Engineering Electromagnetics Essentials

Chapter 10

Waveguide resonator: analytic appreciation by equivalent transmission-line approach

Objective

To analyse and characterise waveguide resonators

Topics dealt with

Transmission line theory as an approach to study waveguide resonators

Distributed transmission line parameters

Telegrapher's equations

Distortionless transmission line

Input impedance of a transmission line terminated in a load impedance

Transmission line theory as applied to a duplexer of a radar system

Transmission line theory as applied to a radome of an antenna

Characteristic impedance of a transmission line

Reflection coefficient and voltage standing wave ratio (VSWR) of a transmission line

Finding the load impedance of a transmission line from the shift of its standing-wave pattern when a short replaces the load impedance

Theory of Smith chart and its applications to transmission line problems

Closed-ended resonator analysed by transmission line theory

Cylindrical waveguide

Inability of a hollow-pipe waveguide to support a TEM mode

Power flow and power loss in a waveguide

Power loss per unit area, power loss per unit length and attenuation constant therefrom

Background

Maxwell's equations (Chapter 5), electromagnetic boundary conditions at conductor-dielectric interface (Chapter 7), basic concepts of propagation of electromagnetic waves through a waveguide (Chapter 9) and those of circuit theory

Difficulties of storing microwave energy at microwave frequencies in conventional tank circuit consisting of an inductor in parallel with a capacitor:

- Radiative loss due to part sizes becoming comparable to the operating wavelength
- Increased resistive loss due to skin effect
- Difficulty of fabricating inductors and capacitors due to their tiny sizes

↓

Difficulties alleviated by storing energy in a microwave resonator formed out of a designed length of a hollow metal-pipe waveguide:

- Closed-ended waveguide resonator

- Open-ended waveguide resonator

How to find the length of the waveguide—closed-ended or open-ended–that would make it a resonator?

We have found this length by treating a waveguide resonator as equivalent to a transmission line —an alternative to the field theory approach that uses Maxwell's equations, wave equation, electromagnetic boundary conditions, Poynting theorem, etc.

Therefore, in what follows, we present the fundamentals of transmission line theory (that is in vogue for the analysis of a transmission line like a two-wire line or a coaxial cable).

Transmission line theory

Let us present here the fundamentals of transmission line theory to be used for treating a waveguide resonator. It becomes convenient to treat a transmission line with the help of distributed transmission line parameters to represent an infinitesimal length of it.

Distributed transmission line parameters *L*, *C*, *R* and *G* to represent an infinitesimal length of a transmission line:

Series inductance per unit length, *L* ,of the line accounting for the energy stored in electric field

Shunt capacitance per unit length, *C,* of the line accounting for the energy stored in electric field

Series resistance per unit length, *R,* of the line accounting for the power losses in the conductors

Shunt conductance per unit length, *G,* of the line accounting for the power losses in the dielectric, if present in the region between the conductors

By choosing the infinitesimal length of the line $\Delta z \to 0$, we can treat the individual elements of line section of infinitesimal length Δz as lumped circuit elements and apply the laws of the circuit theory such as Kirchoff's laws for the analysis of the line.

(See the *Note* to follow)

Note:

In the definition of the distributed line parameters representing a transmission line, the following is implied:

(i) Effect of the size of a circuit element relative to the operating wavelength is not ignored as could be done at low frequencies;

(ii) Effect of the finite time of travel of signal along the electric circuit is taken into account in the analysis for frequencies greater than a kilohertz while assigning electrical quantities such as the voltage, current, resistance, capacitance, etc;

(iii) Due to the transit-time effect, there will be a non-zero reactive drop for a lossless transmission line even though its resistive voltage drop could be zero;

(iv) Higher the operating frequencies, the manifestation of the transit-time effect in the reactance effect becomes more pronounced;

(v) Current I and voltage V continuously vary from point to point along the length of the line, warranting the use of a cascade of individual 'discrete' elemental line sections, each of infinitesimal length Δz, chosen small compared to the operating wavelength λ;

(vi) By choosing $\Delta z \to 0$, the individual elements of line section of infinitesimal length Δz of the distributed transmission line model can be treated as the lumped circuit elements, thereby allowing the application of the laws of the circuit theory such as Kirchoff's laws for the analysis of the line.

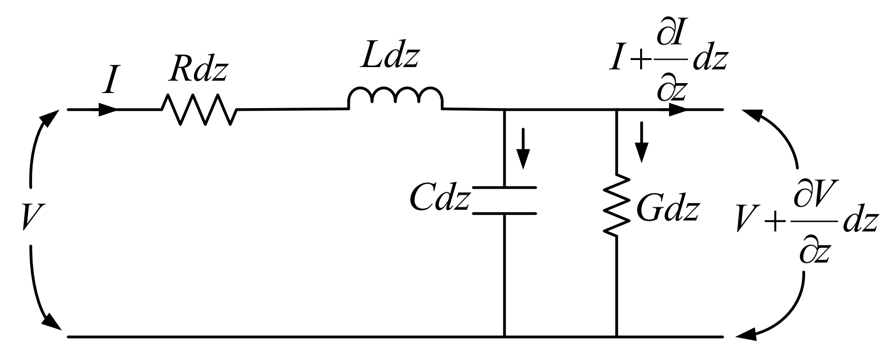
Rdz → Series resistance due to the finite conductivity of the conductor used in making the transmission line, accounting for ohmic power loss in the conductor

Ldz → Series inductance associated with the magnetic flux due to the current through the transmission line that links with the conductors making up the line, which takes into account the energy stored in the magnetic field

Cdz → Shunt capacitance between the conductors of the transmission line because of the existence of an electric field between the conductors insulated by a dielectric medium

Gdz → Shunt conductance between the conductors of the transmission line, often referred to as leakage conductance or 'leakance', which manifests itself because of leakage current flowing between the conductors through the insulating dielectric if it is not perfect, thus accounting for the dielectric power loss in the insulator

Representation of an infinitesimal length of an element of a transmission line by distributed line parameters →



Voltage at the input end of the transmission line of infinitesimal length $= V$

Voltage at the output end of the transmission line of infinitesimal length $= V + (\partial V / \partial z)dz$

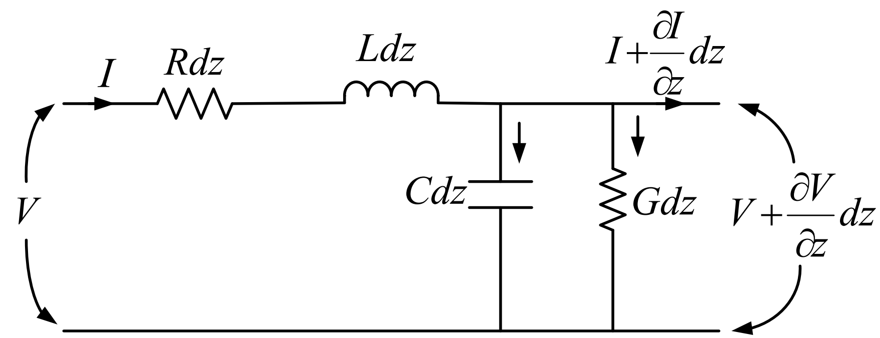
Current at the input end of the transmission line of infinitesimal length $= I$

Current at the output end of the transmission line of infinitesimal length $= (I + \partial I / \partial z)dz$

$$V - \left(V + \frac{\partial V}{\partial z} dz \right) = I(Rdz) + (Ldz)\frac{\partial I}{\partial t}$$

$$\frac{\partial V}{\partial z} = -\left(IR + L\frac{\partial I}{\partial t} \right)$$

$$\frac{\partial V}{\partial z} = -(R + j\omega L)I$$



$$\frac{\partial}{\partial t} = j\omega$$

In view of time dependence: $\exp(j\omega t)$

$$I = I + \frac{\partial I}{\partial z} dz + Cdz\frac{\partial V}{\partial t} + GdzV$$

$$\frac{\partial I}{\partial z} = -\left(GV + C\frac{\partial V}{\partial t} \right)$$

$$\frac{\partial I}{\partial z} = -(G + j\omega C)V$$

Telegrapher's equations

$$\frac{\partial V}{\partial z} = -(R + j\omega L)I \quad \text{(rewritten)}$$

$$\frac{\partial I}{\partial z} = -(G + j\omega C)V \quad \text{(rewritten)}$$

Taking partial derivative

$$\frac{\partial^2 V}{\partial^2 z} = -(R + j\omega L)\frac{\partial I}{\partial z} = (R + j\omega L)(G + j\omega C)V$$

$$\frac{\partial^2 V}{\partial^2 z} = (R + j\omega L)(G + j\omega C)V$$

$$\gamma^2 = (R + j\omega L)(G + j\omega C)$$

$$\frac{\partial^2 V}{\partial^2 z} = \gamma^2 V$$

Solving

$$V = A\exp(-\gamma z) + B\exp(\gamma z)$$

Invoking time dependence $\exp(j\omega t)$

$$V = [A\exp(-\gamma z) + B\exp\gamma z]\exp(j\omega t)$$

$$V = [A\exp(-\gamma z) + B\exp(\gamma z)]\exp(j\omega t)$$ (rewritten)

$$\gamma^2 = (R + j\omega L)(G + j\omega C)$$ (rewritten)

$$\downarrow$$

$$\gamma = [(R + j\omega L)(G + j\omega C)]^{1/2}$$

$$\gamma = \alpha + j\beta \longrightarrow$$

$$\downarrow$$

$$\alpha + j\beta = [(R + j\omega L)(G + j\omega C)]^{1/2}$$

$$\downarrow$$

$$\theta_1 = \tan^{-1}\frac{\omega L}{R}, \quad \theta_2 = \tan^{-1}\frac{\omega C}{G} \longrightarrow$$

$$\alpha + j\beta = [(R^2 + \omega^2 L^2)^{1/2}\exp(j\theta_1)(G^2 + \omega^2 C^2)^{1/2}\exp(j\theta_2)]^{1/2}$$

$$= [(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/4}\exp j\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$\downarrow$$

$$\alpha + j\beta = [(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/4}\left[\cos\left(\frac{\theta_1 + \theta_2}{2}\right) + j\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\right]$$

$$V = A\exp(-\alpha z)\exp j(\omega t - \beta z) + B\exp(\alpha z)\exp j(\omega t + \beta z)$$

(expression for the voltage on the transmission line)

$$\alpha + j\beta = [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/4}\left[\cos\left(\frac{\theta_1+\theta_2}{2}\right) + j\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right]$$ (rewritten)

Equating real part

Equating imaginary part

$$\alpha = [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/4}\cos\left(\frac{\theta_1+\theta_2}{2}\right) \qquad \beta = [(R^2+\omega^2L^2)(G^2+\omega^2C^2)]^{1/4}\sin\left(\frac{\theta_1+\theta_2}{2}\right)$$

Let us next find an expression for the current on the transmission line.

$$V = [A\exp(-\gamma z) + B\exp\gamma z]\exp(j\omega t)$$

(expression for the voltage on the transmission line) (recalled)

$$\frac{\partial V}{\partial z} = -\gamma A\exp(j\omega t)\exp(-\gamma z) + \gamma B\exp(j\omega t)\exp(\gamma z) \qquad \frac{\partial V}{\partial z} = -(R+j\omega L)I$$ (recalled)

$$I = -\frac{\dfrac{\partial V}{\partial z}}{R+j\omega L} = -\frac{-\gamma A\exp(j\omega t)\exp(-\gamma z) + \gamma B\exp(j\omega t)\exp(\gamma z)}{R+j\omega L}$$

$$I = -\frac{\frac{\partial V}{\partial z}}{R + j\omega L} = -\frac{-\gamma A \exp(j\omega t)\exp(-\gamma z) + \gamma B \exp(j\omega t)\exp(\gamma z)}{R + j\omega L}$$ (rewritten)

$$I = \frac{\gamma}{R + j\omega L}[A \exp(-\gamma z) - B \exp(\gamma z)]\exp(j\omega t)$$

$$\gamma = [(R + j\omega L)(G + j\omega C)]^{1/2}$$

$$I = \frac{\gamma}{R + j\omega L}[A \exp(-\gamma z) - B \exp(\gamma z)]\exp(j\omega t)$$

$$I = \frac{[A \exp(-\gamma z) - B \exp(\gamma z)]\exp(j\omega t)}{\left(\frac{R + j\omega L}{G + j\omega C}\right)^{1/2}}$$

$$Z_0 = \left(\frac{R + j\omega L}{G + j\omega C}\right)^{1/2}$$

(Characteristic impedance of the line)

$$I = \frac{A \exp(-\gamma z) - B \exp(\gamma z)}{Z_0} \exp(j\omega t)$$

(expression for the current on the transmission line)

Distortionless transmission line

The information or signal that we transmit through a transmission line is composed, in general, of a number of frequency components. For distortionless transmission, two conditions need to be satisfied:
(i) all the frequency components constituting the signal should simultaneously reach the same receiving point on the line, which also implies that the phase velocity of the wave, v_{ph}, supported by the line should be constant with frequency, or in other words, the line should be 'dispersion-free' and
(ii) the attenuation constant α of the line due to line losses, if any, should be the same for all the frequency components of the signal.

Does a lossless transmission line ($R = G = 0$) satisfy the distortionless condition?

Lossless line

$$R = G = 0 \rightarrow \left.\begin{aligned} \theta_1 &= \tan^{-1}\frac{\omega L}{R} \\ \theta_2 &= \tan^{-1}\frac{\omega C}{G} \end{aligned}\right\} \rightarrow \left.\begin{aligned} \theta_1 &= \tan^{-1}\frac{\omega L}{0} \\ \theta_2 &= \tan^{-1}\frac{\omega C}{0} \end{aligned}\right\} \rightarrow \theta_1 = \theta_2 = \tan^{-1}\infty = \pi/2$$

(recalled)

$$\cos(\pi/2) = 0$$

$$\alpha = [(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/4}\cos\left(\frac{\theta_1 + \theta_2}{2}\right) \rightarrow \alpha = [\omega^2 L^2)(\omega^2 C^2)]^{1/4}\cos(\pi/2) = 0$$

(recalled)

$$\sin(\pi/2) = 1$$

$$\beta = [(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/4}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) \rightarrow \beta = [\omega^2 L^2)(\omega^2 C^2)]^{1/4}\sin(\pi/2) = \omega(LC)^{1/2}$$

(recalled)

For a lossless line (R = G = 0) we obtained earlier

$$\alpha = 0$$

$$v_{\text{ph}} = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \longleftarrow \beta = \omega (LC)^{1/2}$$

The attenuation constant α and phase velocity v_{ph} of the line are both constant with frequency, suggesting that the lossless line (R = G = 0) is a distortionless line.

Can you show that if the distributed line parameters of a lossy transmission line satisfy the relation *RC* = *LG*, then the line becomes a distortionless line?

$$\alpha = [(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/4} \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$\alpha =$$

$$\sqrt{RG}[(1 + \frac{\omega^2 L^2}{R^2})(1 + \frac{\omega^2 C^2}{G^2})]^{1/4} \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$\beta = [(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/4} \sin\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$\beta =$$

$$\omega\sqrt{LC}[1 + \frac{R^2}{\omega^2 L^2})(1 + \frac{G^2}{\omega^2 C^2})]^{1/4} \sin\left(\frac{\theta_1 + \theta_2}{2}\right)$$

$$\frac{L}{R}=\frac{C}{G} \longleftarrow RC = LG \longrightarrow \frac{R}{L}=\frac{G}{C}$$

(condition)

$$\alpha =$$

$$\sqrt{RG}[(1+\frac{\omega^2 L^2}{R^2})(1+\frac{\omega^2 C^2}{G^2})]^{1/4}\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

(recalled)

$$\alpha = \sqrt{RG}(1+\frac{\omega^2 L^2}{R^2})^{1/2}\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$\beta =$$

$$\omega\sqrt{LC}[1+\frac{R^2}{\omega^2 L^2})(1+\frac{G^2}{\omega^2 C^2})]^{1/4}\sin\left(\frac{\theta_1+\theta_2}{2}\right)$$

(recalled)

$$\beta = \omega\sqrt{LC}(1+\frac{R^2}{\omega^2 L^2})^{1/2}\sin\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$\frac{L}{R} = \frac{C}{G} \quad \leftarrow \quad RC = LG$$

(condition)

$$\left.\begin{aligned} \theta_1 &= \tan^{-1}\frac{\omega L}{R} \\ \theta_2 &= \tan^{-1}\frac{\omega C}{G} \end{aligned}\right\} \longrightarrow \theta_1 = \theta_2$$

(recalled)

$$\left.\begin{aligned} \sin\theta_1 &= \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} \\ \cos\theta_1 &= \frac{R}{\sqrt{R^2 + \omega^2 L^2}} \end{aligned}\right\} \longleftarrow$$

$$\alpha = \sqrt{RG}\left(1 + \frac{\omega^2 L^2}{R^2}\right)^{1/2} \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$

(recalled)

$$\downarrow$$

$$\alpha = \sqrt{RG}\left(1 + \frac{\omega^2 L^2}{R^2}\right)^{1/2} \cos\theta_1$$

$$\beta = \omega\sqrt{LC}\left(1 + \frac{R^2}{\omega^2 L^2}\right)^{1/2} \sin\left(\frac{\theta_1 + \theta_2}{2}\right)$$

(recalled)

$$\downarrow$$

$$\beta = \omega\sqrt{LC}\left(1 + \frac{R^2}{\omega^2 L^2}\right)^{1/2} \sin\theta_1$$

$$\left.\begin{aligned}\sin\theta_1 &= \frac{\omega L}{\sqrt{R^2+\omega^2 L^2}}\\ \cos\theta_1 &= \frac{R}{\sqrt{R^2+\omega^2 L^2}}\end{aligned}\right\}$$

$$\alpha = \sqrt{RG}\left(1+\frac{\omega^2 L^2}{R^2}\right)^{1/2}\cos\theta_1$$ (recalled)

$$\downarrow$$

$$\alpha = \sqrt{RG}\left(1+\frac{\omega^2 L^2}{R^2}\right)^{1/2}\frac{R}{\sqrt{R^2+\omega^2 L^2}} = \sqrt{RG}$$

$$\beta = \omega\sqrt{LC}\left(1+\frac{R^2}{\omega^2 L^2}\right)^{1/2}\sin\theta_1$$ (recalled)

$$\downarrow$$

$$\beta = \omega\sqrt{LC}\left(1+\frac{R^2}{\omega^2 L^2}\right)^{1/2}\frac{\omega L}{\sqrt{R^2+\omega^2 L^2}} = \omega\sqrt{LC}$$

$$\downarrow$$

$$v_{\rm ph} = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

The attenuation constant α and phase velocity $v_{\rm ph}$ of the line are both constant with frequency, thereby suggesting that that the lossy transmission line can be made distortionless if the distributed line parameters satisfies the relation $RC = LG$.

Reflection coefficient of a transmission line

Let a transmission line of length *l* be terminated in load impedance Z_l. We consider the source end to be located at z = = –*l* and the load end at *z* = 0.

$$V = [A\exp(-\gamma z) + B\exp(\gamma z)]\exp(j\omega t)$$

(recalled)

$$I = \frac{A\exp(-\gamma z) - B\exp(\gamma z)}{Z_0}\exp(j\omega t)$$

(recalled)

$$z = 0$$

$$V\big|_{z=0} = (A+B)\exp(j\omega t)$$

$$I\big|_{z=0} = \frac{A-B}{Z_0}\exp(j\omega t)$$

$$Z_l = \frac{V}{I}\bigg|_{z=0} = \frac{A+B}{\dfrac{A-B}{Z_0}} = \frac{A+B}{A-B}Z_0$$

(load impedance)

$$\frac{Z_l}{Z_0} = \frac{A+B}{A-B}$$

$$\Gamma_l = \frac{B}{A} = \frac{Z_l - Z_0}{Z_l + Z_0}$$

(reflection coefficient)

Input impedance of a transmission line

$$V = [A\exp(-\gamma z) + B\exp\gamma z]\exp(j\omega t)$$

$$I = \frac{A\exp(-\gamma z) - B\exp(\gamma z)}{Z_0}\exp(j\omega t)$$

$$z = -l$$

$$V\Big|_{z=-l} = [A\exp(\gamma l) + B\exp(-\gamma l)]\exp(j\omega t)$$

$$I\Big|_{z=-l} = \frac{A\exp(\gamma l) - B\exp(-\gamma l)}{Z_0}\exp(j\omega t)$$

$$Z_{in} = \frac{V}{I}\Big|_{z=-l} = Z_0\frac{A\exp(\gamma l) + B\exp(-\gamma l)}{A\exp(\gamma l) - B\exp(-\gamma l)}$$ (input impedance)

$$Z_{in} = Z_0\frac{\exp(\gamma l) + \frac{B}{A}\exp(-\gamma l)}{\exp(\gamma l) - \frac{B}{A}\exp(-\gamma l)}$$

$$Z_{in} = Z_0 \frac{\exp(\gamma l) + \frac{B}{A}\exp(-\gamma l)}{\exp(\gamma l) - \frac{B}{A}\exp(-\gamma l)}$$ (rewritten) ⟵ $$\frac{B}{A} = \frac{Z_l - Z_0}{Z_l + Z_0}$$ (recalled)

↓

$$Z_{in} = Z_0 \frac{\exp(\gamma l) + \frac{Z_l - Z_0}{Z_l + Z_0}\exp(-\gamma l)}{\exp(\gamma l) - \frac{Z_l - Z_0}{Z_l + Z_0}\exp(-\gamma l)} = Z_0 \frac{(Z_l + Z_0)\exp(\gamma l) + (Z_l - Z_0)\exp(-\gamma l)}{(Z_l + Z_0)\exp(\gamma l) - (Z_l - Z_0)\exp(-\gamma l)}$$

↓ ⟵ Rearranging terms

$$Z_{in} = Z_0 \frac{Z_l[\exp(\gamma l) + \exp(-\gamma l)] + Z_0[\exp(\gamma l) - \exp(-\gamma l)]}{Z_l[\exp(\gamma l) - \exp(-\gamma l)] + Z_0[\exp(\gamma l) + \exp(-\gamma l)]}$$ ⟵ $$\left.\begin{aligned} \cosh(\gamma l) &= \frac{\exp(\gamma l) + \exp(-\gamma l)}{2} \\ \sinh(\gamma l) &= \frac{\exp(\gamma l) - \exp(-\gamma l)}{2} \end{aligned}\right\}$$

↓

$$Z_{in} = Z_0 \frac{Z_l \cosh(\gamma l) + Z_0 \sinh(\gamma l)}{Z_l \sinh(\gamma l) + Z_0 \cosh(\gamma l)}$$ (input impedance)

$$Z_{in} = Z_0 \frac{Z_l \cosh(\gamma l) + Z_0 \sinh(\gamma l)}{Z_l \sinh(\gamma l) + Z_0 \cosh(\gamma l)}$$ (input impedance) (rewritten)

↓ ← Dividing by $\cosh(\gamma l)$

$$Z_{in} = Z_0 \frac{Z_l + Z_0 \tanh(\gamma l)}{Z_l \tanh(\gamma l) + Z_0}$$ (input impedance)

↓ ← Rearranging terms

$$Z_{in} = Z_0 \frac{Z_l + Z_0 \tanh(\gamma l)}{Z_0 + Z_l \tanh(\gamma l)}$$ (input impedance)

(general expression for the input impedance of a transmission line which)

Let us next read this general expression for the input impedance the line for a lossless line.

Input impedance of a lossless transmission line:

Let us take a lossless line (*R* = *G* = 0) for which we obtained earlier:

$\gamma = \alpha + j\beta$ (recalled) $\longleftarrow$ $\alpha = 0$ (lossless line)

$$\beta = \omega(LC)^{1/2}$$

$$\alpha + j\beta = [(R + j\omega L)(G + j\omega C)]^{1/2}$$

$\downarrow$

$\gamma = j\beta$ (lossless line)

$\downarrow$

$$Z_{in} = Z_0 \frac{Z_l + Z_0 \tanh(j\beta l)}{Z_0 + Z_l \tanh(j\beta l)} \quad \text{(lossless line)}$$

$\downarrow$ $\longleftarrow$ $\tanh j\beta l = j \tan \beta l$

$$Z_{in} = Z_0 \frac{Z_l + jZ_0 \tan(\beta l)}{Z_0 + jZ_l \tan(\beta l)} \quad \text{(lossless line)}$$

Input impedance of a lossless transmission line that is short-circuited at the load end:

$$Z_{in} = Z_0 \frac{Z_l + jZ_0 \tan(\beta l)}{Z_0 + jZ_l \tan(\beta l)}$$ (lossless line) (recalled)

$\downarrow \leftarrow Z_l = 0$

$$Z_{in} = jZ_0 \tan \beta l$$ (short-circuited lossless transmission line at the load end)

Input impedance of a lossless transmission line that is open-ended at the load end:

Dividing the numerator and denominator each by Z_l

$\downarrow$

$$Z_{in} = Z_0 \frac{1 + j\dfrac{Z_0}{Z_l}\tan(\beta l)}{\dfrac{Z_0}{Z_l} + j\tan(\beta l)}$$ $\longleftarrow$ Putting for an open-ended line $Z_l = \infty$

$\downarrow$

$$Z_{in} = Z_0 \frac{1}{j\tan(\beta l)} = -jZ_0 \frac{1}{\tan(\beta l)} = -jZ_0 \cot(\beta l)$$

(lossless transmission line open-ended at the load end)

Quarter-wave transformer and its use in a radar duplexer:

$Z_{in} = jZ_0 \tan \beta l$ (short-circuited lossless transmission line at the load end)

$$\left.\begin{aligned} \beta &= \frac{2\pi}{\lambda} \\ l &= \frac{\lambda}{4} \end{aligned}\right\}$$ (quarter-wave transmission line: $l = \lambda/4$)

$$Z_{in} = jZ_0 \tan \beta l = jZ_0 \tan[(\frac{2\pi}{\lambda})(\frac{\lambda}{4})] = jZ_0 \tan(\frac{\pi}{2})] = (jZ_0)(\infty) = \infty$$

(short-circuited lossless quarter-wave transmission line of length $l = \lambda/4$ at the load end)

Thus the input impedance of a short-circuited lossless quarter-wave transmission is found to be infinity.

$$Z_{in} = Z_0 \frac{1}{j\tan(\beta l)} = -jZ_0 \frac{1}{\tan(\beta l)} = -jZ_0 \cot(\beta l)$$

(lossless transmission line open-ended at the load end)

$$\left.\begin{aligned} \beta &= \frac{2\pi}{\lambda} \\ l &= \frac{\lambda}{4} \end{aligned}\right\}$$

(quarter-wave transmission line: $l = \lambda/4$)

$$Z_{in} = -jZ_0 \cot \beta l = -jZ_0 \cot[(\frac{2\pi}{\lambda})(\frac{\lambda}{4})] = -jZ_0 \cot(\frac{\pi}{2})] = (-jZ_0)(0) = 0$$

(lossless quarter-wave transmission line open-ended at the load end)

Thus the input impedance of a lossless quarter-wave transmission line open-ended at the load end is found to be zero.

Transmission line theory as applied to a duplexer of a radar system

In a radar system, the signal power is sent to a target in pulses and the echo signal is received between pulses.

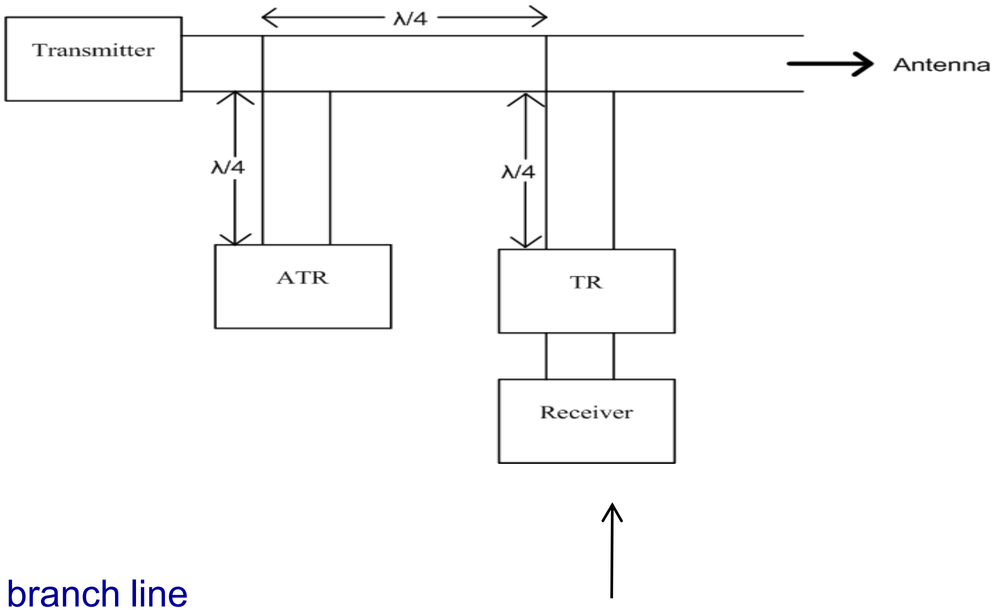
The duplexer in the radar permits the use of a single antenna in both transmitting and receiving modes of the radar. In the transmitting mode the duplexer allows the transmitter to send signal power to the antenna for radiation while protecting the receiver from the transmitted power and, in the receiving mode, allows the signal power to be received by the receiver.

In the transmitting mode the duplexer allows the transmitter to send signal power to the antenna for radiation while protecting the receiver from the transmitted power

In the receiving mode the duplexer allows the signal power to be received by the receiver.

<u>Branch-type radar duplexer</u>:

Transmit-receive (TR) and anti-transmit-receive (ATR) switches (both, typically, gas-discharge type) are turned on (by gas ionisation discharge) when the radar sends signal power pulses and turned off (by gas deionisation) between pulses. TR and the ATR switches primarily disconnect the receiver in the transmitting mode and disconnect the transmitter in the receiving mode.



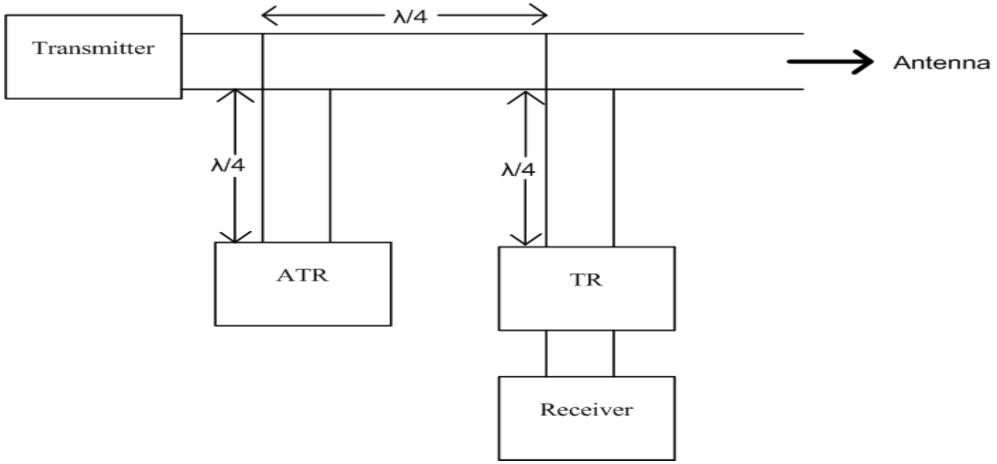
Branch-type radar duplexer

TR switch is located toward the antenna end in the branch line at a quarter wavelength ($\lambda/4$) distance from the main line the antenna end. It is terminated in the receiver.

ATR switch is located toward the transmitter end in the branch line at a quarter wavelength ($\lambda/4$) distance from the main line. It is separated from TR switch by a distance of $\lambda/4$ measured on the main line.

Input impedance of a short-circuited lossless quarter-wave ($\lambda/4$) transmission is infinity. (recalled)

Input impedance of a lossless quarter-wave ($\lambda/4$) transmission line open-ended at the load end is zero. (recalled)



Transmitting mode:

(i) TR switch turns on and short circuits the transmitted thereby preventing it from entering the receiver. (ii) Branch line containing TR switch with its terminating end shorted presents infinite impedance at the main line $\lambda/4$ away from it and does not impede power flow in the main line to the antenna. (Iii) Similarly, ATR switch also turns on and presents infinite impedance at the main line and does not impede power flow in the main line to the antenna (as in (ii) above).

Receiving mode:

(i) ATR switch turns off and makes the branch line containing it open ended thereby presenting zero impedance at a point $\lambda/4$ away on the main line. (ii) TR switch turns off and similarly presents zero impedance at a point on the main line. (iii) Received power sees infinite impedance toward the transmitter end at a point on the main line where it is connected to the branch line containing TR switch since this point is $\lambda/4$ distance away from the zero impedance point on the main line connected to ATR switch via the branch line (see (i) above). (iv) Since received power sees zero impedance point toward receiver (as in (i) above) and infinite impedance toward transmitter (see (iii) above), and it goes to the receiver instead of transmitter taking the lower impedance path.

Transmission line theory as applied to a radome of an antenna

Let us exemplify the transmission line theory to determine the thickness of a radome that is used to protect an antenna while for this purpose by first appreciating that the input impedance of a half-wave lossless transmission line is the same as the terminating load impedance of the line and hence finding the thickness of a radome, treated as a lossless transmission line, made of a fibre glass of dielectric constant 4.9 and taking the operating frequency as 3 GHz.

$$Z_{in} = Z_0 \frac{Z_l + jZ_0 \tan \beta l}{Z_0 + jZ_l \tan \beta l}$$ (recalled) (lossless line)

Here, l corresponds to radome thickness

$$\left.\begin{aligned} & l = \lambda / 2 \\ & \beta l = (2\pi / \lambda) l = (2\pi / \lambda)(\lambda / 2) = \pi \\ & \tan(\beta l) = \tan \pi = 0 \end{aligned}\right\}$$

$$Z_{in} = Z_0 \frac{Z_l}{Z_0} = Z_l$$

(input impedance of a half-wave line treated here as a lossless line becoming equal to the load impedance)

From the physical point of view, taking the thickness of the radome as $\lambda/2$ will ensure a path difference of the wave reflected from the radome as λ, thereby making it in anti-phase with the incident wave and hence causing the cancellation of the incident and reflected waves resulting in the matching of the radome to the incident wave.

$$Z_{in} = Z_0 \frac{Z_l}{Z_0} = Z_l \quad \text{(rewritten)}$$

We have found the input impedance of a half-wave line as equal to the load impedance. This concept may be used to find the thickness of the radome. In the present context, if we take the thickness of the radome treated as a transmission line of length $l = \lambda/2$, then since the load impedance is here the free-space intrinsic impedance η_0, we obtain

$$Z_{in} = Z_l = \eta_0 \qquad \text{(input and load ends of the radome each being both a free space medium)}$$

In the present example

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu_0\varepsilon}} = \frac{2\pi}{\omega\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{2\pi}{2\pi f\sqrt{\mu_0\varepsilon_0\varepsilon_r}}$$

$$= \frac{1}{f\sqrt{\varepsilon_r}}\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{c}{f}\frac{1}{\sqrt{\varepsilon_r}} \longleftarrow \left.\begin{array}{l}\varepsilon_r = 4.9 \\ f = 3\ \text{GHz} = 3\times10^9\ \text{Hz}\end{array}\right\} \text{(given)}$$

$$\downarrow$$

$$\lambda = \frac{c}{f}\frac{1}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{3\times10^9}\frac{1}{\sqrt{4.9}} = 0.04517\,\text{m} = 45.17\,\text{mm}$$

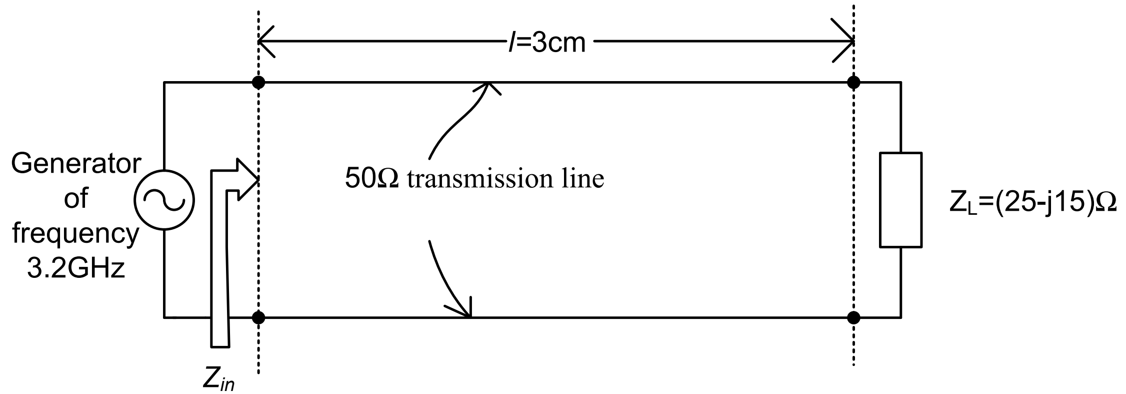
$$\downarrow$$

$$\lambda/2 = 45.17/2 = 22.59\,\text{mm}$$

(required thickness of radome)

Let us further numerically appreciate the problem of finding the input impedance of a 3-cm long lossless transmission line of characteristic resistance 50 Ω, which operates at 3.2 GHz and is terminated in load impedance Z_l = 25-j15 Ω.

$$\left.\begin{aligned} Z_0 &= 50\,\Omega \\ l &= 3\,\text{cm} \\ f &= 3.2\,\text{GHz} = 3.2\times10^9\ \text{Hz} \\ Z_l &= 25 - j15\ \Omega \end{aligned}\right\}\ \text{(given)}$$



↓

$$Z_{in} = Z_0 \frac{Z_l + jZ_0 \tan\beta l}{Z_0 + jZ_l \tan\beta l} \quad \longleftarrow \quad \left.\begin{aligned} \lambda &= c/f = (3\times10^{10})/(3.2\times10^9) = 9.375\ \text{cm} \\ \beta l &= (2\pi/\lambda)l = (2\pi/9.375)(3) = 2.01 \end{aligned}\right\}$$

(recalled)

↓

$$Z_{in} = 50\,\frac{25 - j15 + j(50)(\tan 2.01)}{50 + j(25 - j15)(\tan 2.01)}$$

Separating the real and imaginary parts →

$$Z_{in} = 109.5 - j13.6\,\Omega$$

Characteristic impedance of a transmission line

$$Z_{in} = Z_0 \frac{A\exp(\gamma l) + B\exp(-\gamma l)}{A\exp(\gamma l) - B\exp(-\gamma l)}$$ (input impedance) (recalled)

$\downarrow$ $\leftarrow$ $l = \infty$ (infinitely long line)

(the second term of each of the numerator and the denominator becoming significantly less than the first term)

$$Z_{in} = \left.\frac{V}{I}\right|_{z=-l} = Z_0 \frac{A\exp(\gamma l)}{A\exp(\gamma l)} = Z_0$$ (infinitely long line)

The characteristic impedance of a transmission line may be thus identified as the input impedance of an infinitely long line ($l = \infty$) .

Next let us find the value of the input impedance of a transmission line of finite length that is terminated in the characteristic impedance.

$$Z_{in} = Z_0 \frac{Z_l + Z_0 \tanh(\gamma l)}{Z_l \tanh(\gamma l) + Z_0}$$ (input impedance) (recalled)

$\downarrow \leftarrow$ $Z_l = Z_0$ (terminating load impedance being taken as the characteristic impedance Z_0) $\longrightarrow$ $Z_0 = \left(\frac{R + j\omega L}{G + j\omega C} \right)^{1/2}$

(characteristic impedance defined earlier in terms of the line parameters)

$$Z_{in} = Z_0 \frac{Z_0 + Z_0 \tanh(\gamma l)}{Z_0 \tanh(\gamma l) + Z_0} = Z_0$$

Thus the characteristic impedance of a transmission line may be identified as the terminating load impedance of the line that makes the input impedance of the line equal to the load impedance.

Voltage standing wave ratio (VSWR) of a transmission line

The superposition of the forward and backward waves on a transmission line will give rise to standing waves. Consequent to such superposition, the magnitude of the line voltage becomes alternately maximum and minimum down the length of the line. We can then define a quantity called the voltage standing-wave ratio (VSWR) as the ratio of the magnitude of the line voltage maximum to the magnitude of the line voltage minimum as $\mathrm{VSWR} = |V_{\max}|/|V_{\min}|$.

$$V = [A\exp(-j\beta z) + B\exp(j\beta z)]\exp(j\omega t)$$

$\gamma = \alpha + j\beta$ (recalled) $\longleftarrow$ $\alpha = 0$ (lossless line)

$\downarrow$

$\gamma = j\beta$ (lossless line)

$\downarrow$

$V = [A\exp(-\gamma z) + B\exp\gamma z]\exp(j\omega t)$ (recalled) $\longrightarrow$ $V = [A\exp(-j\beta z) + B\exp(j\beta z)]\exp(j\omega t)$

$$V = [A\exp(-j\beta z) + B\exp(j\beta z)]\exp(j\omega t)$$ (rewritten)

Dividing by $A\exp(-j\beta z)$

Rearranging terms

$$\frac{V}{A\exp(-j\beta z)} = [1 + \frac{B\exp(j\beta z)}{A\exp(-j\beta z)}]\exp(j\omega t) \longrightarrow \frac{V}{A} = [1 + \frac{B}{A}\exp(j2\beta z)]\exp j(\omega t - \beta z)$$

Defining the load end of the line as the reference point $z = 0$

Normalised potential at a distance $z = -d$ from the load end:

$$\Gamma_l = \frac{B}{A}$$ (recalled)

(reflection coefficient)

$$\frac{V}{A} = [1 + \frac{B}{A}\exp(-j2\beta d)]\exp j(\omega t + \beta d)$$

$$\frac{V}{A} = [1 + \Gamma_l \exp(-j2\beta d)]\exp j(\omega t + \beta d)$$

$$\frac{V}{A} = [1 + \Gamma_l \exp(-j2\beta d)]\exp j(\omega t + \beta d)$$ (rewritten)

$$\downarrow \quad \leftarrow \quad \Gamma_l = |\Gamma_l| \exp(j\theta)$$

(reflection coefficient expressed in terms of its magnitude $|\Gamma_l|$ and phase θ)

$$\frac{V}{A} = [1 + |\Gamma_l| \exp j(\theta - 2\beta d)]\exp j(\omega t + \beta d)$$

$$\downarrow$$

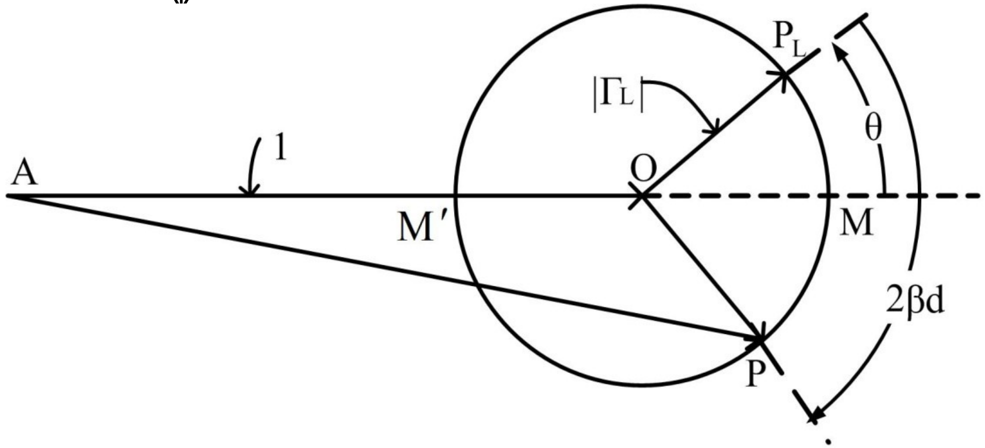
$$\frac{|V|}{A} = 1 + |\Gamma_l| \exp j(\theta - 2\beta d)$$

$\leftarrow$ In general a complex quantity assuming real values for specific values of $\theta - 2\beta d$

(magnitude of normalised voltage amplitude)

$$\frac{|V|}{A} = 1 + |\Gamma_l| \exp\, j(\theta - 2\beta d)$$ (magnitude of normalised voltage amplitude) (rewritten)

Can be represented by $\overrightarrow{AP}$ in the following vector diagram:



$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$$

$\overrightarrow{AO}$ has magnitude unity and is directed along the reference $\theta = 0$.

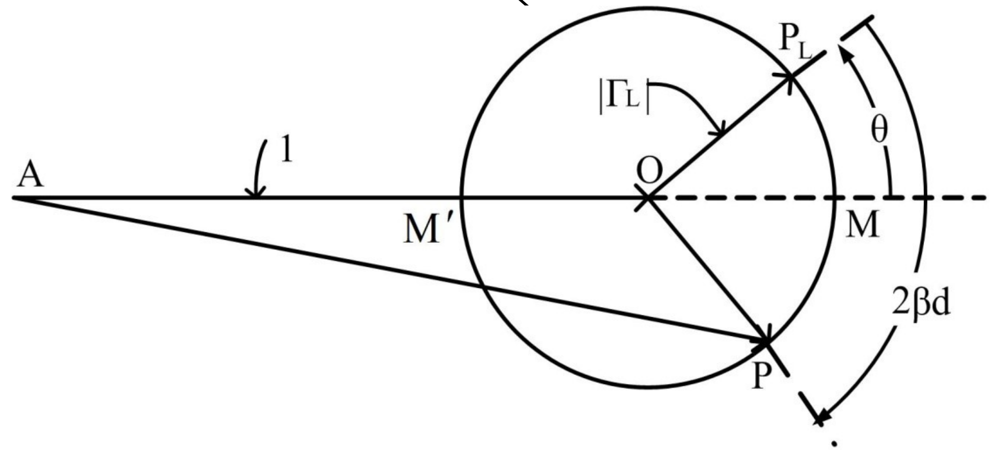
$\overrightarrow{OP}$ has magnitude $|\Gamma_l|$ and its direction depends on the location of the point P on a circle of radius $|\Gamma_l|$ which also depends on the distance *d* of P from the load.

In order to reach the point P_L we can rotate the vector $\overrightarrow{OP}$ through an angle θ from its reference $\theta = 0$ anticlockwise with its base fixed at O around the circle of radius $|\Gamma_l|$.
In order to reach the point P we can then rotate the same vector around the same circle now clockwise from its position P_L through an angle $2\beta d$.

The length of the vector $\overrightarrow{AP}$ representing the magnitude of normalised voltage amplitude $|V|/A$ takes on maximum value AM = $1+|\Gamma_l|$ for $\theta-2\beta d=0,2\pi,4\pi,\ldots.$

The length of the vector $\overrightarrow{AP}$ representing the magnitude of normalised voltage amplitude $|V|/A$ takes on minimum value AM/ = $1-|\Gamma_l|$ for $\theta-2\beta d=\pi,3\pi,5\pi,\ldots.$

$$\text{VSWR}=\frac{\text{AM}}{\text{AM}'}=\frac{|V_{\max}|}{|V_{\min}|}=\frac{1+|\Gamma_l|}{1-|\Gamma_l|}$$



$$2\beta\Delta d=2\pi$$

Δd representing the distance between consecutive maxima = distance between consecutive minima

$$\beta=\frac{2\pi}{\lambda}$$

$$\Delta d=\frac{\pi}{\beta}$$

$$\Delta d=\frac{\lambda}{2}$$

In a simple illustrative example let us find the reflection coefficient and VSWR of a transmission line for three situations of line termination: (i) line terminated in a short (ii) line open-ended at the load end and (iii) line terminated in characteristic impedance Z_0.

(i) Line terminated in a short:

$$Z_l = 0$$

$$\downarrow$$

$$\Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1 \longrightarrow |\Gamma_l| = 1 \longrightarrow \text{VSWR} = \frac{1+|\Gamma_l|}{1-|\Gamma_l|} = \frac{1+1}{1-1} = \frac{2}{0} = \infty$$

(ii) Line open-ended at the load end:

$$Z_l = \infty$$

$$\downarrow$$

$$\Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0} = \frac{1 - \frac{Z_0}{Z_l}}{1 + \frac{Z_0}{Z_l}} = \frac{1 - \frac{Z_0}{\infty}}{1 + \frac{Z_0}{\infty}} = \frac{1-0}{1+0} = 1 \longrightarrow |\Gamma_l| = 1 \longrightarrow \text{VSWR} = \frac{1+|\Gamma_l|}{1-|\Gamma_l|} = \frac{1+1}{1-1} = \frac{2}{0} = \infty$$

(iii) Line terminated in characteristic impedance:

$$Z_l = Z_0$$

$$\downarrow$$

$$\Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = \frac{0}{2Z_0} = 0 \longrightarrow |\Gamma_l| = 0 \longrightarrow \text{VSWR} = \frac{1+|\Gamma_l|}{1-|\Gamma_l|} = \frac{1+0}{1-0} = 1$$

In another example let us calculate the reflection coefficient and the VSWR a transmission line of characteristic resistance 50 Ω if it is terminated in complex impedance of 25 + *j*100 Ω.

$$\Gamma_l = \frac{25 + j100 - 50}{25 + j100 + 50} = \frac{-25 + j100}{75 + j100} \longleftarrow \Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0} \longleftarrow \left.\begin{array}{l} Z_l = 25 + j100\,\Omega \\ Z_0 = 50\,\Omega \end{array}\right\} \text{(given)}$$

$$\downarrow$$

$$\Gamma_l = \frac{-25 + j100}{75 + j100} \times \frac{75 - j100}{75 - j100}$$

$$= \frac{-25\times 75 + j25\times 100 + j100\times 75 + 100\times 100}{(75)^2 + (100)^2} = \Gamma_l = 0.8 + j0.36\Omega$$

$$\downarrow$$

$$|\Gamma_l| = \sqrt{(0.8)^2 + (0.36)^2} = 0.877 \longrightarrow \text{VSWR} = \frac{1 + |\Gamma_l|}{1 - |\Gamma_l|} = \frac{1 + 0.877}{1 - 0.877} = 15.3$$

In yet another illustrative example let us show that, for a transmission line supporting a standing wave, the impedance at the voltage maximum is Z_{max} = (Z_0)(VSWR) and the impedance at the voltage minimum is Z_{min} = Z_o/ VSWR.

$$\frac{|V|}{A} = 1 + |\Gamma_l| \exp j(\theta - 2\beta d)$$ (recalled)

(magnitude of normalised voltage amplitude)

$$\frac{|V_{\max}|}{A} = 1 + |\Gamma_l|$$

$$(\text{corresponding to } \theta - 2\beta d = 0, 2\pi, 4\pi, \ldots.)$$

$$\frac{|V_{\min}|}{A} = 1 - |\Gamma_l|$$

$$(\text{corresponding to } \theta - 2\beta d = \pi, 3\pi, 5\pi, \ldots.)$$

$$Z_{max} = \frac{|V_{max}|}{|I_{min}|} \longleftarrow \left.\begin{aligned} \frac{|V_{max}|}{A} &= 1+|\Gamma_l| \\ \frac{|I_{min}|}{\frac{A}{Z_0}} &= 1-|\Gamma_l| \end{aligned}\right\}$$

(recalled)

$$\text{VSWR} = \frac{1+|\Gamma_l|}{1-|\Gamma_l|}$$

$$Z_{max} = \frac{|V_{max}|}{|I_{min}|} = Z_0 \frac{1+|\Gamma_l|}{1-|\Gamma_l|} = (Z_0)(\text{VSWR})$$

$$Z_{min} = \frac{|V_{min}|}{|I_{max}|} = Z_0 \frac{1}{\frac{1+|\Gamma_l|}{1-|\Gamma_l|}} \longleftarrow \left.\begin{aligned} \frac{|V_{min}|}{A} &= 1-|\Gamma_l| \\ \frac{|I_{max}|}{\frac{A}{Z_0}} &= 1+|\Gamma_l| \end{aligned}\right\}$$

(recalled)

$$\text{VSWR} = \frac{1+|\Gamma_l|}{1-|\Gamma_l|}$$

$$Z_{min} = \frac{|V_{min}|}{|I_{max}|} = Z_0 \frac{1}{\frac{1+|\Gamma_l|}{1-|\Gamma_l|}} = \frac{Z_0}{\text{VSWR}}$$

Finding the load impedance of a transmission line from the study of its standing-wave patterns

In this method, the distance of the first voltage minimum from the load end or alternatively the spatial shift of the voltage minimum of the standing-wave pattern on the length of a transmission line when a short replaces the complex impedance at the terminating load end is interpreted to find the load impedance.

$Z_l = 0$ (for a short at the load end)

$$\Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0}$$ (reflection coefficient in terms of load impedance recalled)

$$\exp(j\theta) = -1$$

$$\Gamma_l = \frac{0 - Z_0}{0 + Z_0} = -1 \longrightarrow \Gamma_l = |\Gamma_l|\exp(j\theta) \longrightarrow |\Gamma_l| = 1;\ \theta = \pi$$

(reflection coefficient in terms of its amplitude and phase θ)

(for a short at the load end)

$$\theta - 2\beta d = \pi, 3\pi, 5\pi, \ldots.$$

(condition for voltage minimum at a point on the line that is distant *d* from the terminating load end)

$$\pi - 2\beta d' = \pi, 3\pi, 5\pi, \ldots.$$ ($\theta = \pi$ for a short at the load end)

(condition for voltage minimum at a point on the line that is distant d' from the terminating load replaced by a short)

$$\theta = \pi + 2\beta(d - d') = \pi + 2\beta\Delta d \longleftarrow d - d' = \Delta d$$ (shift in minima when the terminating load is replaced by a short)

(taking the difference between the above two conditions)

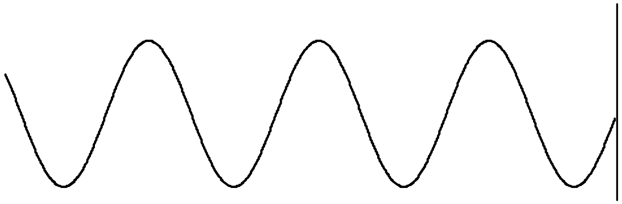
$$d - d' = \Delta d$$

$$\theta = \pi + 2\beta(d - d') = \pi + 2\beta\Delta d$$

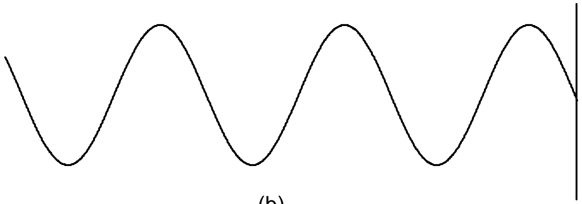
(rewritten)

Δd is interpreted as positive when the voltage standing-wave pattern shifts towards the load end when a short replaces the complex load at the terminating end ($d' < d$).
This also implies that the first voltage extremum from the load end is the voltage minimum at a distance Δd for a complex load, the first voltage minimum however shifting to the location of the short when the latter replaces the load (Fig. (a)).

Δd is interpreted as negative when the voltage standing-wave pattern shifts towards the source end when a short replaces the complex load at the terminating end ($d' > d$).
This also implies that the first voltage extremum from the load end is the maximum at a distance Δd for a complex load which would shift to the location of the source such that a minimum coincides with the short when the latter replaces the load (Fig. (b)).



(a)



(b)

$$\text{VSWR} = \frac{1+|\Gamma_l|}{1-|\Gamma_l|} \longrightarrow \frac{\text{VSWR}}{1} = \frac{1+|\Gamma_l|}{1-|\Gamma_l|} \longrightarrow \frac{\text{VSWR}-1}{\text{VSWR}+1} = \frac{(1+|\Gamma_l|)-(1-|\Gamma_l|)}{(1+|\Gamma_l|)+(1-|\Gamma_l|)} = \frac{2|\Gamma_l|}{2} = |\Gamma_l|$$

(recalled)

$$\downarrow$$

$$\Gamma_l = |\Gamma_l| \exp(j\theta)$$

$$\downarrow \qquad \exp(j\theta) = \cos\theta + j\sin\theta$$

$$\Gamma_l = \frac{\text{VSWR}-1}{\text{VSWR}+1} \exp(j\theta)$$

$$\longleftarrow$$

$$\Gamma_l = \left(\frac{\text{VSWR}-1}{\text{VSWR}+1}\right)(\cos\theta + j\sin\theta)$$

$$\theta = \pi + 2\beta\Delta d$$ (recalled)

$$\downarrow$$

$$\Gamma_l = \left(\frac{\text{VSWR}-1}{\text{VSWR}+1}\right)(\cos(\pi + 2\beta\Delta d) + j\sin(\pi + 2\beta\Delta d))$$

$$\downarrow$$

$$\Gamma_l = -\left(\frac{\text{VSWR}-1}{\text{VSWR}+1}\right)(\cos 2\beta\Delta d + j\sin 2\beta\Delta d)$$

$$\Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0} \longrightarrow \frac{\Gamma_l}{1} = \frac{Z_l - Z_0}{Z_l + Z_0} \longrightarrow \frac{\Gamma_l + 1}{\Gamma_l - 1} = \frac{(Z_l - Z_0) + (Z_l + Z_0)}{(Z_l - Z_0) - (Z_l + Z_0)} = \frac{2Z_l}{-2Z_0} = -\frac{Z_l}{Z_0}$$

(recalled)

$$\Gamma_l = -\left(\frac{\text{VSWR} - 1}{\text{VSWR} + 1}\right)(\cos 2\beta\Delta d + j\sin 2\beta\Delta d)$$

(recalled)

Put together

$$\left.\begin{aligned} Z_l &= \frac{1+\Gamma_l}{1-\Gamma_l} Z_0 \\ \Gamma_l &= -\left(\frac{\text{VSWR} - 1}{\text{VSWR} + 1}\right)(\cos 2\beta\Delta d + j\sin 2\beta\Delta d) \end{aligned}\right\}$$

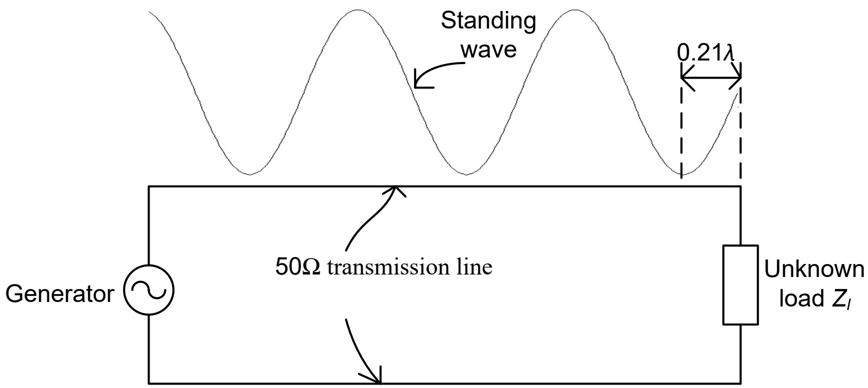
We can use the above expression for the load impedance Z_l in conjunction with the above expression for Γ_l to find the load impedance.

The approach provides the theory of measurement of the load impedance terminating a transmission line. The method involves finding experimentally (i) VSWR of the line and (ii) the shift Δd in the minima of the standing-wave pattern which is also equal to the distance of the first minimum from the load end. However, it is easier to find the shift in the minima of the standing-wave pattern than the distance of the first minimum from the load end.

As an illustration, let us find the complex impedance terminating a transmission line of characteristic impedance 50 Ω that exhibits VSWR = 1.5 and gives the first extremum of the voltage standing-wave pattern as a minimum located at a distance of 0.21λ from the load end.

$\Delta d = 0.21\lambda$ (given)

$$2\beta\Delta d = (2)(2\pi/\lambda)(0.21\lambda) = 0.84\pi$$



$$\left.\begin{aligned}\cos 2\beta\Delta d &= \cos 0.84\pi = \cos 151.2^0 = -0.8763 \\ \sin 2\beta\Delta d &= \sin 0.84\pi = \sin 151.2^0 = 0.4818\end{aligned}\right\}$$

$$\Gamma_l = -\left(\frac{\text{VSWR}-1}{\text{VSWR}+1}\right)(\cos 2\beta\Delta d + j\sin 2\beta\Delta d) \ \text{(recalled)} \longleftarrow \quad \text{VSWR} = 1.5$$

$$= -\left(\frac{1.5-1}{1.5+1}\right)(-0.8763 + j0.4818)$$

$$\Gamma_l = -\left(\frac{1.5-1}{1.5+1}\right)(-0.8763 + j0.4818)$$ (rewritten)

Simplifies to

$$\Gamma_l = 0.175 - j0.096 \longrightarrow Z_l = \frac{1+\Gamma_l}{1-\Gamma_l} Z_0$$ (recalled)

$$Z_l = \frac{1+(0.175-j0.096)}{1-(0.175-j0.096)} \times 50 = \frac{1.175-j0.096}{0.825+j0.096} \times 50$$

$$= \frac{1.175-j0.096}{0.825+j0.096} \times 50 = \frac{1.175-j0.096}{0.825+j0.096} \times \frac{0.825-j0.096}{0.825-j0.096} \times 50$$

$$= \frac{(1.175-j0.096)(0.825-j0.096)}{(0.825)^2+(0.096)^2} \times 50 = \frac{0.9602-j0.1920}{0.6898} \times 50 = 69.6 - j13.9\,\Omega$$

Smith chart: theory and application to transmission line problems

Smith chart due to Phillip H. Smith:

- an elegant graphical tool for high frequency circuit applications, is popularly used to solve transmission line problems
- similar to a slide rule used for carrying out calculations involving addition, subtraction, multiplication, division, logarithmic and trigonometric functions, etc.
- can be applied to transmission line problems, for instance, to state the value of VSWR from the value of the complex load impedance terminating a lossless line of known characteristic resistance and find the input impedance of the line if its length is also given

In what follows, let us outline the theory of Smith chart describing the basic features of the chart and some of its applications.

Readers can find more applications of the chart in Section 10.1.8 of Chapter 10 of the book.

Constant-$|\Gamma_l|$ or VSWR circle of Smith chart

$$\frac{Z_l}{Z_0} = \frac{A+B}{A-B}$$

(recalled)

$$Z_0 = \left(\frac{R + j\omega L}{G + j\omega C}\right)^{1/2}$$

(Characteristic impedance of a line)

$$R = G = 0$$

(for a lossless line)

$$Z_0 = \left(\frac{R + j\omega L}{G + j\omega C}\right)^{1/2}$$

$$Z_0 = \left(\frac{L}{C}\right)^{1/2} = R_0, \text{say}$$

(Characteristic resistance of a lossless line)

Dividing the numerator and denominator of the right hand side by the same quantity

$$z_l = \frac{Z_l}{R_0} = \frac{A+B}{A-B}$$

(load impedance normalised with respect to load resistance)

$$z_l = \frac{1 + \dfrac{B}{A}}{1 - \dfrac{B}{A}}$$

$$z_l = \frac{1 + \frac{B}{A}}{1 - \frac{B}{A}}$$ (rewritten) ← $$\frac{B}{A} = \Gamma_l = |\Gamma_l| \exp j\theta$$ (reflection coefficient in terms of its amplitude and phase)

(recalled)

↓

$$z_l = \frac{1 + |\Gamma_l| \exp(j\theta)}{1 - |\Gamma_l| \exp(j\theta)} = r + jx, \text{say}$$ → $$\frac{1 + |\Gamma_l| \exp(j\theta)}{1 - |\Gamma_l| \exp(j\theta)} = \frac{r + jx}{1}$$

↓

$$|\Gamma_l| \exp\ j\theta = \frac{r + jx - 1}{r + jx + 1}$$ ← $$\frac{[1 + |\Gamma_l| \exp(j\theta)] - [1 - |\Gamma_l| \exp(j\theta)]}{[1 + |\Gamma_l| \exp(j\theta)] + [1 - |\Gamma_l| \exp(j\theta)]} = \frac{(r + jx) - 1}{(r + jx) + 1}$$

↓

$$\Gamma_l = |\Gamma_l| \exp\ j\theta = \frac{r + jx - 1}{r + jx + 1} = u + jv$$ (reflection coefficient in terms of its real part *u* and imaginary part *v*)

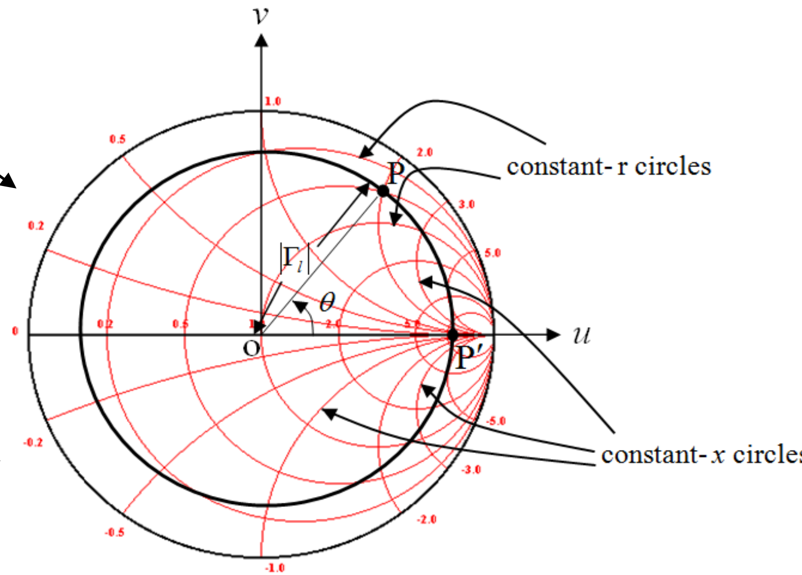
$$\Gamma_l = |\Gamma_l| \exp j\theta = \frac{r + jx - 1}{r + jx + 1} = u + jv$$

(reflection coefficient in terms of its real part *u* and imaginary part *v*) (rewritten)

The projection of the length OP on the abscissa and ordinate represent the real part *u* and the imaginary part *v* of the complex reflection coefficient Γ_l respectively.

A point P on the reflection coefficient Γ_l plane of the Smith chart indicating its magnitude $|\Gamma_l|$ represented by the length OP and the phase θ represented by the angle between OP and *u* axis, O being the origin of coordinates.

$$|\Gamma_l| = \mathrm{OP} = \mathrm{OP}'$$



(complex reflection coefficient Γ_l plane of Smith chart in the domain of $|\Gamma_l| \le 1$)

Can be read as the intercept OP/ with the *u* axis of a circle of radius $|\Gamma_l| = \mathrm{OP}$ called the 'constant-$|\Gamma_l|$ circle' drawn by the user such that it passes through the point P. The length $\mathrm{OP}' = |\Gamma_l|$ can be read as the magnitude of the reflection coefficient of the line on a horizontal scale provided on a commercial Smith chart. The phase θ of the reflection coefficient can be read on a scale provided on the periphery of the chart.

Thus Smith chart is generated on the reflection coefficient Γ_l plane which can also be referred to as the VSWR plane since these two quantities are related as: $\mathrm{VSWR} = (1 + |\Gamma_l|)/(1 - |\Gamma_l|)$. Therefore, the constant-$|\Gamma_l|$ circle can also be referred to as the 'constant-VSWR circle'.

Constant-*r* and constant-*x* circles of Smith chart

$$z_l = \frac{Z_l}{R_0} = r + jx$$ (defined earlier)

(normalised load impedance in terms of its real part *r* and imaginary part *x*)

$$\frac{r + jx - 1}{r + jx + 1} = u + jv = \Gamma_l$$ (recalled)

(real part *r* and imaginary part *x* of normalised load impedance related to real part u and imaginary part v of reflection coefficient)

↓ ← Multiplying the left hand side by a quantity that is equal to unity having the same numerator and denominator

$$\frac{r - 1 + jx}{r + 1 + jx} \times \frac{r + 1 - jx}{r + 1 - jx} = u + jv$$

↓ ← After a little algebra

$$\frac{r^2 + r - jxr - r - 1 + jx + jxr + jx + x^2}{(r+1)^2 + x^2} = \frac{r^2 - 1 + 2jx + x^2}{(r+1)^2 + x^2}$$

$$= \frac{r^2 - 1 + x^2}{(r+1)^2 + x^2} + j\frac{2x}{(r+1)^2 + x^2} = u + jv$$

$$\frac{r^2-1+x^2}{(r+1)^2+x^2}+j\frac{2x}{(r+1)^2+x^2}=u+jv$$ (rewritten)

↓ ← Equating the real and imaginary parts

$$u=\frac{r^2-1+x^2}{(r+1)^2+x^2}$$ (real part)

$$v=\frac{2x}{(r+1)^2+x^2}$$ (imaginary part)

→ Thus, we have expressed the real (*u*) and imaginary (*v*) parts of reflection coefficient Γ_l in terms of the real (*r*) and imaginary (*x*) parts of the normalised load impedance z_l .

Similarly, let us express in what follows next the real (*r*) and imaginary (*x*) parts of the normalised load impedance z_l in terms of the real (*u*) and imaginary (*v*) parts of reflection coefficient Γ_l .

Let us now proceed to express the real (r) and imaginary (x) parts of the normalised load impedance z_l in terms of the real (u) and imaginary (v) parts of reflection coefficient Γ_l.

$$\frac{r+jx-1}{r+jx+1} = u + jv \quad \text{(recalled)} \longrightarrow \frac{r+jx-1}{r+jx+1} = \frac{u+jv}{1}$$

By algebraic manipulation

$$\frac{(r+jx+1)+(r+jx-1)}{(r+jx+1)-(r+jx-1)} = \frac{1+(u+jv)}{1-(u+jv)}$$

By simplification

$$r + jx = \frac{1+u+jv}{1-u-jv}$$

Multiplying the right hand side by a quantity that is equal to unity having the same numerator and denominator

$$r + jx = \frac{1+u+jv}{1-u-jv} \times \frac{1-u+jv}{1-u+jv}$$

Simplifies to

$$r + jx = \frac{1-u^2-v^2+2jv}{(1-u)^2+v^2}$$

Equating the real and imaginary parts

$$r = \frac{1-u^2-v^2}{(1-u)^2+v^2}$$

$$x = \frac{2v}{(1-u)^2+v^2}$$

Thus we have expressed the real (r) and imaginary (x) parts of the normalised load impedance z_l in terms of the real (u) and imaginary (v) parts of reflection coefficient Γ_l.

$$r = \frac{1 - u^2 - v^2}{(1-u)^2 + v^2}$$ (rewritten)

$$x = \frac{2v}{(1-u)^2 + v^2}$$ (rewritten)

↓ ← With a little algebraic manipulation → ↓

$$(u - \frac{r}{r+1})^2 + v^2 = \frac{1}{(r+1)^2}$$

(equation representing a constant-*r* circle on the reflection coefficient plane of Smith chart)

(values of *u* and *v* varying from point to point for a constant value *r*)

$$(u-1)^2 + (v - \frac{1}{x})^2 = \frac{1}{x^2}$$

(equation representing a constant-*x* circle on the reflection coefficient plane of Smith chart)

(values of *u* and *v* varying from point to point for a constant value *x*)

However, before proceeding further with the above two equations representing constant-*r* and constant-x circles respectively, for the benefit to readers, let us provide in what follows next the necessary algebraic steps for the deduction of these equations.

Necessary algebraic steps for deduction of the two equations representing constant-*r* and constant-*x* circles respectively

$$r = \frac{1-u^2-v^2}{(1-u)^2+v^2}$$ (recalled)

By cross multiplication →

$$r[(1-u)^2+v^2] = 1-u^2-v^2$$

↓

$$ru^2 - 2ur + r + u^2 - 1 + v^2(1+r) = 0$$

Adding the quantity 1/(1+r) to both sides →

$$ru^2 - 2ur + r + u^2 - 1 + v^2(1+r) + \frac{1}{1+r} = \frac{1}{1+r}$$

↓

$$ru^2 - 2ur + (r - 1 + \frac{1}{1+r}) + u^2 + v^2(1+r) = \frac{1}{1+r}$$

→

$$ru^2 - 2ur + \frac{r^2}{1+r} + u^2 + v^2(1+r) = \frac{1}{1+r}$$

↓

$$u^2(1+r) - 2ur + \frac{r^2}{1+r} + v^2(1+r) = \frac{1}{1+r}$$

←

$$(1+r)\left(u^2 - 2u\frac{r}{1+r} + \frac{r^2}{(1+r)^2} + v^2\right) = \frac{1}{1+r}$$

Dividing by (1 + *r*) ↓

$$u^2 - 2u\frac{r}{1+r} + \frac{r^2}{(1+r)^2} + v^2 = \left(\frac{1}{1+r}\right)^2$$

Combining the first three terms →

$$\left(u - \frac{r}{r+1}\right)^2 + v^2 = \left(\frac{1}{1+r}\right)^2$$

(equation representing the constant-*r* circle)

By cross multiplication and rearrangement of terms

$$x = \frac{2v}{(1-u)^2 + v^2} \quad \text{(recalled)} \quad \longrightarrow \quad x[(1-u)^2 + v^2] - 2v = 0$$

Multiplying both sides by *x*

$$x^2[(1-u)^2 + v^2] - 2vx = 0$$

Adding the quantity 1 to both sides

$$x^2[(u-1)^2 + v^2] - 2vx + 1 = 1$$

By rearrangement of terms

$$x^2(u-1)^2 + x^2v^2 - 2vx + 1 = 1$$

Combining the first three terms

$$x^2(u-1)^2 + (xv-1)^2 = 1$$

Dividing by x^2

$$(u-1)^2 + (v - \frac{1}{x})^2 = \frac{1}{x^2}$$

(equation representing the constant-*x* circle)

Let us next discuss in what follows the features of constant-*r* and constant-*x* circles of Smith chart with the help of the equations representing them derived here.

Features of constant-*r* and constant-*x* circles of Smith chart with the help of the equations representing them

$$(u-\frac{r}{r+1})^2+v^2=\frac{1}{(r+1)^2}$$ (recalled)

(equation representing a constant-*r* circle on the reflection coefficient plane of Smith chart)

↓ ← Examining the equation

Features of constant-*r* circles of Smith chart:

$$\text{radius}=\pm\frac{1}{r+1}$$

$$u_{\text{intercept}}=1,\ \frac{r-1}{r+1};\ v_{\text{intercept}}=\sqrt{\frac{1-r}{1+r}},\ -\sqrt{\frac{1-r}{1+r}}$$

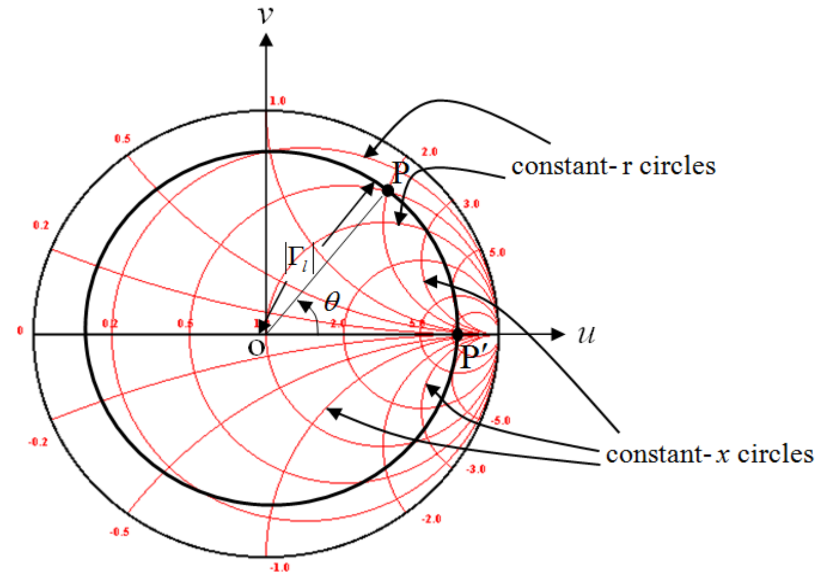
↑ $v=0$

↑ $u=0$

(constant-*r* circle)

$$\left.\begin{aligned}&\text{radius} = \pm\frac{1}{r+1}\\&u_{\text{intercept}} = 1, \frac{r-1}{r+1}; v_{\text{intercept}} = \sqrt{\frac{1-r}{1+r}}, -\sqrt{\frac{1-r}{1+r}}\end{aligned}\right\}$$

(rewritten) (constant-*r* circle)



(complex reflection coefficient Γ_l plane of Smith chart in the domain of $|\Gamma_l| \le 1$)

(1) The constant-$|\Gamma_l|$ circles are concentric all having the common centre at *u* = 0, *v* = 0.

(1) In general, there are two intercepts of a constant-*r* circle with *u* axis and two intercepts with *v* axis.

(2) The value of *r* referring to a constant-*r* circle is near the left intercept on the chart.

(3) The constant *r* = 0 circle is the outermost circle of radius = 1. It is also referred to as the constant $|\Gamma_l| = 1$ circle. All the points of relevance in the chart are located within this circle. The two intercepts of this circle each on *u* and *v* axes are $u_{\text{intercept}} = \pm 1; v_{\text{intercept}} = \pm 1$.

(4) All the constant-*r* circles meet *u* axis at a common point at the extreme right point of the axis corresponding to $u_{\text{intercept}} = 1$.

(5) With the increase of the value of *r*, the radius of the constant-*r* circle decreases and in the limit *r* = ∞ the value of the radius of the circle becomes zero, which means that the constant *r* = ∞ circle shrinks to a point on *u* axis $(u_{\text{intercept}} = 1)$.

(6) The constant $r = 0$ and the constant $r = \infty$ circles pass through the extreme left and right points of the real u axis respectively.

(7) The VSWR of the transmission line is the value of r of the constant-r circle (read on the real u axis) passing through the positive intercept of the constant-$|\Gamma_l|$ circle with the real u axis of the chart.

Explanation:

$$\text{OP}' = u_{\text{intercept}} = \frac{r-1}{r+1} \qquad |\Gamma_l| = \text{OP} = \text{OP}'$$

$$\downarrow$$

$$|\Gamma_l| = \frac{r-1}{r+1} \longrightarrow \frac{1}{|\Gamma_l|} = \frac{r+1}{r-1}$$

$$\downarrow$$

$$\frac{1+|\Gamma_l|}{1-|\Gamma_l|} = \frac{(r+1)+(r-1)}{(r+1)-(r-1)} = \frac{(r+1)+(r-1)}{(r+1)-(r-1)} = \frac{2r}{2} = r \qquad \frac{1+|\Gamma_l|}{1-|\Gamma_l|} = \text{VSWR} \text{ (recalled)}$$

$$\downarrow$$

$$\text{VSWR} = r$$

(8) The values of the impedance maxima and minima are the values of *r* of the constant-*r* circles (read on the real *u* axis) intersecting with the extreme right and left intercepts of the constant-$|\Gamma_l|$ circle with the real *u* axis.

(9) The extreme left and right points of the $|\Gamma_l| = 1$ circle (constant *r* = 0 circle) represent the short circuit and open circuits at the load end respectively.

(10) The VSWR of the points on the chart representing the short and open circuits at the load end is infinity, noting that $|\Gamma_l| = 1$ holds good for each of these terminating loads.

(11) The constant-*r* circles with negative *r* values are of no relevance to the Smith chart since they correspond to inadmissible values of both $u_{\text{intercept}}$ and $v_{\text{intercept}}$, each falling outside the domain of the reflection coefficient.

Features of constant-*x* circles of Smith chart:

$$(u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \frac{1}{x^2} \qquad \text{(recalled)}$$

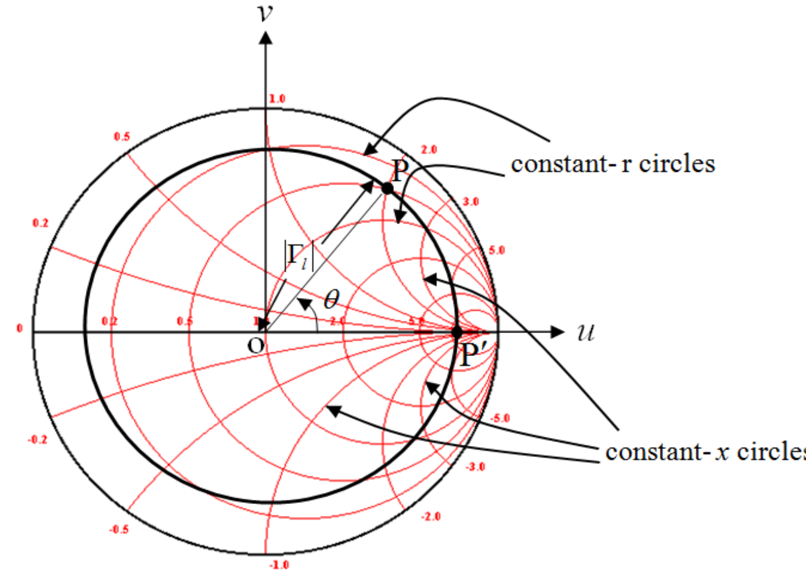
(equation representing the constant-*x* circle)

↓ ← Examining the equation

$$\left.\begin{array}{l} \text{radius} = \pm\dfrac{1}{x} \\[2ex] u_{\text{intercept}} = 1;\ v_{\text{intercept}} = \sqrt{\dfrac{1}{x^2} - 1} + \dfrac{1}{x}, -\sqrt{\dfrac{1}{x^2} - 1} + \dfrac{1}{x} \\[1ex] \quad\uparrow \qquad\qquad\quad \uparrow \\ \quad v = 0 \qquad\qquad u = 0 \end{array}\right\} \quad \text{(constant-}x\text{ circle)}$$

$$\left.\begin{aligned} &\text{radius} = \pm\frac{1}{x} \\ &u_{\text{intercept}} = 1;\ v_{\text{intercept}} = \sqrt{\frac{1}{x^2}-1}+\frac{1}{x}, -\sqrt{\frac{1}{x^2}-1}+\frac{1}{x} \end{aligned}\right\}$$

(rewritten) (constant-*r* circle)



(complex reflection coefficient Γ_l plane of Smith chart in the domain of $|\Gamma_l| \le 1$)

(1) The values of the constant-*x* circles are displayed near these circles at the periphery of the outermost (*r* = 0) constant-*r* circle.

(2) The radius of the constant-*x* circle increases with the decrease of the value of *x* and the radius becomes infinite for the constant *x* = 0 circle, which in fact becomes a straight line coinciding with *u* axis.

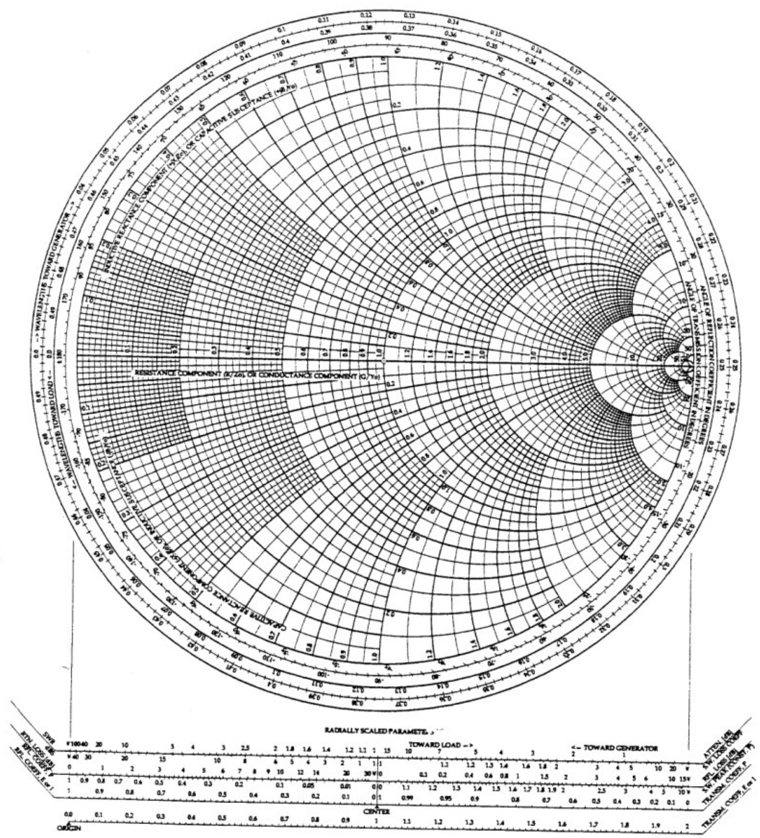
(3) All the constant-*x* circles meet the real *u* axis of the chart at a common point which is the extreme right point of *u* axis. $(u_{\text{intercept}} = 1)$.

(4) The radius of the constant-*x* circle decreases with the increase of the value of *x,* shrinking to zero for the constant-*x* = ∞ circle at the extreme right point on the *u* axis of the chart $(u_{\text{intercept}} = 1)$.

(5) The constant-*x* circles of relevance to the Smith chart correspond to the admissible values of $u_{\text{intercept}}$ and $v_{\text{intercept}}$ falling within the domain of $|\Gamma_l| \le 1$. This, in fact, allows only the family of ‘arcs’ rather than the full circles to be displayed on the chart.

(6) The constant-*x* circles for the positive and negative *x* values lie respectively on the positive and negative imaginary *v* regions of the chart which appear as the mirror images of each other across *u* axis for the same though opposite values of *x*.

The Smith Chart



Source: Emeloid Company, Hillside, New Jersey

In an earlier example we had calculated the magnitude of reflection coefficient of a transmission line of characteristic resistance 50 Ω as 0.877 and its VSWR as 15.3 when it is terminated in complex impedance of 25 + *j*100 Ω using the expressions $\Gamma_l = (Z_l - Z_0)/(Z_l + Z_0)$ and $\mathrm{VSWR} = (1+|\Gamma_l|)/(1-|\Gamma_l|)$. Let us now use Smith chart to find the same quantities of the same line.

$$\left.\begin{array}{l} Z_l = 25 + j100\,\Omega \\ Z_0 = 50\,\Omega \end{array}\right\}\text{(given)}$$

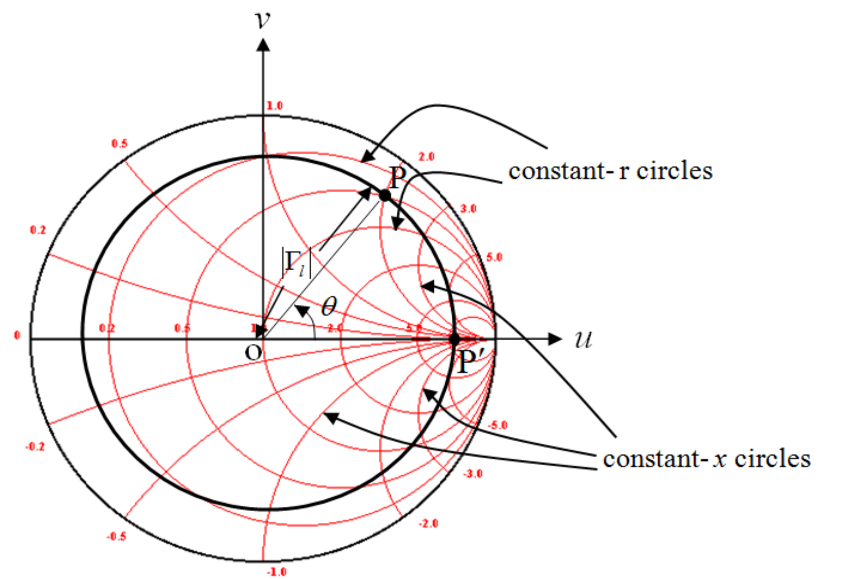
$$z_l (= Z_l / Z_0) = (25 + j100)/50 = 0.5 + j2.0 \longleftarrow$$

$$\downarrow$$

$$\left.\begin{array}{l} r = 0.5 \\ x = 2.0 \end{array}\right\}$$

$$\downarrow$$

We locate the point P as the point of intersection between the *r* = 0.5 and *x* = 0 circles on Smith chart. Next, we draw a circle taking the origin (*u* = 0, *v* =0) as its centre such that it passes through the point P intersecting *u* axis at the point P$^{/}$. →



We can then read the length $\mathrm{OP}'(=\mathrm{OP})$ by superposing it on the horizontal scale provided at the bottom of commercially available Smith chart. This gives the value of the magnitude of reflection coefficient as $|\Gamma_l| = \mathrm{OP}' \cong 0.87$, which agrees with the value calculated earlier from the formula of the reflection coefficient in terms of the load impedance and characteristic impedance. The value of VSWR can be read as $\mathrm{VSWR} \cong 15.2$ (the value of *r* displayed on the scale of the chart), being the value of *r* referring to the constant-*r* circle passing through P$^{/}$ (as explained earlier while describing the features of constant-*r* circle). This value of VSWR also agrees to that calculated earlier using the formula of the VSWR in terms of the magnitude of the reflection coefficient.

Location of the load impedance of a transmission line on Smith chart

$$z_l = \frac{Z_l}{R_0} = r + jx$$

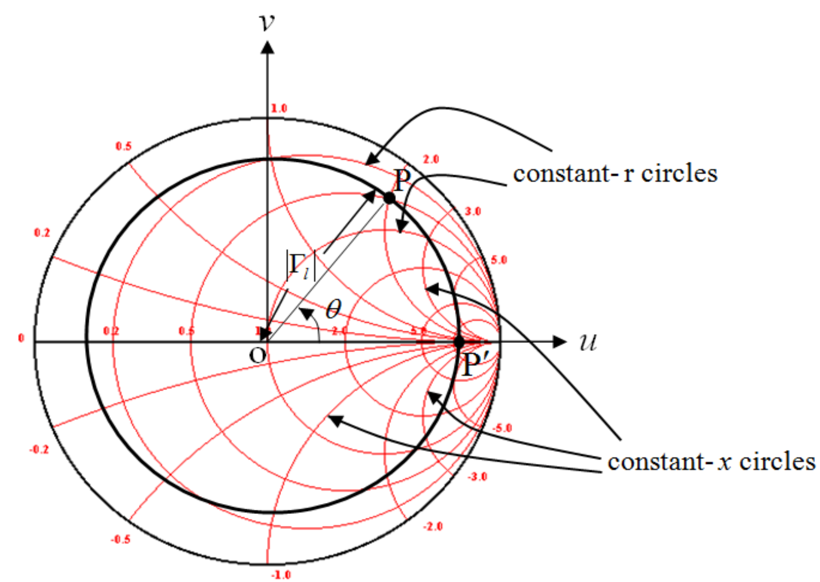
(load impedance normalised with respect to characteristic resistance expressed in terms of its real part *r* and imaginary part *x*)

Locate the point of intersection between constant-*r* and constant-*x* circles now that the values of *r* and *x* have been identified.

We can read the values of *r* on *u* axis and those of x on the periphery of the outermost constant-*r* circle.

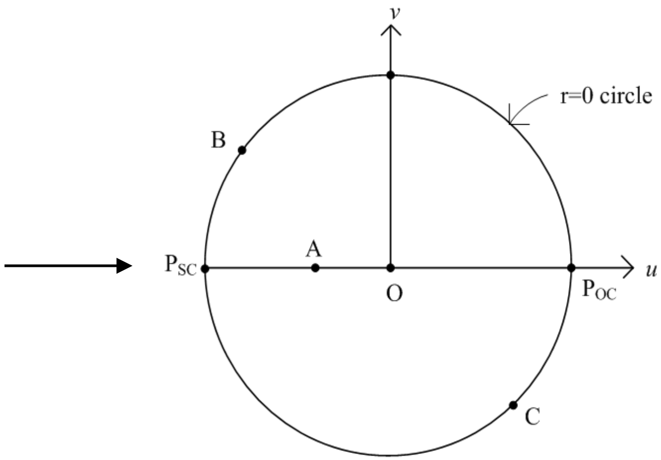
In the accompanying figure, the point P being the intersection between constant *r* = 0.5 and constant *x* = 2 circles, can be made to represent the normalised load impedance $z_l = Z_l/R_0 = 0.5 + j2$. Therefore, P can be made to represent the load impedance of a line of characteristic resistance 50 Ω (typical):

$Z_l = R_0(0.5 + j2) = 50 \times (0.5 + j2) = 25 + j100\ \Omega$.



Representation of purely resistive, purely inductive and purely capacitive, short-circuit, open-circuit, and matched loads:

Load at terminating end	Representing point on chart
Purely resistive	A (typical)
Purely inductive	B (typical)
Purely capacitive	C (typical)
Short circuit	P_{SC}
Open circuit	P_{OC}
matched	O



Explanation:

(i) For purely resistive load, $x = 0$ and the point A lies on u axis which coincides with $x = 0$ line (see constant-x circle features discussed earlier).

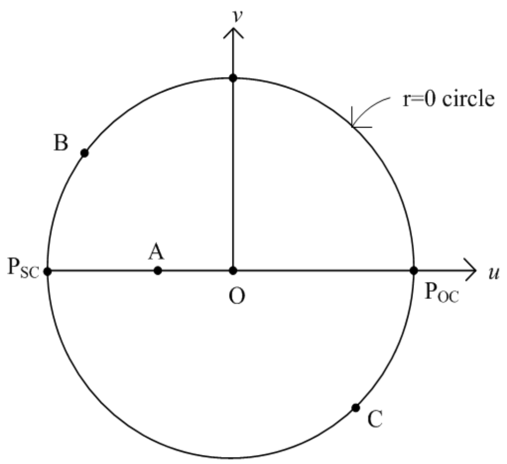
(ii) For purely inductive load, $r = 0$ and the point B lies on constant $r = 0$ circle (outermost constant-r circle) such that constant-x circle referring to a positive value of x passes through B.

(iii) For purely capacitive load too, r = 0 and the point C lies on constant r = 0 circle (outermost constant-r circle) such that constant-x circle referring to a negative value of x passes through C.

(iv) For the line terminated in a short circuit, r = 0 and x = 0 and the point P_{SC} lies on the extreme left end point on u axis.

(v) For the line terminated in an open circuit, $r = \infty$ and $x = \infty$ and the point P_{OC} lies on the extreme right end point on u axis. (We recall the constant-r and constant-x circle features that the constant r = 0 and the constant $r = \infty$ circles pass through the extreme left and right points of the real u axis respectively; and also that in the limit x = 0 circle becomes a straight line coinciding with u axis and that constant-$x = \infty$ circle shrinks to the the extreme right point on u axis).

(vi) For matched load $|\Gamma_l| = 0$ and the point O is the origin u = 0, v = 0, which is also the centre of constant $|\Gamma_l|$ circles.



Location of the input impedance of a transmission line on Smith chart

$$Z_{in} = Z_0 \frac{A\exp(\gamma l) + B\exp(-\gamma l)}{A\exp(\gamma l) - B\exp(-\gamma l)}$$ (input impedance) (recalled)

↓ ← Putting $\gamma = j\beta$ for a lossless line

$$z_{in} = \frac{Z_{in}}{Z_0} = \frac{A\exp(j\beta l) + B\exp(-j\beta l)}{A\exp(j\beta l) - B\exp(-j\beta l)}$$ (input impedance normalised with respect to characteristic impedance)

↓ ← Dividing the numerator and denominator of the right hand side by $A\exp(j\beta l)$ and remembering $B/A = \Gamma_l$

$$z_{in} = \frac{1 + \frac{B}{A}\exp(-2j\beta l)}{1 - \frac{B}{A}\exp(-2j\beta l)} = \frac{1 + \Gamma_l \exp(-2j\beta l)}{1 - \Gamma_l \exp(-2j\beta l)}$$

↓ ← $\Gamma_l = |\Gamma_l|\exp(j\theta)$ (recalled)

$$z_{in} = \frac{1 + |\Gamma_l|\exp(j\theta)\exp(-j2\beta l)}{1 - |\Gamma_l|\exp(j\theta)\exp(-j2\beta l)} = \frac{1 + |\Gamma_l|\exp j(\theta - 2\beta l)}{1 - |\Gamma_l|\exp j(\theta - 2\beta l)}$$

$$z_{in} = \frac{1+|\Gamma_l|\exp j(\theta - 2\beta l)}{1-|\Gamma_l|\exp j(\theta - 2\beta l)}$$ (rewritten)

(input impedance)

↓

Impedance at a distance *l* from the load end)

$$z_l = \frac{1+|\Gamma_l|\exp(j\theta)}{1-|\Gamma_l|\exp(j\theta)}$$ (recalled)

(load impedance)

↓

Impedance at the load end

Two expressions are identical but for the phase factor exp ($-j\beta l$).

↓

We had earlier showed how to locate the point P, say, to represent the normalised load impedance z_l.

Let θ be the angle of P measured from *u* axis (θ = 0) moving 'anticlockwise' around the constant-$|\Gamma_l|$ circle.

Now, we have to move from P through angle $2\beta l$ 'clockwise' to reach a new point representing the impedance of the line at a distance *l* from the load end, which is the same as the input impedance of a line of length *l*.

The values of *r* and *x* referring to the constant-*r* and constant-*x* circles passing through this new point give the real and imaginary parts of the input impedance z_{in} normalised with respect to characteristic resistance R_0.

We can then obtain the input impedance Z_{in} by multiplying z_{in} by R_0.

The above method of finding the input impedance of a transmission line using Smith chart has been further illustrated in an example to follow.

Note on wavelength scale displayed around the periphery of the outermost constant-*r* circle

$$l = p\lambda$$

$$\beta = 2\pi/\lambda \longrightarrow 2\beta l = (4\pi/\lambda)l \longrightarrow 2\beta l = (4\pi/\lambda)l = 4\pi p$$

$$p = 1/2$$

$$l = p\lambda = \lambda/2$$

$$2\beta l = 4\pi p = 2\pi$$

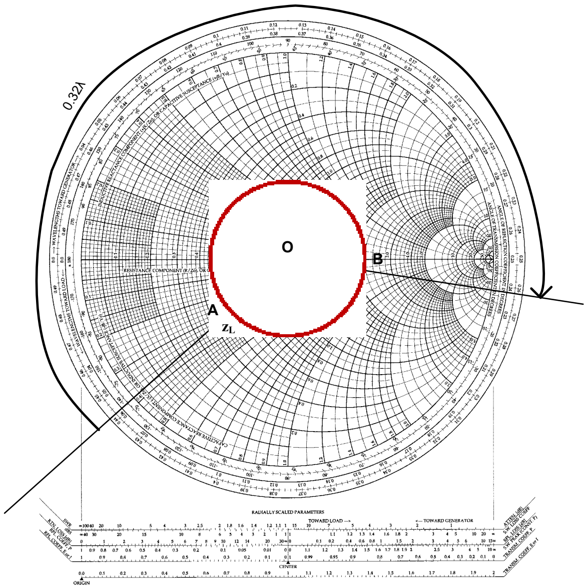
- One rotation through an angle 2π round the periphery of the outermost constant-*r* circle (constant $r = 0$ circle) of the chart corresponds to the length of the transmission line of half wavelength ($\lambda/2$).

- The distance in wavelengths is indicated on the periphery of the outermost constant-*r* circle.

- The extreme left and right points of the circle on the real *u* axis of the chart respectively refer to 0 and 0.25 times the wavelength on the distance scale of the length of the transmission line.

Let us retake, however now using Smith chart, the problem of finding the input impedance of a 3-cm long lossless transmission line of characteristic resistance 50 Ω, which operates at 3.2 GHz and is terminated in load impedance Z_l = 25-j15 Ω.

$$\left.\begin{aligned} Z_0 &= 50\,\Omega \\ l &= 3\,\text{cm} \\ f &= 3.2\,\text{GHz} = 3.2\times10^{9}\,\text{Hz} \\ Z_l &= 25 - j15\,\Omega \end{aligned}\right\} \text{(given)}$$



$$z_l = Z_l / Z_0 = Z_l / R_0 = (25 - j15)/50 = 0.5 - j0.3$$

$$\lambda = c/f = (3\times10^{10})/(3.2\times10^{9}) = 9.375\ \text{cm}$$

Length of the line in number of wavelengths ⟶ $l/\lambda = 3/9.375 = 0.32$

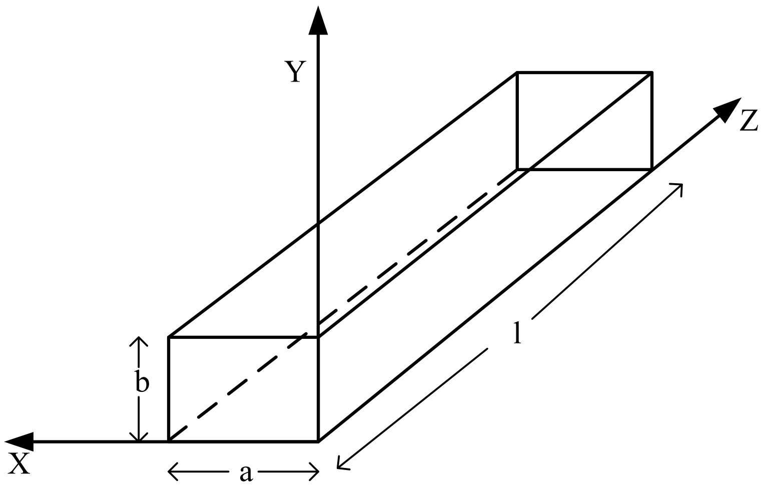
• Locate the point A of intersection between the constant r = 0.5 and constant x = 0.3 circles.

• Move 0.32λ (wavelengths) clockwise towards generator around the constant-$|\Gamma_l|$ circle through the point A to reach the point B. Identify the constant-r and constant-x circles passing through the point B and note that these circles are assigned the values: r = 2.2 and x = - 0.3 respectively.

$$z_{in}(= Z_{in}/Z_0) = 2.2 - j0.3 \longrightarrow Z_{in} = (2.2 - j0.3)(Z_0) = (2.2 - j0.3)(50) = 110 - j15\ \Omega$$

The value so obtained by Smith chart is very close to that calculated earlier using formulae.

Closed-ended resonator



A closed-ended rectangular waveguide resonator ($a \times b \times l$), which is a rectangular waveguide of broad dimension *a* and narrow dimension *b* and whose length is *l* and which has six conducting walls located at $x = 0$ and $x = a$; $y = 0$ and $y = b$; and $z = 0$ and $z = l$.

Let us use the theory of transmission line to the rectangular waveguide with both of its ends closed by conducting walls to form a closed-ended waveguide resonator.

$Z_{in} = jZ_0 \tan \beta l$ (recalled) ← $Z_{in} = 0$

(input impedance of the line short-circuited at the load end)

(input impedance of the line if its input end is short-circuited)

$$\tan \beta l = 0 \longrightarrow \beta l = p\pi \quad (p = 1, 2, 3, \ldots)$$

(closed-ended resonator)

$$\omega_c = c\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

(Waveguide cutoff frequency) (See Chapter 9)

$$\omega^2 - \beta^2 c^2 - \omega_c{}^2 = 0$$

Dispersion relation for both TE and TM modes

The length *l* of the waveguide resonates at angular frequency ω_r.

$$\omega_r{}^2 - \left(\frac{p\pi}{l}\right)^2 c^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] c^2 = 0$$

$$\omega_r = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{l}\right)^2\right]^{1/2} c$$

p is the mode number in addition to *m* and *n*

(angular resonant frequency of closed-ended rectangular waveguide resonator)

We can similarly use the theory of transmission line to the rectangular waveguide with both of its ends open to form an open-ended waveguide resonator.

$$Z_{in} = -jZ_0 \cot \beta l$$ (recalled) ← $$Z_{in} = \infty$$

(input impedance of the line open-ended at the load end)

↓

$$\cot \beta l = \infty$$

↓

$$\tan \beta l = 0$$ → $$\beta l = p\pi \quad (p = 1, 2, 3, \ldots)$$ → Condition identical with that obtained for closed-ended resonator

(open-ended resonator)

↓ ← Following the same steps as taken for closed-ended rectangular waveguide resonator

$$\omega_r = \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 + \left(\frac{p\pi}{l} \right)^2 \right]^{1/2} c$$ → Expression identical to that obtained for closed-ended resonator

(angular resonant frequency of open-ended rectangular waveguide resonator)

In an illustrative example let us find the resonant frequency of a rectangular cavity which has the dimensions 3 cm (broad), 2 cm (narrow) and 5 cm (linear) and which is excited in the mode TE_{101}.

$m = 1$, $n = 0$, $p = 1$ (given)

$$\omega_r = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{l}\right)^2\right]^{1/2} c$$

$$f_r = \frac{\omega_r}{2\pi} = \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{l^2}\right)^{1/2} c$$

$a = 3$ cm, $b = 2$ cm, $l = 5$ cm (given)

$$f_r = \frac{1}{2}\left[(\frac{100}{3})^2 + (\frac{100}{5})^2\right]^{1/2} \times 3 \times 10^8 \text{ Hz}$$

$$f_r = 58.3 \times 10^8 \text{ Hz} = 5.83 \times 10^9 \text{ Hz} = 5.83 \text{ GHz}$$

Fields in a waveguide resonator

A standing wave is formed in a waveguide resonator due to the combination of forward and backward waves caused by reflection from the conducting walls perpendicular to axial z direction.

$$H_z = H_{z0} \cos\frac{\pi}{a} x \exp j(\omega t - \beta z)$$

(recalled) ← Axial magnetic field in a rectangular waveguide excited in TE_{10} mode (recalled from Chapter 9) (representing only the forward wave propagating along *z*)

← $H_{z0} = S$ (symbol for field amplitude H_{z0} changed so for the sake of convenience)

← Keeping understood the time dependence $\exp(j\omega t)$

$$H_z = S \exp(-j\beta z) \cos\left(\frac{\pi}{a} x\right)$$

(forward wave)

(*S* standing for the field amplitude of the forward wave)

Similarly for a backward wave: $H_z = S' \exp(j\beta z) \cos\left(\frac{\pi}{a} x\right)$

(*S*′ standing for the field amplitude of the backward wave)

Combining the contribution of the forward and backward waves:

$$H_z = [S \exp(-j\beta z)) + S' \exp(j\beta z)] \cos\left(\frac{\pi}{a} x\right)$$

$$H_z = [S\exp(-j\beta z) + S'\exp(j\beta z)]\cos\left(\frac{\pi}{a}x\right)$$ (rewritten)

T20: Thank you!

We can establish a relation between *S* and *S*′ with the help of electromagnetic boundary condition at the conducting wall at *z* = 0

$\vec{a}_n =$ Unit vector at the conducting surface directed from the conducting wall to dielectric/free-space medium

$\vec{E}_2 =$ Electric field vector in dielectric/free-space medium

$$\vec{a}_n \times \vec{E}_2 = 0$$

(electromagnetic boundary condition recalled (see Chapters 7 and 9) at the interface between conducting and dielectric/free-space media)

$$\left.\begin{aligned}\vec{a}_n &= \vec{a}_z \\ \vec{E}_2 &= E_y\vec{a}_y\end{aligned}\right\}$$

At the waveguide wall *z* = 0

$$\vec{a}_z \times \vec{a}_y = -\vec{a}_x$$

$$\vec{a}_z \times E_y\vec{a}_y = 0 \quad (z=0)$$

$$-\vec{a}_x E_y = 0 \quad (z=0)$$

$$E_y\big|_{z=0} = 0$$

(for all values of *x*)

$$E_y = \frac{j\omega\mu_0}{k^2-\beta^2}\frac{\partial H_z}{\partial x}$$

(relation recalled from Chapter 9)

$$\frac{\partial H_z}{\partial x} = 0 \quad (z=0)$$

(for all values of *x*)

$$H_z = [S\exp(-j\beta z) + S'\exp(j\beta z)]\cos\left(\frac{\pi}{a}x\right)$$ (rewritten)

$$\frac{\partial H_z}{\partial x} = -\frac{\pi}{a}\sin\frac{\pi}{a}x[S\exp(-j\beta z) + S'\exp(j\beta z)] \longrightarrow \frac{\partial H_z}{\partial x} = 0 \quad (z = 0)$$ (recalled)

(for all values of *x*)

In view of $z = 0$

$$S + S' = 0 \longrightarrow S' = -S$$

$$H_z = [S\exp(-j\beta z) + S'\exp(j\beta z)]\cos\left(\frac{\pi}{a}x\right)$$

$$H_z = S[\exp(-j\beta z) - \exp(j\beta z)]\cos\left(\frac{\pi}{a}x\right)$$

$$\exp(\pm j\varphi) = \cos\varphi \pm j\sin\varphi$$

(trigonometrical relation)

$$H_0 = -2jS$$

$$H_z = H_0\cos\left(\frac{\pi}{a}x\right)\sin\beta z$$

$$H_z = H_0 \cos\left(\frac{\pi}{a}x\right)\sin\beta z$$ (rewritten) $\rightarrow$ $$\frac{\partial H_z}{\partial x} = -H_0\frac{\pi}{a}\sin\frac{\pi}{a}x\sin\beta z$$

$$E_y = \frac{j\omega\mu_0}{k^2-\beta^2}\frac{\partial H_z}{\partial x}$$

$$E_y = -\frac{\pi}{a}\frac{j\omega\mu_0}{k^2-\beta^2}H_0\sin\left(\frac{\pi}{a}x\right)\sin\beta z$$

$m = 1, n = 0$ (TE_{10} mode) $\rightarrow$ $$k_c = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2}$$ $\rightarrow$ $$k_c = \frac{\pi}{a}$$

$$k^2 - \beta^2 = k_c{}^2$$

$$k^2 - \beta^2 = (\pi/a)^2$$

Boundary condition at the conducting wall at $z = l$

$$E_y\big|_{z=l} = 0$$

(for all values of x)

$$E_y = -\frac{j\omega\mu_0 a}{\pi}H_0\sin\left(\frac{\pi}{a}x\right)\sin\beta z\exp(j\omega t)$$

$$\sin\beta l = 0 \longrightarrow \beta l = p\pi \quad (p = 1, 2, 3, \ldots)$$

$$\sin \beta l = 0 \longrightarrow \beta l = p\pi \quad (p = 1, 2, 3, \ldots)$$ (rewritten)

The condition exactly agrees to what has been derived earlier using transmission line theory

$$\beta = \frac{p\pi}{l} \quad (p = 1, 2, 3, \ldots.)$$

$$l = \frac{p\pi}{\beta} \quad (p = 1, 2, 3, \ldots.)$$

$$\beta = 2\pi / \lambda_g$$ (phase propagation β in terms of guide wavelength λ_g)

$$l = p\frac{\lambda_g}{2} \quad (p = 1, 2, 3, \ldots.)$$

(length of the resonator excited in TE_{10p} mode)

$$H_z = H_0 \cos\left(\frac{\pi}{a}x\right)\sin\beta z$$

$$H_z = H_0 \cos\left(\frac{\pi}{a}x\right)\sin(\frac{p\pi}{l})z$$

(TE_{10p} mode)

$$E_y = -\frac{j\omega\mu_0 a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin\beta z$$

$$E_y = -\frac{j\omega\mu_0 a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin(\frac{p\pi}{l})z$$

(TE_{10p} mode)

$$E_y = -\frac{j\omega\mu_0 a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin(\frac{p\pi}{l})z$$ (TE_{10p} mode) (rewritten)

↓

$$\frac{\partial E_y}{\partial z} = -\frac{j\omega\mu_0 a}{\pi}\frac{p\pi}{l} H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{p\pi}{l}z\right)$$

↓ ← Substituting in the following Maxwell's equation recalled from Chapter 9

$$\frac{\partial E_y}{\partial z} = j\omega\mu_0 H_x$$ we get → $$H_x = -p\frac{a}{l} H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{p\pi}{l}z\right)$$

(TE_{10p} mode)

Putting, in the field expressions deduced, *p* = 1 and invoking time dependence: $\exp(j\omega t)$ →

$$\left.\begin{aligned} H_z &= H_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t) \\ E_y &= -\frac{j\omega\mu_0 a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t) \\ H_x &= -\frac{a}{l} H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{l}z\right)\exp(j\omega t) \end{aligned}\right\}$$

(TE_{101}-mode field expressions of a rectangular waveguide resonator)

Quality factor of a waveguide resonator

Energy stored (time-averaged) in electric field:

$$W_E = \frac{1}{2}\varepsilon_0 \oint_\tau \vec{E}.\vec{E}\, d\tau$$ (static electric field energy) (recalled from Chapter 8)

↓ ← Time averaging for time-varying fields

$$\left.\begin{aligned} H_z &= H_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t) \\ E_y &= -\frac{j\omega\mu_0 a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t) \\ H_x &= -\frac{a}{l} H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{l}z\right)\exp(j\omega t) \end{aligned}\right\}$$

(TE_{101} mode)

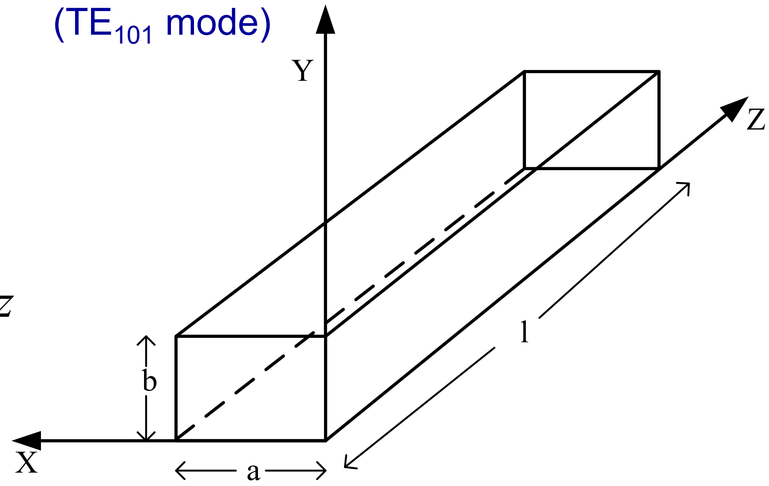
$$W_E = \frac{1}{2}\varepsilon_0 (\oint_\tau \vec{E}.\vec{E})_{\text{time-average}} d\tau = \frac{1}{2}\varepsilon_0 (\oint_\tau E_y \vec{a}_y . E_y \vec{a}_y)_{\text{time-average}} d\tau$$

$$= \frac{1}{2}\varepsilon_0 (\oint_\tau E_y^{\ 2})_{\text{time-average}} d\tau$$

$$W_E = \frac{1}{2}\varepsilon_0 (\oint_\tau E_y{}^2)_{\text{time-average}} d\tau$$ (rewritten)

$$d\tau = dxdydz$$

$$E_y = -\frac{j\omega\mu_0 a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t)$$ (TE_{101} mode)



← An extra factor of 1/2 introduced for time averaging

$$W_E = \frac{\mu_0{}^2\varepsilon_0\omega^2 a^2}{4\pi^2} H_0{}^2 \int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx \int_0^b dy \int_0^l \sin^2\left(\frac{\pi}{l}z\right)dz$$

← Making use of relation: $\sin^2\varphi = \frac{1}{2} - \frac{\cos 2\varphi}{2}$

← Evaluating the integral

← Making use of relations: $\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx = \frac{a}{2}$, $\int_0^b dy = b$ and $\int_0^l \sin^2\left(\frac{\pi}{a}x\right)dx = \frac{l}{2}$

← $\omega = 2\pi f$

$$W_E = \frac{1}{4}\mu_0{}^2\varepsilon_0 a^3 bl f^2 H_0{}^2$$ (TE_{101} mode)

(average energy stored in electric field)

Energy stored (time-averaged) in magnetic field:

$$W_B = \frac{1}{2}\mu_0 \oint_\tau \vec{H}.\vec{H}\, d\tau$$

(static magnetic field energy) (recalled from Chapter 8)

← Time averaging for time-varying fields

$$\left.\begin{aligned} H_z &= H_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t) \\ E_y &= -\frac{j\omega\mu_0 a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t) \\ H_x &= -\frac{a}{l} H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{l}z\right)\exp(j\omega t) \end{aligned}\right\}$$

(TE_{101} mode)

$$W_B = \left(\frac{1}{2}\mu_0 \oint_\tau (H_x\vec{a}_x + H_y\vec{a}_y)\cdot(H_x\vec{a}_x + H_y\vec{a}_y)\, d\tau\right)_{\text{time-average}}$$

$$W_B = \left(\frac{1}{2}\mu_0 \oint_\tau (H_x\vec{a}_x + H_y\vec{a}_y)\cdot(H_x\vec{a}_x + H_y\vec{a}_y)\, d\tau\right)_{\text{time-average}}$$

$$= \frac{1}{2}\mu_0 \oint_\tau (H_x{}^2 + H_z{}^2)_{\text{time-average}}\, d\tau$$

$$d\tau = dxdydz$$

← An extra factor of 1/2 introduced for time averaging

$$W_B = \frac{1}{4}\mu_0 H_0{}^2 \left[\left(\frac{a^2}{l^2}\right)\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\int_0^b dy\int_0^l \cos^2\left(\frac{\pi}{l}z\right)dz + \int_0^a \cos^2\left(\frac{\pi}{a}x\right)dx\int_0^b dy\int_0^l \sin^2\left(\frac{\pi}{l}z\right)dz\right]$$

$$W_B = \frac{1}{4}\mu_0 {H_0}^2 \left[\left(\frac{a^2}{l^2}\right) \int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx \int_0^b dy \int_0^l \cos^2\left(\frac{\pi}{l}z\right)dz + \int_0^a \cos^2\left(\frac{\pi}{a}x\right)dx \int_0^b dy \int_0^l \sin^2\left(\frac{\pi}{l}z\right)dz \right]$$

(rewritten) (TE$_{101}$ mode)

← Making use of relation: $\sin^2\varphi = \frac{1}{2} - \frac{\cos 2\varphi}{2}, \quad \cos^2\varphi = \frac{1}{2} + \frac{\cos 2\varphi}{2}$

← Making use of relations: $\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx = \frac{a}{2}, \ \int_0^b dy = b, \ \int_0^l \cos^2\left(\frac{\pi}{a}x\right)dx = \frac{l}{2}, \int_0^l \sin^2\left(\frac{\pi}{a}x\right)dx = \frac{l}{2}$

$$W_B = \frac{\mu_0}{16} abl\left(\frac{a^2}{l^2} + 1\right){H_0}^2$$

$$W_E = \frac{1}{4}{\mu_0}^2 \varepsilon_0 a^3 bl f^2 {H_0}^2 \quad \leftarrow \quad f = f_r$$

(recalled)

$$W_E = \frac{1}{4}{\mu_0}^2 \varepsilon_0 a^3 bl {f_r}^2 {H_0}^2$$

$$f_r = \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{l^2}\right)^{1/2} c = \frac{1}{2(\mu_0\varepsilon_0)^{1/2}}\left(\frac{1}{a^2} + \frac{1}{l^2}\right)^{1/2}$$

(resonant frequency of TE$_{101}$ mode)
(recalled)

$$W_E = \frac{\mu_0}{16} abl\left(\frac{a^2}{l^2} + 1\right){H_0}^2$$

$$W_B = W_E = \frac{\mu_0}{16} abl\left(1 + \frac{a^2}{l^2}\right){H_0}^2$$

(average energies stored in electric and magnetic fields are equal)

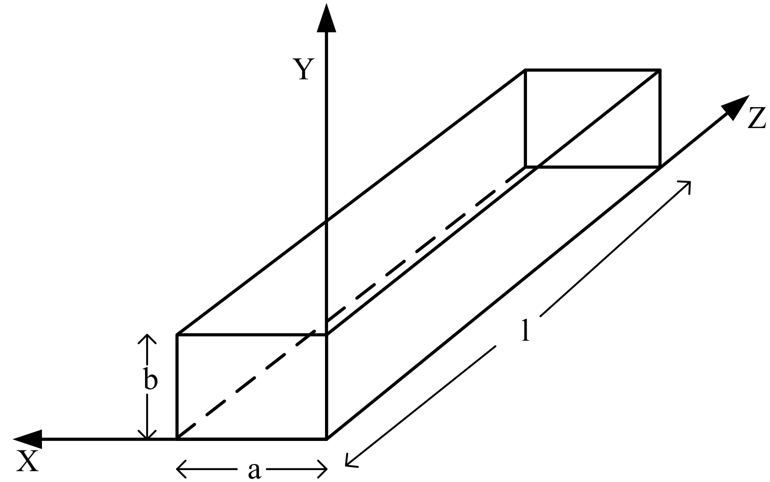
Total energy (time-averaged) stored in a resonant cavity

$$W_E = W_B = \frac{\mu_0}{16} abl\left(1+\frac{a^2}{l^2}\right)H_0^{\,2} \quad \text{(rewritten)}$$

↓

$$W = W_E + W_B = 2W_E = 2W_B = \frac{\mu_0}{8} abl\left(1+\frac{a^2}{l^2}\right)H_0^{\,2}$$

(total time-averaged energy stored in resonant cavity)



Power dissipated in the conducting walls of a resonant cavity

In finding the power dissipated in the conducting walls of a rectangular waveguide resonator, we can use the same approach as followed in finding the power dissipated in the conducting walls of a rectangular waveguide in Chapter 9.

$$P_{\text{LA}} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^{\,*}$$

(power loss in a conducting wall in terms of the surface current density developed at surface of the conducting wall and surface resistance)

$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$$

(boundary condition at a conducting wall)

↑

$\vec{H}_2 =$ magnetic field inside waveguide

$\vec{a}_n =$ unit vector at the conducting wall →

$$\left.\begin{aligned}
\vec{a}_n\big|_{x=0} &= \vec{a}_x \quad \text{(right wall)}\\
\vec{a}_n\big|_{x=a} &= -\vec{a}_x \quad \text{(left wall)}\\
\vec{a}_n\big|_{y=0} &= \vec{a}_y \quad \text{(bottom wall)}\\
\vec{a}_n\big|_{y=b} &= -\vec{a}_y \quad \text{(top wall)}\\
\vec{a}_n\big|_{z=0} &= \vec{a}_z \quad \text{(front wall)}\\
\vec{a}_n\big|_{z=l} &= -\vec{a}_z \quad \text{(back wall)}
\end{aligned}\right\}$$

Magnetic field has two components

$$H_z = H_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t)$$

$$E_y = -\frac{j\omega\mu_0 a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t)$$

$$H_x = -\frac{a}{l} H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{l}z\right)\exp(j\omega t)$$

(recalled)

(TE$_{101}$-mode field expressions of a rectangular waveguide resonator)

$$\vec{H}_2 = H_x\vec{a}_x + H_z\vec{a}_z$$

$$\vec{a}_n \times \vec{H}_2 = \vec{J}_s$$

(boundary condition)

$$\vec{J}_s\Big|_{x=0} = \vec{a}_x \times (H_x\vec{a}_x + H_z\vec{a}_z) = -H_z\vec{a}_y \qquad \text{(right wall)}$$

$$\vec{J}_s\Big|_{x=a} = -\vec{a}_x \times (H_x\vec{a}_x + H_z\vec{a}_z) = H_z\vec{a}_y \qquad \text{(left wall)}$$

$$\vec{J}_s\Big|_{y=0} = \vec{a}_y \times (H_x\vec{a}_x + H_z\vec{a}_z) = -H_x\vec{a}_z + H_z\vec{a}_x \qquad \text{(bottom wall)}$$

$$\vec{J}_s\Big|_{y=b} = -\vec{a}_y \times (H_x\vec{a}_x + H_z\vec{a}_z) = H_x\vec{a}_z - H_z\vec{a}_x \qquad \text{(top wall)}$$

$$\vec{J}_s\Big|_{z=0} = \vec{a}_z \times (H_x\vec{a}_x + H_z\vec{a}_z) = H_x\vec{a}_y \qquad \text{(front wall)}$$

$$\vec{J}_s\Big|_{z=l} = -\vec{a}_z \times (H_x\vec{a}_x + H_z\vec{a}_z) = -H_x\vec{a}_y \qquad \text{(backwall)}$$

(wall surface current densities in terms of magnetic field components)

$$\left.\begin{aligned} H_z &= H_0 \sin\left(\frac{\pi}{l} z\right)\exp(j\omega t) \\ H_x &= 0 \end{aligned}\right\} \quad \text{(right wall; } x = 0)$$

$$\left.\begin{aligned} H_z &= -H_0 \sin\left(\frac{\pi}{l} z\right)\exp(j\omega t) \\ H_x &= 0 \end{aligned}\right\} \quad \text{(left wall; } x = a)$$

$$\left.\begin{aligned} H_z &= H_0 \cos\left(\frac{\pi}{a} x\right)\sin\left(\frac{\pi}{l} z\right)\exp(j\omega t) \\ H_x &= -\frac{a}{l} H_0 \sin\left(\frac{\pi}{a} x\right)\cos\left(\frac{\pi}{l} z\right)\exp(j\omega t) \end{aligned}\right\} \quad \text{(bottom wall; } y = 0)$$

$$\left.\begin{aligned} H_z &= H_0 \cos\left(\frac{\pi}{a} x\right)\sin\left(\frac{\pi}{l} z\right)\exp(j\omega t) \\ H_x &= -\frac{a}{l} H_0 \sin\left(\frac{\pi}{a} x\right)\cos\left(\frac{\pi}{l} z\right)\exp(j\omega t) \end{aligned}\right\} \quad \text{(top wall; } y = b)$$

$$\left.\begin{aligned} H_z &= 0 \\ H_x &= -\frac{a}{l} H_0 \sin\left(\frac{\pi}{a} x\right)\exp(j\omega t) \end{aligned}\right\} \quad \text{(front wall; } z = 0)$$

$$\left.\begin{aligned} H_z &= 0 \\ H_x &= \frac{a}{l} H_0 \sin\left(\frac{\pi}{a} x\right)\exp(j\omega t) \end{aligned}\right\} \quad \text{(back wall; } z = l)$$

(magnetic field components to be put in the expressions for surface current densities at resonator walls)

We have obtained immediately above

(i) the expressions for the magnetic field components and

(ii) the expressions for the surface current densities in terms of magnetic field components both at each of the waveguide resonator walls.

We can substitute (i) in to (ii).

$$\vec{J}_s\Big|_{x=0} = -H_0 \sin\left(\frac{\pi}{l}z\right)\exp(j\omega t)\vec{a}_y \qquad \text{(right wall)}$$

$$\vec{J}_s\Big|_{x=a} = -H_0 \sin\left(\frac{\pi}{l}z\right)\exp(j\omega t)\vec{a}_y \qquad \text{(left wall)}$$

$$\vec{J}_s\Big|_{y=0} = \frac{a}{l}H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{l}z\right)\exp(j\omega t)\vec{a}_z + H_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t)\vec{a}_x \qquad \text{(bottom wall)}$$

$$\vec{J}_s\Big|_{y=b} = -\frac{a}{l}H_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{l}z\right)\exp(j\omega t)\vec{a}_z - H_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{l}z\right)\exp(j\omega t)\vec{a}_x \qquad \text{(top wall)}$$

$$\vec{J}_s\Big|_{z=0} = -\frac{a}{l}H_0 \sin\left(\frac{\pi}{a}x\right)\exp(j\omega t)\vec{a}_y \qquad \text{(front wall)}$$

$$\vec{J}_s\Big|_{z=l} = -\frac{a}{l}H_0 \sin\left(\frac{\pi}{a}x\right)\exp(j\omega t)\vec{a}_y \qquad \text{(backwall)}$$

(wall surface current densities to be used later for deriving the expressions for power lost per unit area at resonator walls)

Let us next proceed to derive the expressions for power loss per unit area at resonator walls.

In order to derive the expressions for power loss or dissipated per unit area at resonator walls, let us recall the expression for power loss per unit area at a conducting surface P_{LA} in terms of the surface resistance and surface current density deduced in Chapter 8 and already used in Chapter 9 with respect to waveguide walls.

↓

$$P_{\mathrm{LA}} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^* \quad \text{(recalled from Chapter 9)}$$

The expression for $\vec{J}_s^*$ can be obtained from that of $\vec{J}_s$ already obtained earlier simply by replacing the factor exp($j\omega t$) with exp–($j\omega t$). Hence we obtain

$$(\vec{J}_s \cdot \vec{J}_s^*)\Big|_{x=0} (\text{right wall}) = (\vec{J}_s \cdot \vec{J}_s^*)\Big|_{x=a} (\text{left wall})$$

$$= H_0^{\,2} \sin^2\left(\frac{\pi}{l} z\right)$$

$$(\vec{J}_s \cdot \vec{J}_s^*)\Big|_{y=0} (\text{bottom wall}) = (\vec{J}_s \cdot \vec{J}_s^*)\Big|_{y=b} (\text{top wall})$$

$$= \left(\frac{a}{l}\right)^2 H_0^{\,2} \sin^2\left(\frac{\pi}{a} x\right) \cos^2\left(\frac{\pi}{l} z\right) + H_0^{\,2} \cos^2\left(\frac{\pi}{a} x\right) \sin^2\left(\frac{\pi}{l} z\right)$$

$$(\vec{J}_s \cdot \vec{J}_s^*)\Big|_{z=0} (\text{front wall}) = (\vec{J}_s \cdot \vec{J}_s^*)\Big|_{z=l} (\text{back wall})$$

$$= \left(\frac{a}{l}\right)^2 H_0^{\,2} \sin^2\left(\frac{\pi}{a} x\right)$$

(expressions to be substituted in the expressions for power loss per unit area at the surfaces of the waveguide walls)

Substitute the expression for $\vec{J}_s \cdot \vec{J}_s^*$ already derived in the expression for power loss per unit area

↓

$$P_{\text{LA}} = \frac{1}{2} R_S \vec{J}_s \cdot \vec{J}_s^* \quad \text{(recalled)}$$

↓

$$P_{\text{LA}}(\text{right wall}) = P_{\text{LA}}(\text{left wall})$$
$$= \frac{1}{2} R_S H_0^{\,2} \sin^2\left(\frac{\pi}{l} z\right)$$
$$P_{\text{LA}}(\text{bottom wall}) = P_{\text{LA}}(\text{top wall})$$
$$= \frac{1}{2} R_S \left[\left(\frac{a}{l}\right)^2 H_0^{\,2} \sin^2\left(\frac{\pi}{a} x\right) \cos^2\left(\frac{\pi}{l} z\right) + H_0^{\,2} \cos^2\left(\frac{\pi}{a} x\right) \sin^2\left(\frac{\pi}{l} z\right) \right]$$
$$P_{\text{LA}}(\text{front wall}) = P_{\text{LA}}(\text{back wall})$$
$$= \frac{1}{2} R_S \left(\frac{a}{l}\right)^2 H_0^{\,2} \sin^2\left(\frac{\pi}{a} x\right)$$

(power loss per unit area at resonator walls)

Next, we are going to use these expressions for power loss per unit area at the surfaces of the resonator walls to find the power loss over the entire wall surfaces.

Let us now use these expressions for power loss per unit area P_{LA} at the surfaces of the resonator walls to find the power loss over the entire wall surfaces.

For this purpose, with reference to a resonator wall, with the help the expression for P_{LA} , let us find the power loss over an infinitesimal area of the wall and then integrate it to find the power loss P_L over the entire area of the wall.

$$P_{\mathrm{LA}}(\text{right wall}) = \frac{1}{2} R_S H_0^{\,2} \sin^2\left(\frac{\pi}{l} z\right)$$

↓

$$P_{\mathrm{L}}(\text{right wall}) = \int_{\text{right wall}} P_{\mathrm{LA}}(\text{right wall})\, dy dz$$

↓

$$P_{\mathrm{L}}(\text{right wall}) = \frac{1}{2} R_S H_0^{\,2} \int_0^b dy \int_0^l \sin^2\left(\frac{\pi}{l} z\right) dz$$

← Making use of the relation:

$$\int_0^b dy = b, \int_0^l \sin^2\left(\frac{\pi}{a} x\right) dx = \frac{l}{2}$$

$$P_{\mathrm{L}}(\text{right wall}) = \frac{1}{2} R_S H_0^{\,2} (b)\left(\frac{l}{2}\right)$$

$$P_{\mathrm{LA}}(\text{left wall}) = \frac{1}{2} R_S H_0^{\,2} \sin^2\left(\frac{\pi}{l} z\right)$$

↓

$$P_{\mathrm{L}}(\text{left wall}) = \int_{\text{right wall}} P_{\mathrm{LA}}(\text{left wall})\, dy dz$$

↓

$$P_{\mathrm{L}}(\text{left wall}) = \frac{1}{2} R_S H_0^{\,2} \int_0^b dy \int_0^l \sin^2\left(\frac{\pi}{l} z\right) dz$$

← Making use of the relation:

$$\int_0^b dy = b, \int_0^l \sin^2\left(\frac{\pi}{a} x\right) dx = \frac{l}{2}$$

$$P_{\mathrm{L}}(\text{left wall}) = \frac{1}{2} R_S H_0^{\,2} (b)\left(\frac{l}{2}\right)$$

$$P_{\mathrm{LA}}(\text{top wall}) = \frac{1}{2}R_S\left[\left(\frac{a}{l}\right)^2 H_0^{\,2}\sin^2\left(\frac{\pi}{a}x\right)\cos^2\left(\frac{\pi}{l}z\right) + H_0^{\,2}\cos^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{l}z\right)\right]$$

↓

$$P_{\mathrm{L}}(\text{top wall}) = \int_{\text{bottom wall}} P_{\mathrm{LA}}(\text{top wall})\,dxdz$$

↓

$$P_{\mathrm{L}}(\text{top wall}) = \frac{1}{2}R_S H_0^{\,2}\left[\left(\frac{a}{l}\right)^2 \int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx \int_0^l \cos^2\left(\frac{\pi}{l}z\right)dz + \int_0^a \cos^2\left(\frac{\pi}{a}x\right)dx \int_0^l \sin^2\left(\frac{\pi}{l}z\right)dz\right]$$

↓ ← Using integral values as done earlier

$$P_{\mathrm{L}}(\text{top wall}) = \frac{1}{2}R_S H_0^{\,2}\left(\frac{a}{l}\right)\left(\frac{l}{2}\right)\left[\left(\frac{a}{l}\right)^2 + 1\right]$$

$$P_{\mathrm{LA}}(\text{bottom wall}) = \frac{1}{2}R_S\left[\left(\frac{a}{l}\right)^2 H_0^{\,2}\sin^2\left(\frac{\pi}{a}x\right)\cos^2\left(\frac{\pi}{l}z\right) + H_0^{\,2}\cos^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{l}z\right)\right]$$

↓

$$P_{\mathrm{L}}(\text{bottom wall}) = \int_{\text{bottom wall}} P_{\mathrm{LA}}(\text{bottom wall})\,dxdz$$

↓

$$P_{\mathrm{L}}(\text{bottom wall}) = \frac{1}{2}R_S H_0^{\,2}\left[\left(\frac{a}{l}\right)^2 \int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx \int_0^l \cos^2\left(\frac{\pi}{l}z\right)dz + \int_0^a \cos^2\left(\frac{\pi}{a}x\right)dx \int_0^l \sin^2\left(\frac{\pi}{l}z\right)dz\right]$$

↓ ← Using integral values as done earlier

$$P_{\mathrm{L}}(\text{bottom wall}) = \frac{1}{2}R_S H_0^{\,2}\left(\frac{a}{l}\right)\left(\frac{l}{2}\right)\left[\left(\frac{a}{l}\right)^2 + 1\right]$$

$$P_{LA}(\text{front wall}) = \frac{1}{2} R_S \left(\frac{a}{l}\right)^2 H_0^{\,2} \sin^2\left(\frac{\pi}{a}x\right)$$

$$P_{L}(\text{front wall}) = \int_{\text{frontwall}} P_{LA}(\text{front wall})\,dxdy$$

$$P_{L}(\text{front wall}) = \frac{1}{2} R_S H_0^{\,2} \left(\frac{a}{l}\right)^2 \int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx \int_0^b dy$$

← Using integral values as done earlier

$$P_{L}(\text{front wall}) = \frac{1}{2} R_S H_0^{\,2} \left(\frac{a}{l}\right)^2 \left(\frac{a}{l}\right)(b)$$

$$P_{LA}(\text{back wall}) = \frac{1}{2} R_S \left(\frac{a}{l}\right)^2 H_0^{\,2} \sin^2\left(\frac{\pi}{a}x\right)$$

$$P_{L}(\text{back wall}) = \int_{\text{frontwall}} P_{LA}(\text{back wall})\,dxdy$$

$$P_{L}(\text{backwall}) = \frac{1}{2} R_S H_0^{\,2} \left(\frac{a}{l}\right)^2 \int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx \int_0^b dy$$

← Using integral values as done earlier

$$P_{L}(\text{back wall}) = \frac{1}{2} R_S H_0^{\,2} \left(\frac{a}{l}\right)^2 \left(\frac{a}{l}\right)(b)$$

$$P_L = P_L(\text{right wall}) + P_L(\text{left wall}) + P_L(\text{top wall}) + P_L(\text{bottom wall}) + P_L(\text{front wall}) + P_L(\text{back wall})$$

(power losses in all the six walls of a rectangular waveguide resonator summed up)

$$P_L = R_S H_0^{\,2} \left(\frac{a}{2}\right)\left(\frac{a^2}{l}\left(\frac{b}{l}+\frac{1}{2}\right) + l\left(\frac{b}{a}+\frac{1}{2}\right)\right)$$

(TE_{101} mode)

$$W = W_E + W_B = 2W_E = 2W_B = \frac{\mu_0}{8} abl\left(1 + \frac{a^2}{l^2}\right) H_0^{\ 2}$$

(total time-averaged energy stored in resonant cavity)

$$P_L = R_S H_0^{\ 2}\left(\frac{a}{2}\right)\left(\frac{a^2}{l}\left(\frac{b}{l} + \frac{1}{2}\right) + l\left(\frac{b}{a} + \frac{1}{2}\right)\right)$$

(power losses in all the six walls of a rectangular waveguide resonator summed up)

Let us see how these two expressions derived for a rectangular waveguide resonator excited in TE101 mode can be used to obtain an expression for the quality factor *Q* of the resonator defined as follows.

W = time-averaged energy stored

$W_L\big|_{T=\omega/2\pi}$ = energy lost per cycle, that is, in the wave time period $T = 2\pi/\omega$

$$Q = 2\pi \frac{W}{W_L\big|_{T=\omega/2\pi}}$$

$$W_L\big|_{T=2\pi/\omega} = P_L T = P_L \frac{2\pi}{\omega}$$

$$T = \frac{2\pi}{\omega}$$

$$Q\left(= 2\pi \frac{W}{P_L \frac{2\pi}{\omega}}\right) = \omega \frac{W}{P_L}$$

$$W = \frac{\mu_0}{8} abl\left(1+\frac{a^2}{l^2}\right){H_0}^2$$

(recalled)

$$P_{\mathrm{L}} = R_S {H_0}^2\left(\frac{a}{2}\right)\left(\frac{a^2}{l}\left(\frac{b}{l}+\frac{1}{2}\right)+l\left(\frac{b}{a}+\frac{1}{2}\right)\right)$$

(recalled)

$$Q\left(=2\pi\frac{W}{P_L\dfrac{2\pi}{\omega}}\right)=\omega\frac{W}{P_L}$$

$$Q\left(=\omega\frac{W}{P_L}\right)=\omega\frac{\dfrac{\mu_0}{8}abl\left(1+\dfrac{a^2}{l^2}\right)}{R_S\left(\dfrac{a}{2}\right)\left(\dfrac{a^2}{l}\left(\dfrac{b}{l}+\dfrac{1}{2}\right)+l\left(\dfrac{b}{a}+\dfrac{1}{2}\right)\right)}$$

(TE_{101} mode)

$$Q\left(=\omega\frac{W}{P_L}\right)=\omega\frac{\frac{\mu_0}{8}abl\left(1+\frac{a^2}{l^2}\right)}{R_S\left(\frac{a}{2}\right)\left(\frac{a^2}{l}\left(\frac{b}{l}+\frac{1}{2}\right)+l\left(\frac{b}{a}+\frac{1}{2}\right)\right)}$$

(TE_{101} mode) (re-written)

$$f_r=\frac{\omega_r}{2\pi}=\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{l^2}\right)^{1/2}c$$

(TE_{101} mode) (recalled)

← Interpreted at resonant frequency $\omega=\omega_r=2\pi f_r$

← $c=1/(\mu_0\varepsilon_0)^{1/2}$

$$Q=\omega_r\frac{\frac{\mu_0}{8}abl\left(1+\frac{a^2}{l^2}\right)}{R_S\left(\frac{a}{2}\right)\left(\frac{a^2}{l}\left(\frac{b}{l}+\frac{1}{2}\right)+l\left(\frac{b}{a}+\frac{1}{2}\right)\right)}$$

← $\omega_r=2\pi f_r=\frac{\pi}{(\mu_0\varepsilon_0)^{1/2}}\left(\frac{1}{a^2}+\frac{1}{l^2}\right)^{1/2}$

$$Q=\frac{\pi}{(\mu_0\varepsilon_0)^{1/2}}\left(\frac{1}{a^2}+\frac{1}{l^2}\right)^{1/2}\frac{\frac{\mu_0}{8}abl\left(1+\frac{a^2}{l^2}\right)}{R_S\left(\frac{a}{2}\right)\left(\frac{a^2}{l}\left(\frac{b}{l}+\frac{1}{2}\right)+l\left(\frac{b}{a}+\frac{1}{2}\right)\right)}$$

← $\eta_0=(\mu_0/\varepsilon_0)^{1/2}$

$$Q=\frac{\pi\eta_0}{4R_S}\frac{2b(a^2+l^2)^{3/2}}{al(a^2+l^2)+2b(a^3+l^3)}$$

(quality factor of a rectangular waveguide resonator)

(TE_{101} mode)

$$Q = \frac{\pi\eta_0}{4R_S}\frac{2b(a^2+l^2)^{3/2}}{al(a^2+l^2)+2b(a^3+l^3)}$$

(TE$_{101}$ mode) (rewritten)

$$R_S = \frac{1}{\sigma\delta} = \left(\frac{\pi f_r \mu_0}{\sigma}\right)^{1/2} \qquad \delta = \sqrt{\frac{1}{\pi f \mu_0 \sigma}}$$

(both recalled from Chapter 6 for a good conductor at relatively high frequencies: $f = f_r$)

$$f_r = \frac{\omega_r}{2\pi} = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{l^2}\right)^{1/2} c$$

(TE$_{101}$ mode) (recalled)

The quality factor of a waveguide resonator depends on the dimensions of the resonator and on the conductivity of the material used in making it.

Loaded quality factor

In an actual application, some part of energy stored in a cavity is coupled out from the cavity to an external load and the cavity thereby becomes loaded. For such a loaded cavity, the power lost from the loaded cavity $P_{L,\text{loaded}}$ consists of

(i) the ohmic power loss $P_{L,\text{ohmic}}$ due the finite conductivity of the material making the cavity and

(ii) the power $P_{L,\text{loaded}}$ that couples out from the cavity to the load.

$$P_{L,\text{loaded}} = P_{L,\text{ohmic}} + P_{L,\text{ext}}$$

$$P_{L,\text{loaded}} = P_{L,\text{ohmic}} + P_{L,\text{ext}}$$ (rewritten)

$$Q = \omega \frac{W}{P_L}$$ (recalled)

$$P_{L,\text{ohmic}} = \frac{\omega W}{Q_{\text{unloaded}}}$$

$$P_{L,\text{ext}} = \frac{\omega W}{Q_{\text{ext}}}$$

$$P_{L,\text{loaded}} = \frac{\omega W}{Q_{\text{loaded}}}$$

(quality factors: unloaded Q_{unloaded}, external Q_{ext} and loaded Q_{loaded} related to ohmic power loss $P_{L,\text{ohmic}}$, power coupled out to the external load $P_{L,\text{ext}}$ and total power loss $P_{L,\text{loaded}}$ of the loaded cavity, respectively

$$\frac{\omega W}{Q_{\text{loaded}}} = \frac{\omega W}{Q_{\text{unloaded}}} + \frac{\omega W}{Q_{\text{ext}}}$$

$$\frac{1}{Q_{\text{loaded}}} = \frac{1}{Q_{\text{unloaded}}} + \frac{1}{Q_{\text{ext}}}$$

(relation between the loaded, unloaded and external quality factors)

Frequency response of equivalent impedance of the resonator

The waveguide resonator may be represented by a resonant circuit comprising an inductor of inductance L having reactance $j\omega L$, a capacitor of capacitance C having reactance $1/(j\omega C)$, and a resistor of resistance R, all in parallel.

In what follows next, let us find the frequency response of the impedance Z_{eq} of the L, C, R parallel resonant circuit equivalent to the waveguide resonator and hence relate its quality factor to the resonant frequency and the bandwidth $\Delta\omega$ of the resonator.

$$W = \frac{1}{2} C V^2$$ (total energy W stored in the circuit, taken here as the energy in the capacitor which is transferred back and forth between the inductor and the capacitor during each cycle)

$$W = P_L C R \longleftarrow W = \frac{1}{2} C V^2 \longleftarrow P_L = \frac{V^2}{2R}$$

(recalled)

(power loss in the resistor of the parallel equivalent circuit)

$$\downarrow$$

$$Q = \omega_r \frac{W}{P_L}$$

(recalled)

$$\downarrow$$

$$Q = \omega_r C R \longleftarrow C = \frac{1}{L\omega_r^{\ 2}} \longleftarrow \omega_r = \frac{1}{(LC)^{1/2}}$$

(resonant angular frequency of the circuit)

$$\downarrow$$

$$Q = \frac{R}{\omega_r L} \longrightarrow$$

An expression that is going to be put in the expression for the impedance Z_{eq} of the equivalent *L*, *C*, *R* parallel resonant circuit

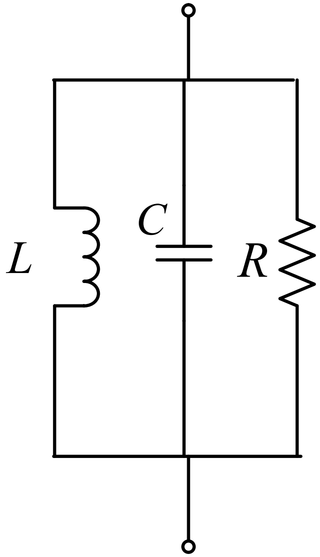
$$\downarrow$$

$$\frac{R}{L} = Q\omega_r$$

$$\frac{1}{Z_{\mathrm{eq}}} = \frac{1}{j\omega L} + \frac{1}{\dfrac{1}{j\omega C}} + \frac{1}{R}$$

$\downarrow$ ← After a little algebra

$$Z_{\mathrm{eq}} = \frac{R}{1 - jQ\dfrac{\omega_r}{\omega}(1 - \omega^2 L C)}$$



$$Z_{\text{eq}} = \frac{R}{1 - jQ\dfrac{\omega_r}{\omega}(1-\omega^2 LC)}$$ (rewritten) ← $$\omega_r = \frac{1}{(LC)^{1/2}}$$

↓

$$Z_{\text{eq}} = \frac{R}{1 - jQ\dfrac{\omega_r}{\omega}\left(1-\dfrac{\omega^2}{\omega_r{}^2}\right)} \longrightarrow Z_{\text{eq}} = \frac{R}{1 + j\dfrac{Q\omega_r}{\omega}\left(\dfrac{\omega^2}{\omega_r{}^2}-1\right)}$$

↓

At resonance $\omega = \omega r$, $Z_{\text{eq}} = R$ and we can normalize Z_{eq} with respect to R that is its value at resonance.

↓

$$\frac{Z_{\text{eq}}}{R} = \frac{1}{1 + j\dfrac{Q}{\omega}\dfrac{(\omega+\omega_r)(\omega-\omega_r)}{\omega_r}}$$

$$\frac{Z_{\text{eq}}}{R} = \frac{1}{1 + j\dfrac{Q}{\omega}\dfrac{(\omega+\omega_r)(\omega-\omega_r)}{\omega_r}}$$

(rewritten)

→ Normalized impedance magnitude and its square

$$\frac{|Z_{\text{eq}}|}{R} = \frac{1}{\left[1 + \left(\dfrac{Q}{\omega}\dfrac{(\omega+\omega_r)(\omega-\omega_r)}{\omega_r}\right)^2\right]^{1/2}}$$

$$\left(\frac{|Z_{\text{eq}}|}{R}\right)^2 = \frac{1}{1 + \left(\dfrac{Q}{\omega}\dfrac{(\omega+\omega_r)(\omega-\omega_r)}{\omega_r}\right)^2}$$

Impedance magnitude and voltage across the circuit

- maximum at resonance
- decreases both below and above resonance.

Circuit power is proportional to the square of voltage across the circuit

Voltage across the circuit is proportional to impedance

→ Circuit power is proportional to the square of impedance

$$\left(\frac{|Z_{\text{eq}}|}{R}\right)^2 = \frac{1}{1 + \left(\dfrac{Q}{\omega}\dfrac{(\omega+\omega_r)(\omega-\omega_r)}{\omega_r}\right)^2} = \frac{1}{2}$$

← Corresponds to half the maximum power at resonant frequency ω_r

$$\frac{1}{1+\left(\frac{Q}{\omega}\frac{(\omega+\omega_r)(\omega-\omega_r)}{\omega_r}\right)^2}=\frac{1}{2}$$ (corresponding to half the maximum power at resonant frequency ω_r)

(recalled)

We expect two solutions ω_1 and ω_2 of around the resonant ω_r.

$$\omega_1=\omega_r+\Delta\omega$$
$$\omega_2=\omega_r-\Delta\omega$$
$$(\Delta\omega<<\omega)$$

$$\omega=\omega_1=\omega_r+\Delta\omega$$ (first solution)

$$\left.\begin{array}{l}\omega_1\approx\omega_r\\ \omega_1+\omega_r\approx 2\omega_r\\ \omega_1-\omega_r\approx\Delta\omega\end{array}\right\}$$

$$\frac{4Q^2}{\omega_r^{\,2}}(\Delta\omega)^2=1$$ (from first solution)

$$\omega=\omega_2=\omega_r-\Delta\omega$$ (second solution)

$$\left.\begin{array}{l}\omega_2\approx\omega_r\\ \omega_2+\omega_r\approx 2\omega_r\\ \omega_2-\omega_r\approx-\Delta\omega\end{array}\right\}$$

$$\frac{4Q^2}{\omega_r^{\,2}}(\Delta\omega)^2=1$$ (from second solution) (relation identical to that obtained taking from first solution

$$\frac{4Q^2}{\omega_r^{\ 2}}(\Delta\omega)^2 = 1$$ (rewritten)

$$Q = \frac{\omega_r}{2\Delta\omega}$$

Interpreting $2\Delta\omega$ as the angular frequency bandwidth

$$Q = \frac{\omega_r}{\text{angular frequency bandwidth}}$$

In terms of circular frequency or simply frequency

$$Q = \frac{f_r}{\text{frequency bandwidth}}$$

Let us take an illustrative example of finding the dimension of a cubical cavity made of copper (σ = 5.8 × 10^8 mho/m) excited in the TE_{101} mode so that it resonates at 15 GHz. Calculate also its quality factor and bandwidth.

$c = 3\times10^8$ m/s (given)

m = 1, *n* = 0, *p* = 1 and *a* = *b* = *l* (given)

$$\omega_r = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{l}\right)^2\right]^{1/2} c$$

$$f_r = \frac{1}{\sqrt{2}\pi}\frac{\pi}{a}c$$

$$a = \frac{1}{\sqrt{2}}\frac{1}{f_r}c$$

$f_r = 15$ GHz $= 15\times10^9$ Hz (given)

$$a(=b=l) = \frac{1}{\sqrt{2}}\frac{1}{f_r}c = \frac{1}{\sqrt{2}}\frac{3\times10^8}{15\times10^9} = 1.414\times10^{-2}\ \text{m}$$

$a = b = l$ (given)

$$Q = \frac{\pi\eta_0}{4R_S}\frac{2b(a^2+l^2)^{3/2}}{al(a^2+l^2)+2b(a^3+l^3)}$$

(TE_{101} mode)

$$Q = \frac{\pi\eta_0}{4R_S}\frac{2a(a^2+a^2)^{3/2}}{aa(a^2+a^2)+2a(a^3+a^3)}$$

$f = f_r = 15\,\text{GHz} = 15\times10^9\,\text{Hz}$ (given)

Simplifies to

$$R_s = \sqrt{\frac{\pi f \mu_0}{\sigma}} = \sqrt{\frac{3.143\times15\times10^9\times4\pi\times10^{-7}}{5.8\times10^7}} = 3.19\times10^{-2}\,\text{ohm}$$

(see Chapter 6 for the expression for R_s used)

$$Q = \frac{\pi\eta_0}{4R_S}\frac{2a(a^2+a^2)^{3/2}}{aa(a^2+a^2)+2a(a^3+a^3)} = \frac{\pi\eta_0}{3\sqrt{2}R_S} = \frac{3.143\times377}{3\times1.414\times3.19\times10^{-2}} = 8756$$

$$Q = \frac{f_r}{\text{frequency bandwidth}}$$

(recalled)

$f = f_r = 15\,\text{GHz} = 15\times10^9\,\text{Hz}$ (given)

$$\text{frequency bandwidth} = \frac{f_r}{Q} = \frac{15\times10^9}{8756} = 1.713\times10^6\,\text{Hz} = 1.713\,\text{MHz}$$

Summarising Notes

√ Waveguide resonator at a defined resonant frequency can be made out of an appropriately chosen length of a waveguide with its both ends either closed by a conductor or kept open.

√ Transmission line theory is an easy approach of treating a waveguide resonator.

√ Basic concepts of transmission line theory have been developed such as

◊ distributed transmission line parameters;

◊ telegrapher's equation;

◊ condition for distortionless transmission;

◊ input impedance of the line terminated in a load impedance;

◊ characteristic impedance of the line;

◊ voltage standing-wave ratio (VSWR) of the line;

◊ Impedance matching such as in Radome for the protection of an antenna and branch-type radar duplexer of a radar system; and

◊ Smith chart: theory and application to transmission line problems to make them simpler.

√ Resonator length has been found by transmission line theory as an integral multiple of half the guide wavelength for both closed-ended and open-ended resonators.

√ Resonant frequency of the waveguide resonator can be found with the help of transmission line theory using the dispersion relation of the waveguide.

√ Field theory can be applied to treat a waveguide resonator as an alternative to transmission line theory.

√ Field solutions and electromagnetic boundary conditions typically for a rectangular waveguide closed-ended resonator has yielded

(i) the same resonator length as predicted by the transmission line theory and

(ii) an additional mode number *p* of the waveguide resonator, to be read with reference to TE_{10}-mode excitation of the waveguide as TE_{10p} mode of the resonator (which may be generalised as TE_{mnp} mode of the resonator with reference to TE_{mn}-mode excitation of the waveguide).

√ Field solution and relevant electromagnetic boundary conditions can be used to obtain

(i) the expression for the time-averaged energy stored in electric and magnetic fields and

(ii) the expression for the power loss in resonator walls.

√ Expression for the quality factor of a resonator in the TE_{101} mode at the resonant frequency, in terms of the resonator dimensions and the surface resistance of the conducting material making the resonator, has been derived with the help of the expressions for the time-averaged energy stored and power loss in the resonator.

√ Relation between the unloaded quality factor, external quality factor and loaded quality factor of a cavity has been obtained keeping in view some part of energy stored in a cavity being coupled out from it to an external load in practice.

√ Quality factor of a cavity resonator may also be expressed in terms of the resonant frequency and bandwidth of the frequency response of half the value of square of the ratio of the magnitude of the equivalent impedance of the resonant circuit comprising an inductor, capacitor and resistor in parallel.

Readers are encouraged to go through Chapter 10 of the book for more topics and more worked-out examples and review questions.