vector outwardly normal to right face BC

$-\vec{a}_n$ = unit vector outwardly normal to left face AD



(b)

ρ_s is the surface charge density of the planar sheet of charge.

$$\oint_S \vec{E}\cdot\vec{a}_n dS = \frac{Q}{\varepsilon}$$ Gauss's law

$$Q = \rho_s S$$

S = area of AD = area of BC

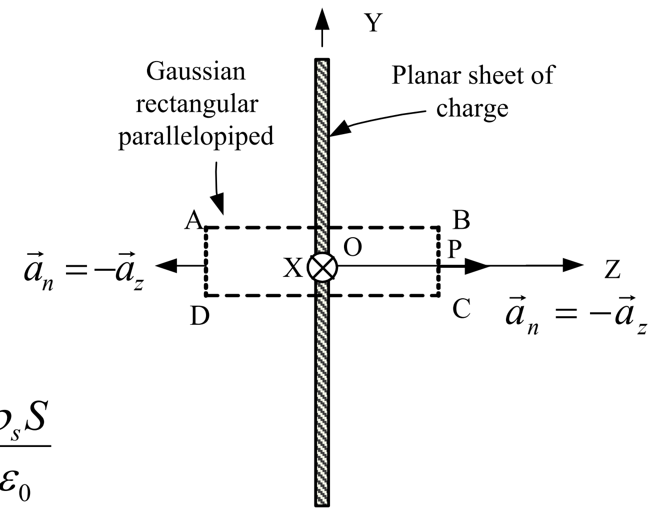
= area of sheet of charge enclosed

$$\int_{S(\text{right face})} E\vec{a}_n\cdot\vec{a}_n dS + \int_{S(\text{left face})} (-E\vec{a}_n)\cdot(-\vec{a}_n dS) = \frac{\rho_s S}{\varepsilon_0}$$

Cross section of a planar sheet of charge and Gaussian volume (rectangular parallelepiped) passing through P

Upper and lower faces of Gaussian parallelepiped do not contribute to the surface integral since the electric field is parallel to these faces being perpendicular to the sheet of charge.

The areas of the faces AD (left) and BC (right) are equal and each much smaller than the area of the large sheet of charge considered. Therefore, the electric field can be regarded as constant over each of these faces and taken outside the integral.



(b)

Cross section of a planar sheet of charge and Gaussian volume (rectangular parallelepiped) passing through P

$$\int_{S\,(\text{right face})} E\vec{a}_n \cdot \vec{a}_n dS + \int_{S\,(\text{left face})} (-E\vec{a}_n)\cdot(-\vec{a}_n dS) = \frac{\rho_s S}{\varepsilon_0}$$

$$\int EdS + \int (EdS) = 2E\int dS) = \frac{\rho_s S}{\varepsilon_0}$$

$$E = \frac{\rho_s}{2\varepsilon_0}$$ (Step 2)

(ρ_s is the surface charge density)

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\vec{a}_n$$ (Step 3)

(Electric field near a large planar sheet of charge)

Find the electric field at a point near a large planar conductor of uniform surface charge density.

Follow the same approach and the symbols as those used to find the electric field near a large planar sheet of charge.

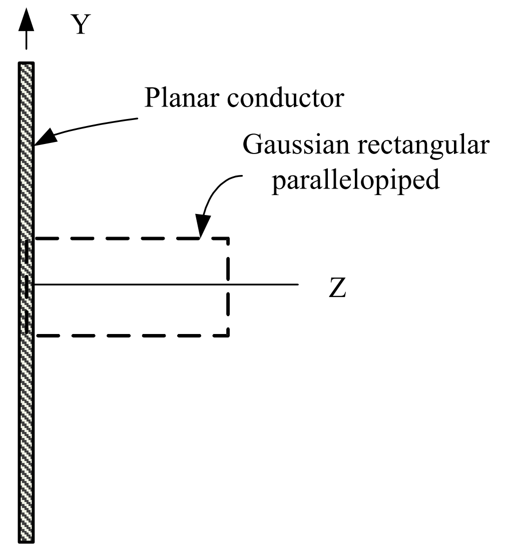
$$\oint_S \vec{E}\cdot\vec{a}_n dS = \frac{Q}{\varepsilon} \quad \text{(Gauss's law)}$$

$$(\varepsilon = \varepsilon_0 = \text{free} = \text{space permittivity})$$

$$\int_{S(\text{right face})} E\vec{a}_n\cdot\vec{a}_n dS + \int_{S(\text{left face})} (-E\vec{a}_n)\cdot(-\vec{a}_n dS) = \frac{\rho_s S}{\varepsilon_0}$$

Left face of parallelepiped passes through the conductor where the electric field *E* is absent and the integrand of the second term is null. Therefore, the second term is null,

$$\int_{S(\text{right face})} E\vec{a}_n\cdot\vec{a}_n dS = \frac{\rho_s S}{\varepsilon_0}$$



Gaussian rectangular parallelepiped enclosure with its left face passing through the conductor and its right face through the point where to find the electric field

$$\int_{S(\text{right face})} E\vec{a}_n \cdot \vec{a}_n dS = \frac{\rho_s S}{\varepsilon_0}$$

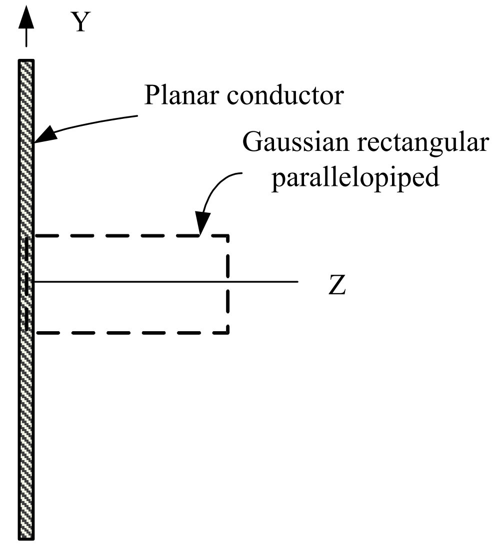
$$\int EdS = E\int dS = ES = \frac{\rho_s S}{\varepsilon_0}$$

$$\downarrow$$

$$E = \frac{\rho_s}{\varepsilon_0} \quad (\rho_s \text{ is the surface charge density})$$

$$\vec{E} = E\vec{a}_n = \frac{\rho_s}{\varepsilon_0}\vec{a}_n$$

(Electric field near a large charged planar conductor)

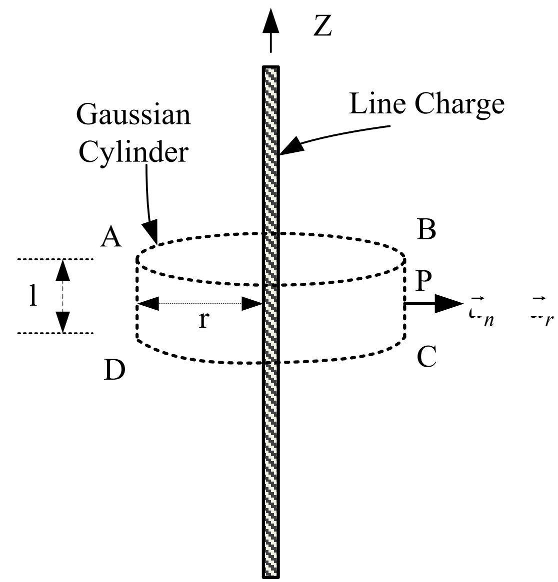


Gaussian rectangular parallelepiped enclosure with its left face passing through the conductor and its right face through the point where to find the electric field

Application of Gauss's law to a problem that enjoys cylindrical symmetry

Find the electric field near a long line charge distribution of uniform line charge density.

Follow the same approach as that used to find the electric field near a large planar sheet of charge.



$$\oint_S \vec{E}\cdot\vec{a}_n dS = \frac{Q}{\varepsilon}$$ (Gauss's law)

$$\vec{E} = E\vec{a}_n = E\vec{a}_r$$

$$\vec{a}_n = \vec{a}_r$$

$$\oint_S \vec{E}\cdot\vec{a}_n dS = \oint_S E\vec{a}_r\cdot\vec{a}_r dS = \oint_S E dS$$

$$\oint_S E dS = \frac{Q}{\varepsilon}$$

$$Q = \rho_l l = \text{charge enclosed}$$

$$(\rho_l = \text{charge per unit length or line charge density})$$

$$\oint_S E dS = \frac{\rho_l l}{\varepsilon}$$

$$\oint_S E dS = \frac{\rho_l l}{\varepsilon} \longleftarrow \oint_S E dS = \frac{\rho_l l}{\varepsilon}$$

$$E \oint_S dS = \frac{\rho_l l}{\varepsilon} \longleftarrow \oint_S dS = 2\pi r l$$

$$E 2\pi r l = \frac{\rho_l l}{\varepsilon} \longleftarrow (\varepsilon = \varepsilon_0 = \text{free} = \text{space permittivi ty})$$

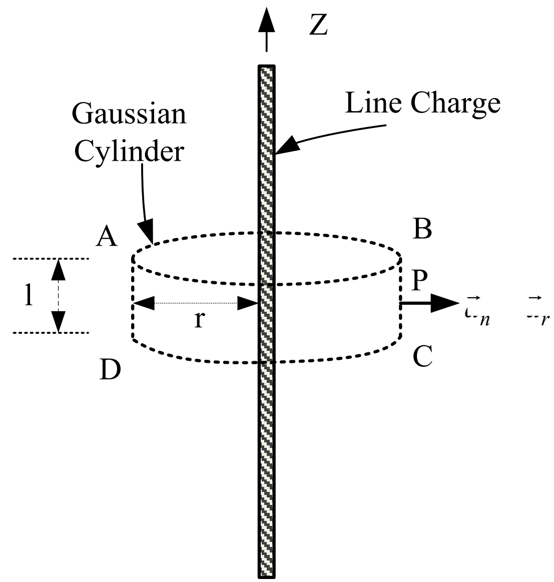
$$\downarrow$$

$$E = \frac{\rho_L}{2\pi\varepsilon_0 r}$$

$$\vec{E} = E\vec{a}_n = E\vec{a}_r$$

$$\downarrow$$

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\vec{a}_n \quad (\rho_l = \text{charge per unit length or line charge density})$$



(Electric field near a long line charge distribution)

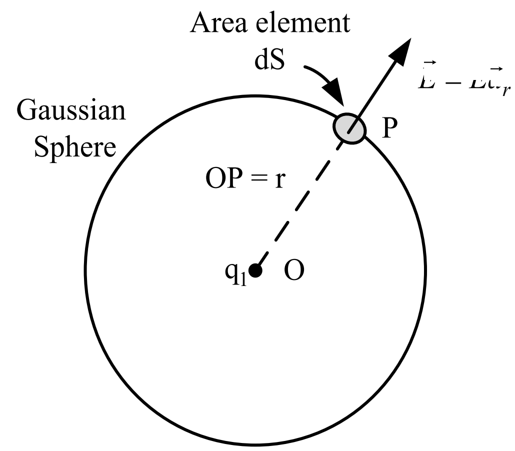
Application of Gauss's law to a problem that enjoys spherical symmetry

Find the electric field due to a point charge.

$$\oint_S \vec{E}\cdot\vec{a}_n dS = \frac{Q}{\varepsilon}$$ (Gauss's law)

$$\vec{E} = E\vec{a}_n = E\vec{a}_r$$

$$\vec{a}_n = \vec{a}_r$$



$$\oint_S \vec{E}\cdot\vec{a}_n dS = \oint_S E\vec{a}_r\cdot\vec{a}_r dS = \oint_S E dS$$

$$\oint_S E dS = \frac{Q}{\varepsilon}$$

$$Q = q_1 = \text{charge enclosed}$$

$$E\oint_S dS = \frac{q_1}{\varepsilon}$$

$$\oint_S dS = 4\pi r^2$$

$$E = \frac{q_1}{4\pi\varepsilon r^2}$$

$$\vec{E} = E\vec{a}_r$$

$$\vec{E} = \frac{q_1}{4\pi\varepsilon r^2}\vec{a}_r$$ (Electric field due to a point charge)

Electric field and potential

Gravitational potential energy

Work required to be done by spending energy to lift a particle to a height from the ground (surface of the earth) against the gravitational force is stored in the form of gravitational potential energy.

The ground surface is taken to be at zero reference gravitational potential.

Electric potential energy

Work required to be done by spending energy to move a charged particle—a point charge Q—from infinity against the force due to an electric field to a point is stored in the form of electric potential energy W .

Taking the zero reference potential taken as the potential at infinity, the potential V at the point is given by

$$V = \frac{W}{Q} \text{ (J/C or V)}$$

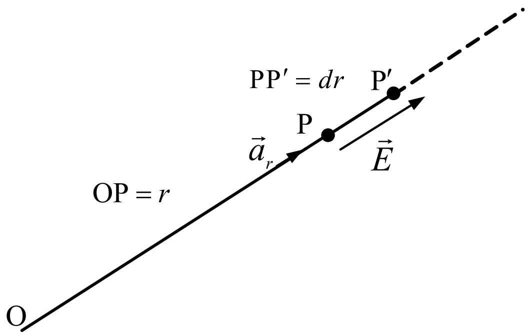
The potential at a point is thus numerically equal to the potential energy of a unit point charge placed at the point.

Potential at a point due to a point charge

$V =$ Potential at the point P due to point charge q at O at a distance r (OP $= r$)

$\vec{E} =$ Electric field at P due to point charge q at O at a distance r (OP $= r$)

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\vec{a}_r$$ ($\vec{a}_r$ is the unit vector directed from O to P)



$\vec{F} =$ Force on the point charge Q supposedly positive at P

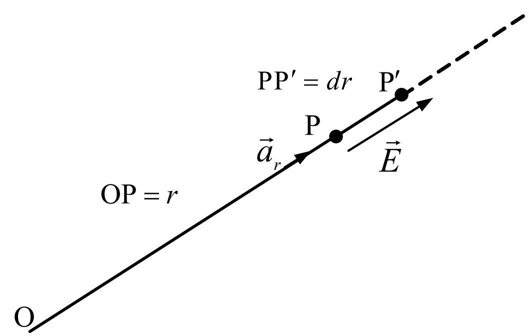
$$\vec{F} = Q\vec{E} = Q\frac{q}{4\pi\varepsilon_0 r^2}\vec{a}_r = F\,\vec{a}_r$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\vec{a}_r$$

$$F = \frac{Qq}{4\pi\varepsilon_0 r^2}$$

$dW =$ Element of work required to be done to move Q from P′ to P through an element of distance dr

$$dW = F dr = \frac{Qq}{4\pi\varepsilon_0 r^2} dr \quad \leftarrow \quad F = \frac{Qq}{4\pi\varepsilon_0 r^2}$$



Work W required to be done to move the point charge Q from infinity up to the point P is

$$W = \int_r^\infty \frac{Qq}{4\pi\varepsilon_0 r^2} dr = \frac{Qq}{4\pi\varepsilon_0} \int_r^\infty \frac{1}{r^2} dr = \frac{Qq}{4\pi\varepsilon_0} \left[-\frac{1}{r} \right]_r^\infty = \frac{Qq}{4\pi\varepsilon_0} \frac{1}{r} = \frac{Qq}{4\pi\varepsilon_0 r}$$

$$V = \frac{W}{Q} = \frac{\dfrac{Qq}{4\pi\varepsilon_0 r}}{Q} = \frac{q}{4\pi\varepsilon_0 r}$$ (Potential at a distance r from the point charge q)

Electric field as negative gradient of potential

A and B are nearby points separated by element of distance AB=*dr*

The potential is *V* at every point on the equipotential *V*.

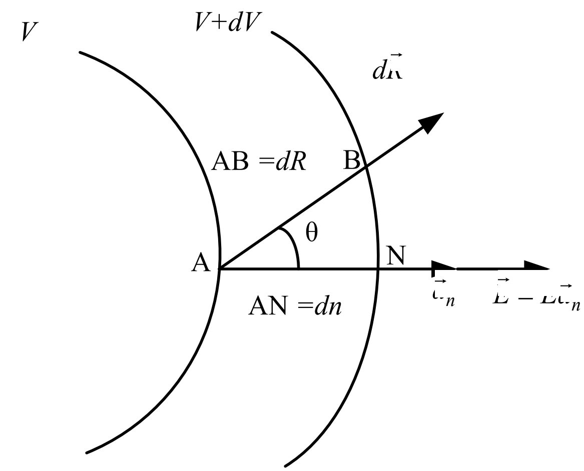
The potential is *V*+*dV* at every point on the equipotential *V*+*dV*.

The same electric field exists at A and B and the same force is experienced by a point charge at A and B.

$\vec{F} = Q\vec{E}$ (Force on a point charge at A or B)

$V_A = V$ (potential at A)

$V_B = V + dV$ (potential at B)



$QV_A = QV$

$QV_B = Q(V + dV)$

$\longleftarrow \quad V = \dfrac{W}{Q}$

$dW = QV_A - QV_B = QV - Q(V + dV) = -QdV$

Work is required to be done to move the charge Q from B to A if $V_A > V_B$ or dV is negative that makes dW positive.

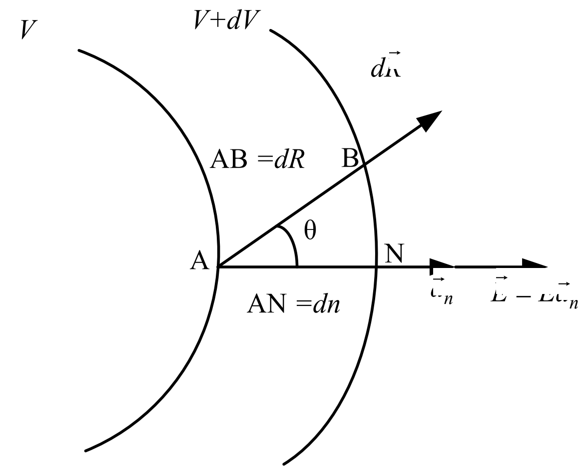
$$dW = -QdV \qquad dW = \vec{F} \cdot d\vec{R}$$

$$-QdV = \vec{F} \cdot d\vec{R} \qquad \vec{F} = Q\vec{E}$$

$$-QdV = Q\vec{E} \cdot d\vec{R}$$

$$dV = -\vec{E} \cdot d\vec{R} \qquad dV = \frac{\partial V}{\partial n} dn$$

$$\frac{\partial V}{\partial n} dn = -\vec{E} \cdot d\vec{R} \qquad dn = dR\cos\theta = \vec{a}_n \cdot d\vec{R}$$



$$\frac{\partial V}{\partial n} \vec{a}_n \cdot d\vec{R} = -\vec{E} \cdot d\vec{R}$$

$$\frac{\partial V}{\partial n} \vec{a}_n = -\vec{E}$$

$$\vec{E} = -\frac{\partial V}{\partial n} \vec{a}_n \qquad \nabla V = \frac{\partial V}{\partial n} \vec{a}_n$$

$$\vec{E} = -\nabla V$$

(Definition of gradient of a scalar here *V*)

Electric field is the negative gradient of potential.

Electric field is normal to an equipotential

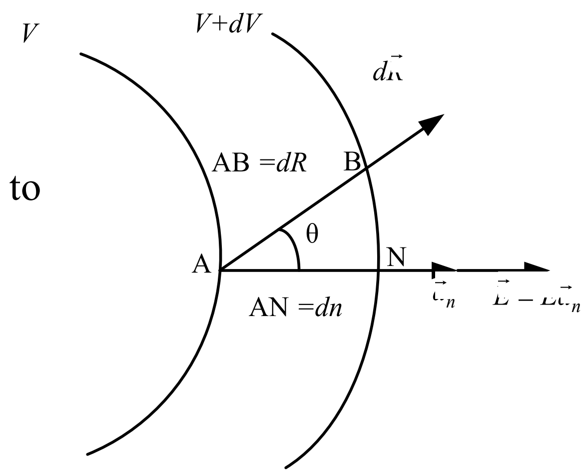
In the limit B→A,

$$dV = -\vec{E}\cdot d\vec{R} = 0$$

$$\vec{E}\cdot d\vec{R} = 0$$

In the limit B→A,

$d\vec{R}$ becomes tangential to the equipotential V at A



$\vec{E}$ becomes normal to $d\vec{R}$ and hence to the equipotential V at A and thus

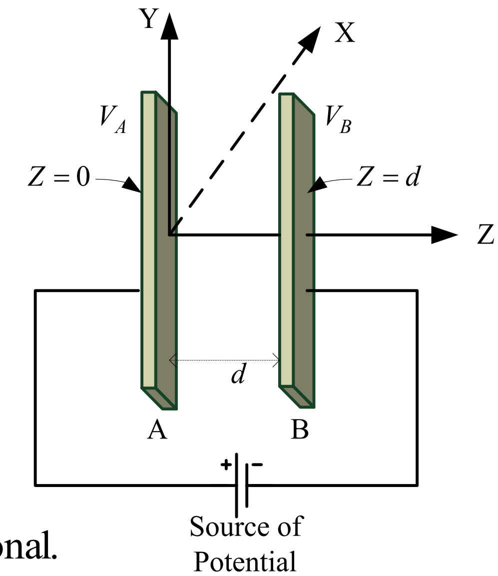
$$\vec{E} = E\vec{a}_n$$

It will be of interest to apply the expression $\vec{E} = -\nabla V$ to find the capacince of capacitors

Capacitance of a parallel-plate capacitor (using the expression for electric field in terms of the gradient of potential)

Let us consider large dimensions of plates compared to the distance between the plates.

The geometry of the problem enjoys rectangular symmetry.



$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z\right)$$

We can take $\partial/\partial z \neq 0; \partial/\partial x = \partial/\partial x = 0$ and the problem becomes one-dimensional.

$$\vec{E} = -\frac{\partial V}{\partial z}\vec{a}_z$$

(electric field inside the capacitor)

$$E = -\frac{\partial V}{\partial z} \quad \text{(magnitude of electric field inside the capacitor)}$$

$$E = -\frac{dV}{dz}$$

V depends only on z.

Integrating

$$\int_{V_B}^{V_A} dV = -\int_{z=d}^{z=0} E\,dz$$

Potential of the plate A at $z = 0$ is V_A

Potential of the plate B at $z = d$ is V_B

Distance between plates = d

$$\int_{V_B}^{V_A} dV = -\int_{z=d}^{z=0} E dz$$

Let us evaluate the left and right hand sides.

$$\int_{V_B}^{V_A} dV = [V]_{V_B}^{V_A} = V_A - V_B$$

(Left hand side)

We obtained earlier the electric field near a chargedconducoras

$$\vec{E} = \frac{\rho_s}{\varepsilon_0}\vec{a}_n.$$

Interpreting the medium to be a dielctric of permittivity ε

$$\vec{E} = \frac{\rho_s}{\varepsilon}\vec{a}_n$$

$$-\int_{z=d}^{z=0} E dz \longleftarrow E = \frac{\rho_s}{\varepsilon}$$

$$-\int_{z=d}^{z=0} E dz = -\int_{z=d}^{z=0} \frac{\rho_s}{\varepsilon} dz = -\frac{\rho_s}{\varepsilon}\int_{z=d}^{z=0} dz = -(\frac{\rho_s}{\varepsilon})(0-d) = \frac{\rho_s}{\varepsilon} d \quad \text{(Right hand side)}$$

Putting the left and right hand sides togetherin the expression

$$V_A - V_B = \frac{\rho_s}{\varepsilon} d$$

Putting the left and right hand sides together in the expression we obtained

$$V_A - V_B = \frac{\rho_s}{\varepsilon} d$$

$$\rho_s = \frac{Q}{A}$$

$Q =$ Charge on each plate

$A =$ Area of each plate

$$V_A - V_B = \frac{\rho_s}{\varepsilon} d = \frac{Q}{\varepsilon A} d$$

Capacitance C of the capacitor becomes

$$C = \frac{Q}{V_A - V_B} = \frac{\varepsilon A}{d}$$

The unit of capacitance is Farad or F and accordingly the unit of permittivity ε is F/m.

(Expression for the capacitance of a parallel-plate capacitor)

Take up the following simple problem

What is the area of the plates required for constructing a parallel-plate capacitor of capacitance 100 pF using two metal plates using a mica spacer of relative permittivity 5.4 that separates the plates by a distance of 10^{-2} cm.

Answer:

$$C = \frac{\varepsilon A}{d}$$

$$\downarrow$$

$$A = \frac{Cd}{\varepsilon} = \frac{Cd}{\varepsilon_0 \varepsilon_r}$$

$$\left.\begin{array}{l} d = 10^{-2}\ \mathrm{cm} = 10^{-4}\ \mathrm{m} \\ \varepsilon_r = 5.4 \\ \mathrm{C} = 100\,\mathrm{pF} = 100 \times 10^{-12}\ \mathrm{F} \end{array}\right\} \text{(given)}$$

$$\downarrow$$

$$A = \frac{Cd}{\varepsilon} = \frac{Cd}{\varepsilon_0 \varepsilon_r} = \frac{100 \times 10^{-12} \times 10^{-4}}{8.854 \times 10^{-12} \times 5.4} = 2.1 \times 10^{-4}\ \mathrm{m}^2 = 2.1\,\mathrm{cm}^2.$$

Capacitance of a coaxial cable (using the expression for electric field in terms of the gradient of potential)

The magnitude of radial electric field in the region between the inner and outer conductors of a coaxial cable can be found using Gauss's law as in the problem of finding the electric field due to a long line charge distribution already dealt with. Thus we get

$$E = \frac{\rho_L}{2\pi\varepsilon_0 r} \quad (a \le r \le b)$$

$a =$ Radius of the inner conductor of the coaxial cable

$b =$ Radius of the outer conductor of the coaxial cable

The geometry of the problem enjoys cylindrical symmetry. The length of the cable is taken large compared to the radial dimensions of the cable. The problem becomes one-dimensional. Thus we have here

$$\partial/\partial r \ne 0; \partial/\partial\theta = \partial/\partial z = 0$$

$$a \le r \le b$$

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\vec{a}_\theta + \frac{\partial V}{\partial z}\vec{a}_z\right)$$

$$\vec{E} = -\frac{\partial V}{\partial r}\vec{a}_r$$

$$E = -\frac{\partial V}{\partial r}$$

$$E = -\frac{\partial V}{\partial r} \longleftarrow E = \frac{\rho_L}{2\pi\varepsilon_0 r} \quad (a \le r \le b)$$

V depends only on r.

$$\frac{dV}{dr} = -\frac{\rho_L}{2\pi\varepsilon_0 r} \longleftarrow \frac{\partial V}{\partial r} = -\frac{\rho_L}{2\pi\varepsilon_0 r}$$

Integrating between the limits $V = V_0$ at $r = a$ and $V = 0$ at $r = b$

$V_0 =$ Potential difference between the inner and outer conductors

$$\int_{V=0}^{V=V_0} dV = -\frac{\rho_L}{2\pi\varepsilon_0} \int_b^a \frac{1}{r}\, dr$$

$$\left[V\right]_0^{V_0} = -\frac{\rho_L}{2\pi\varepsilon_0} \left[\ln r\right]_b^a$$

$$V_0 = -\frac{\rho_L}{2\pi\varepsilon_0} (\ln a - \ln b) = \frac{\rho_L}{2\pi\varepsilon_0} (\ln b - \ln a) = \frac{\rho_L}{2\pi\varepsilon_0} \ln \frac{b}{a}$$

$$\rho_L = \frac{2\pi\varepsilon_0}{\ln \frac{b}{a}} V_0$$

Capacitance per unit length of the cable is obtained by dividing the charge per unit length ρ_L of the conductor by the potential difference V_0 between the conductors

$$\rho_L = \frac{2\pi\varepsilon_0}{\ln\frac{b}{a}} V_0$$

$$C\left(=\frac{\rho_L}{V_0}\right) = \frac{2\pi\varepsilon_0}{\ln\frac{b}{a}} \quad \text{(Capacitance per unit length)}$$

Electric field of a dipole (using the expression for electric field in terms of the gradient of potential)

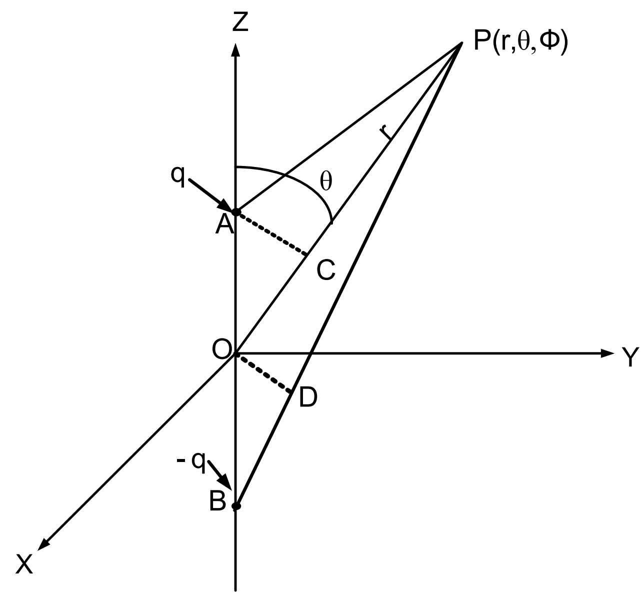
Two equal and opposite point charges separated by a distance constitutes a dipole.

l = AB = Length of the dipole being the distance between the point charges

q = Charge of the point charge at A

–*q* = Charge at the point charge at B

r = Distance of the point P (r,θ,ϕ)—where to find the electric field—from the middle O of the dipole.



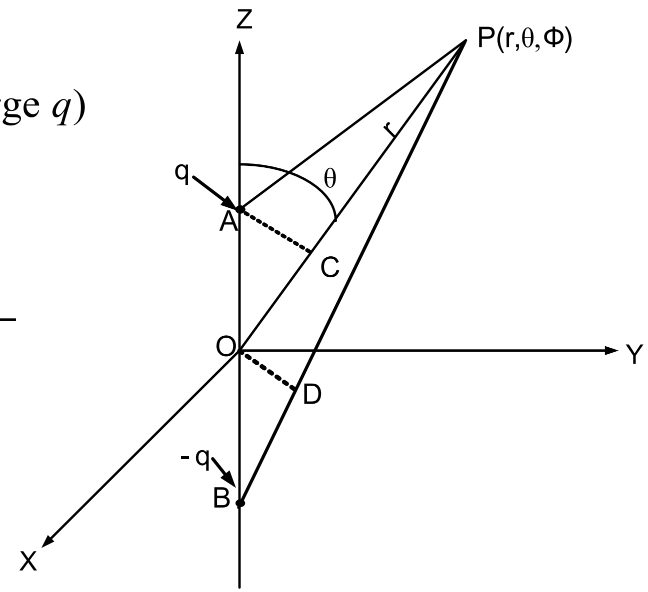
Let us consider a shot dipole for which *l*<<*r*.

The problem enjoys azimuhal symmetry: $\partial / \partial\phi = 0$

$$V = \frac{q}{4\pi\varepsilon_0 r} \quad \text{(Potential at a distance } r \text{ from the point charge } q\text{)}$$

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\mathrm{AP}} - \frac{1}{\mathrm{BP}}\right) \cong \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\mathrm{CP}} - \frac{1}{\mathrm{BP}}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\mathrm{OP}-\mathrm{OC}} - \frac{1}{\mathrm{BD}+\mathrm{DP}}\right)$$



$$\mathrm{DP} \approx \mathrm{OP} = r \quad (r >> l \text{ for a short dipole})$$

$$\mathrm{OC} = \mathrm{OA}\cos\theta = \frac{l}{2}\cos\theta$$

$$\mathrm{BD} = \frac{l}{2}\cos\theta$$

$$\mathrm{DP} \approx \mathrm{OP} = r$$

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r-(l/2)\cos\theta} - \frac{1}{(l/2)\cos\theta + r}\right)$$

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{(l/2)\cos\theta + r - [r-(l/2)\cos\theta]}{[r-(l/2)\cos\theta][(l/2)\cos\theta + r]}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{l\cos\theta}{r^2 - (l^2/4)\cos^2\theta}\right)$$

$$V = \frac{ql\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \qquad (r >> l \text{ for a short dipole})$$

$$p = ql \text{ (dipole moment)}$$

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

Expressing ∇V in spherical-polar coordinates

$p = ql$ (dipole moment)

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi\right)$$

The problem enjoys azimuhal symmetry: $\partial / \partial\phi = 0$

$$\vec{E} = -\left(\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta\right)$$

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\left.\begin{aligned}\frac{\partial V}{\partial r} &= \frac{-2p\cos\theta}{4\pi\varepsilon_0 r^3}\\ \frac{\partial V}{\partial \theta} &= \frac{-p\sin\theta}{4\pi\varepsilon_0 r^2}\end{aligned}\right\}$$

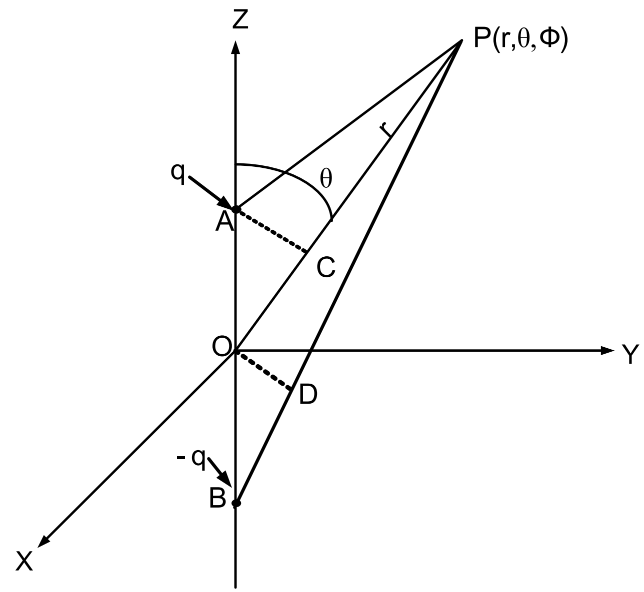
$$\vec{E} = -\left(\frac{-2p\cos\theta}{4\pi\varepsilon_0 r^3}\vec{a}_r + \frac{1}{r}\frac{-p\sin\theta}{4\pi\varepsilon_0 r^2}\vec{a}_\theta\right)$$

$$\vec{E} = -\left(\frac{-2p\cos\theta}{4\pi\varepsilon_0 r^3}\vec{a}_r + \frac{1}{r}\frac{-p\sin\theta}{4\pi\varepsilon_0 r^2}\vec{a}_\theta \right)$$

$$\downarrow$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^3} p(2\cos\theta\vec{a}_r + \sin\theta\vec{a}_\theta)$$

(expression for electric field due to a shot dipole at a distance *r* and at an angle θ from the axis of the dipole in terms of the dipole moment *p*)



Putting $\theta = 0$

$$\vec{E} = \frac{p}{2\pi\varepsilon_0 r^3}\vec{a}_r$$

(expression for electric field due to a shot dipole at a distance *r* on the axis of the dipole ($\theta = 0$) in terms of the dipole moment *p*)

Poisson's and Laplace's equations

Let us take an infinitesimal volume element $\Delta\tau$ over which we can take the value of volume charge density ρ and that of electric displacement *D* to be fairly constant. If we apply Gauss's law to this volume element, we obtain the following approximate relation:

$$\oint_S \vec{D}\cdot\vec{a}_n dS = \int_\tau \rho d\tau \quad \text{(Gauss's law)}$$

$$\oint_S \vec{D}\cdot\vec{a}_n dS \cong \rho\Delta\tau$$

The approximation sign of equality is given since over the volume element $\Delta\tau$, the value of volume charge density ρ and that of electric displacement *D* have been regarded as constant.

$$\frac{\oint_S \vec{D}\cdot\vec{a}_n dS}{\Delta\tau} \cong \rho$$

The relation is made exact and the equality sign is given by taking $\Delta\tau \to 0$ corresponding to the volume element shrinking to a point.

$$\underset{\Delta\tau\to 0}{Lt}\ \frac{\oint_S \vec{D}\cdot\vec{a}_n dS}{\Delta\tau} = \rho$$

The left hand side is the definition of the divergence of a vector here electric displacement.

$$\nabla\cdot\vec{D} = \rho \qquad \vec{D} = \varepsilon\vec{E}$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon}$$

Poisson's equation | **Laplace's equation**

For $\rho = 0$,

$$\nabla\cdot\vec{D} = 0$$

$$\nabla\cdot\vec{E} = 0$$

Laplaciam form of Poisson's and Laplace's equations

With the help of the relation between the electric field in terms of the gradient of electric potential we can find the expression for Poisson's equation in terms of potential.

Putting the volume charge density equal to zero in Poisson's equation so found we get Laplace's equation in electric potential.

We can solve these expressions for potential and subsequently find the electric field from potential with the help of the relation between the electric field in terms of the gradient of potential.

$$\vec{E} = -\nabla V$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$ (Poisson's equation in terms of electric field)

$$\nabla \cdot (\nabla V) = \nabla^2 V$$

$$\nabla \cdot (-\nabla V) = \frac{\rho}{\varepsilon}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$ (Poisson's equation in terms of electric potential)

$$\rho = 0$$

$$\nabla^2 V = 0$$ (Laplace's equation in terms of electric potential)

Electric field in the region between conductors at different potentials from the solution of Laplace's equation in Laplacian form

Take the problem of a parallel-plate capacitor. Find the expression for electric field in the region between plates.

$\nabla^2 V = 0$ (Laplace's equation in terms of electric potential)

$\partial / \partial z \neq 0; \partial / \partial x = \partial / \partial x = 0$

$\nabla^2 V = \partial^2 V / \partial z^2$

$\partial^2 V / \partial z^2 = 0 \leftarrow$ V depends only on r.

$$\frac{d^2 V}{dz^2} = 0$$

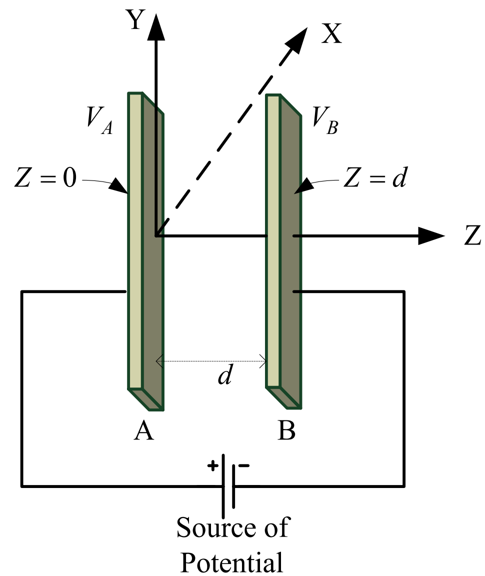
$$\frac{dV}{dz} = A$$

Integration

A is integration constant

$$\int dV = \int A dz$$

$V = Az + B$ *B* is integration constant



$$V = V_A \text{ at } z = 0$$

$$V = Az + B$$

$$B = V_A$$

$$V = V_B \text{ at } z = d$$

$$V_B = Ad + B = Ad + V_A$$

$$A = -\frac{V_A - V_B}{d}$$

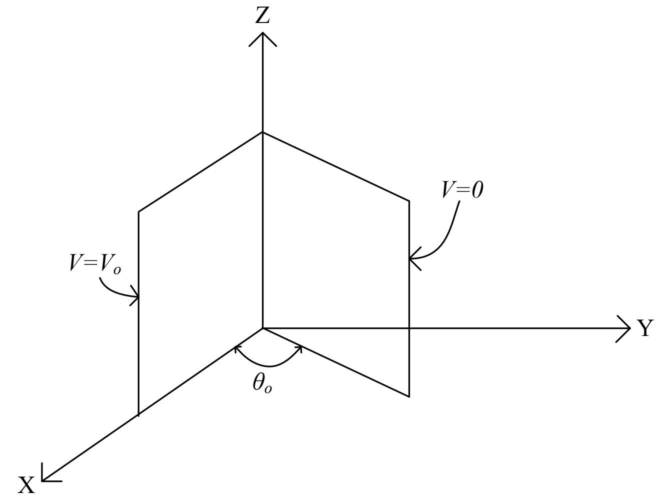
$$V = -\frac{V_A - V_B}{d} z + V_A$$

$$\vec{E} = -\frac{dV}{dz}\vec{a}_z$$

$$\vec{E} = \frac{V_A - V_B}{d}\vec{a}_z$$

(Expression for the electric field in terms of the potential difference $V_A - V_B$ and the distance d between the plates)

Take the problem of a pair of large conducting plates at a difference of potential forming a wedge. Find the expression for electric field in the region between plates.



The problem enjoys cylindrical symmetry, allowing us to treat it in cylindrical coordinates.

V_0 = Potential difference between plates

θ_0 = Wedge angle

$\nabla^2 V = 0$ (Laplace's equation in terms of electric potential)

$\partial / \partial\theta \neq 0; \partial / \partial r = \partial / \partial z = 0$

V depends only on θ.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} = \frac{1}{r^2}\frac{d^2 V}{d\theta^2} = 0$$

$$\frac{d^2 V}{d\theta^2} = 0$$

$$\frac{d^2V}{d\theta^2} = 0$$

Integration

$$\frac{dV}{d\theta} = C_1$$

C_1 is integration constant

$$\int dV = C_1 \int d\theta$$

$$V = C_1\theta + C_2$$

C_2 is integration constant

$V = V_0$ at $\theta = 0$

$$V_0 = C_2$$

$V = 0$ at $\theta = \theta_0$

$$V = C_1\theta + V_0$$

$$0 = C_1\theta_0 + V_0$$

$$C_1 = -\frac{V_0}{\theta_0}$$

$$V = -\frac{V_0}{\theta_0}\theta + V_0$$

$$V = V_0(1 - \frac{\theta}{\theta_0})$$

$$V = V_0\left(1 - \frac{\theta}{\theta_0}\right)$$

$$\partial / \partial\theta \neq 0; \partial / \partial r = \partial / \partial z = 0$$

$$\vec{E} = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta = -\frac{1}{r}\frac{dV}{d\theta}\vec{a}_\theta$$

$$\frac{dV}{d\theta} = -\frac{V_0}{\theta_0}\theta$$

$$\vec{E} = \frac{1}{r}\frac{V_0}{\theta_0}\vec{a}_\theta$$

(Expression for electric field at a point located between a pair of large conducting plates forming a wedge separated by an infinitesimal insulating gap, in terms of the potential difference V_0 between the plates and the wedge angle θ_0)

Summarising Notes

√ Basic concepts of static electric field or electrostatics have been recapitulated in this chapter.

√ Rationalised MKS system has been introduced while writing the expression for the electrostatic force between two point charges called Coulomb's law. The rationalisation factor 4π has been introduced in Coulomb's force expression thereby removing the appearance of the factor 4π from many extensively used expressions derived from Coulomb force expression, for instance, Gauss's law.

√ Coulomb's law has been used to find

◊ electrostatic force on a point charge due to the distribution of other point charges; and

◊ electric field due to a point charge as well as the electric field due to a distribution of point charges.

√ Finding the electric field by Coulomb's law has been illustrated in examples of charge distribution such as line and surface charge distributions.

√ Gauss's law greatly simplifies the finding of electric field in problems that enjoy rectangular, cylindrical or spherical symmetry, which otherwise would become quite involved if Coulomb's law were used instead.

√ Concept of electric potential has been introduced.

√ Expression for the electric field in terms of the gradient of electric potential has been derived and illustrated in examples, for instance, in the problem of finding electric field due to a short dipole as well as in the problem of finding the capacitance of a parallel-plate capacitor and the capacitance of a coaxial cable.

√ Poisson's and Laplace's equations have been derived from first principles and their use illustrated in the problems of finding the electric field in the region between a pair of conductors at different electric potentials by solving Laplace's equation expressed in terms of Laplacian of potential.

√ Poisson's equation in terms of Laplacian of potential has been solved to deduce Child-Langmuir's law for a space-charge-limited diode, which relates the anode current of the diode to the anode potential as well as anode-to-cathode distance of the diode.

Readers are encouraged to go through Chapter 3 of the book for more topics and more worked-out examples and review questions.